HOMEWORK 3 : ANSWERS

(a) In the figure below, $P$ stands for Profit-sharing contract and $R$ for Revenue-sharing contract. $Mi$ stands for Manager of firm $i$.

(b) Number the subgames games 1 to 4 from left to right.

**Subgame 1**: $q_1$ is chosen to maximize $\alpha \Pi_1(q_1,q_2) = \alpha \left(q_1(60-q_1-q_2)-12q_1\right)$ and $q_2$ is chosen to maximize $\alpha \Pi_2(q_1,q_2) = \alpha \left(q_2(60-q_1-q_2)-12q_2\right)$. Solving $\frac{\partial \Pi_1}{\partial q_1} = 0$ and $\frac{\partial \Pi_2}{\partial q_2} = 0$ gives $q_1 = q_2 = 16$. Player 1’s payoff is $(1-\alpha)\Pi_2(16,16) = (1-\alpha)256$ and the same is true for player 2.

**Subgame 2**: $q_1$ is chosen to maximize $\alpha \Pi_1(q_1,q_2) = \alpha \left(q_1(60-q_1-q_2)-12q_1\right)$ and $q_2$ is chosen to maximize $\alpha \Pi_2(q_1,q_2) = \alpha q_2(60-q_1-q_2)$ . Solving $\frac{\partial \Pi_1}{\partial q_1} = 0$ and $\frac{\partial \Pi_2}{\partial q_2} = 0$ gives $q_1 = 12$ and $q_2 = 24$. Player 1’s payoff is $(1-\alpha)\Pi_1(12,24) = (1-\alpha)144$ and player 2’s payoff is $\Pi_2(12,24) - \alpha R_2(12,24) = 288 - \alpha 576$.

**Subgame 3**: this is the same as subgame 2, with the roles reversed. Thus Player 1’s payoff is $288 - \alpha 576$ and Player 2’s payoff is $(1-\alpha)144$. (In this game $q_1$ is chosen to maximize $\alpha R_1(q_1,q_2) = \alpha q_1(60-q_1-q_2)$ and $q_2$ is chosen to maximize $\alpha \Pi_2(q_1,q_2) = \alpha \left(q_2(60-q_1-q_2)-12q_2\right)$). Solving $\frac{\partial R_1}{\partial q_1} = 0$ and $\frac{\partial \Pi_2}{\partial q_2} = 0$ gives $q_1 = 24$ and $q_2 = 12$. Player 1’s is $\Pi_1(24,12) - \alpha R_1(24,12) = 288 - \alpha 576$ and player 2’s payoff is $(1-\alpha)\Pi_2(24,12) = (1-\alpha)144$.)
Subgame 4: $q_1$ is chosen to maximize $\alpha R_1(q_1, q_2) = \alpha q_1(60-q_1-q_2)$ and $q_2$ is chosen to maximize $\alpha R_2(q_1, q_2) = \alpha q_2(60-q_1-q_2)$. Solving $\frac{\partial R_1}{\partial q_1} = 0$ and $\frac{\partial R_2}{\partial q_2} = 0$ gives $q_1 = 20$ and $q_2 = 20$. Player 1’s is $\Pi_1(20,20) - \alpha R_1(20,20) = 160 - \alpha 400$ and similarly for Player 2.

Thus the game reduces to:

![Game Diagram]

The strategic form is

<table>
<thead>
<tr>
<th>Profit contract</th>
<th>Revenue contract</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1-\alpha)256, (1-\alpha)256$</td>
<td>$(1-\alpha)144, 288-\alpha576$</td>
</tr>
<tr>
<td>$288-\alpha576, (1-\alpha)144$</td>
<td>$160-\alpha400, 160-\alpha400$</td>
</tr>
</tbody>
</table>

The Nash equilibria of this game are as follows:

1. If $\alpha < \frac{1}{16}$ then $R$ (a Revenue contract) is a strictly dominant strategy for each player and thus $(R,R)$ is the only Nash equilibrium.
2. If $\alpha = \frac{1}{16}$ then $R$ is a weakly dominant strategy for each player; there are 3 Nash equilibria: $(R,R), (R,P)$ and $(P,R)$.
3. If $\frac{1}{16} < \alpha < \frac{1}{10}$ there are two Nash equilibria: $(R,P)$ and $(P,R)$.
4. If $\alpha = \frac{1}{10}$ then $P$ is a weakly dominant strategy for each player; there are 3 Nash equilibria: $(P,P), (R,P)$ and $(P,R)$.
5. If $\alpha > \frac{1}{10}$ then $P$ is a strictly dominant strategy for each player and thus $(P,P)$ is the only Nash equilibrium.

(c) From the calculations for Subgame 1, we get that in the past each firm had a profit of 256. When $\alpha$ is small, the only equilibrium involves a revenue contract for each manager, yielding an income of at most 160 for each owner. Thus delegation has reduced the owners’ incomes. This is a Prisoners’ Dilemma situation: when one of the firms delegates with a revenue contract then the other must too, giving rise to a Pareto inefficient situation (from the point of view of the owners only).