HOMEWORK 6 : ANSWERS

(a) Suppose there is a signaling equilibrium where type L chooses education level e_L and is paid θ_L and type H chooses education level e_H and is paid θ_H, with $e_L \neq e_H$. Then the following incentive compatibility constraints must be satisfied (the first is type L's and the second type H's):

$$\theta_L - e_L\theta_L \geq \theta_H - e_H\theta_L \qquad (1)$$
$$\theta_H - e_H\theta_H \geq \theta_L - e_L\theta_H \qquad (2)$$

Adding (1) and (2) and simplifying we get that $e_L\left(\theta_H - \theta_L\right) \geq e_H\left(\theta_H - \theta_L\right)$ from which it follows (since $\theta_H - \theta_L > 0$ and $e_L \neq e_H$) that $e_L > e_H$. But then (1) is violated because $\theta_H - e_H\theta_L \underset{\text{since } e_L > e_H}{>} \theta_H - e_L\theta_L \underset{\text{since } \theta_L < \theta_H}{>} \theta_L - e_L\theta_L$.

(b) (b.1) Let us look for a pooling equilibrium where both types choose $e = e^*$. Let $\bar{\theta} = \mu_H\theta_H + (1-\mu_H)\theta_L$. The incentive compatibility constraints are:

$$\bar{\theta} - e^*\theta_L \geq \theta_L - e\theta_L,\ \forall e \in [0,\infty),\ e \neq e^* \qquad (1)$$
$$\bar{\theta} - e^*\theta_H \geq \theta_L - e\theta_H,\ \forall e \in [0,\infty),\ e \neq e^* \qquad (2)$$

Since the RHS of (1) and (2) is decreasing in e we can rewrite (1) as $\bar{\theta} - e^*\theta_L \geq \theta_L$ and (2) as $\bar{\theta} - e^*\theta_H \geq \theta_L$. If the latter is satisfied then so is the former, since $\theta_L < \theta_H$. Thus the necessary and sufficient condition for a pooling equilibrium where both types choose $e = e^*$ is $\bar{\theta} - e^*\theta_H \geq \theta_L$, that is, $e^* \leq \dfrac{\bar{\theta} - \theta_L}{\theta_H}$. Another possibility is a pooling equilibrium where both types choose $e = 0$. In this case the incentive compatibility constraints are as follows (since choosing e such that $0 < e < e^*$ is strictly dominated by choosing $e = 0$ and choosing e such that $e > e^*$ is strictly dominated by choosing $e = e^*$):

$$\theta_L \geq \bar{\theta} - e^*\theta_L \qquad (1')$$
$$\theta_L \geq \bar{\theta} - e^*\theta_H \qquad (2')$$

Since $\theta_L < \theta_H$, if $(1')$ is satisfied then so is $(2')$. Thus a necessary and sufficient condition for a pooling equilibrium where both types choose $e = 0$ is $e^* \geq \dfrac{\bar{\theta}}{\theta_L} - 1$.

(b.2) When $\mu_H = \frac{2}{5}$, $\theta_L = 1$ and $\theta_H = 6$ we get that $\bar{\theta} = \frac{2}{5}6 + \frac{3}{5}1 = 3$ and thus, using the calculations of part (b.1) we conclude that any $e^* \leq \frac{3-1}{6} = \frac{1}{3}$ gives rise to a pooling equilibrium where both types choose $e = e^*$ and any $e^* \geq 2$ gives rise to a pooling equilibrium where both types choose $e = 0$.

(c) (c.1) Let us look for a pooling equilibrium where both types choose $e = e^*$. The incentive compatibility constraints are:

$$\bar{\theta} - e^*\theta_L \geq \theta_L - e\theta_L,\ \forall e \in [0, e^*) \qquad (1a)$$
$$\bar{\theta} - e^*\theta_L \geq \theta_H - e\theta_L,\ \forall e \in [\hat{e}, \infty) \qquad (1b)$$
$$\bar{\theta} - e^*\theta_H \geq \theta_L - e\theta_H,\ \forall e \in [0, e^*) \qquad (2a)$$
$$\bar{\theta} - e^*\theta_H \geq \theta_H - e\theta_H,\ \forall e \in [\hat{e}, \infty) \qquad (2b)$$

Since the RHS of each inequality is decreasing in *e*, we can rewrite them as

$$\bar{\theta} - e^*\theta_L \geq \theta_L \qquad (1a)$$
$$\bar{\theta} - e^*\theta_L \geq \theta_H - \hat{e}\theta_L \qquad (1b)$$
$$\bar{\theta} - e^*\theta_H \geq \theta_L \qquad (2a)$$
$$\bar{\theta} - e^*\theta_H \geq \theta_H - \hat{e}\theta_H \qquad (2b)$$

First of all, note that – since $\theta_L < \theta_H$ – (2a) implies (1a). Thus we only need to consider the remaining inequalities, which can be re-written as follows:

$$(\hat{e} - e^*)\theta_L \geq \theta_H - \bar{\theta} \qquad (1b)$$
$$\bar{\theta} \geq \theta_L + e^*\theta_H \qquad (2a)$$
$$(\hat{e} - e^*)\theta_H \geq \theta_H - \bar{\theta} \qquad (2b)$$

Since $\hat{e} > e^*$ and $\theta_H > \theta_L$, $(\hat{e} - e^*)\theta_H > (\hat{e} - e^*)\theta_L$ and thus (1.b) implies (2.b). Thus we only need to consider the two inequalities

$$(\hat{e} - e^*)\theta_L \geq \theta_H - \bar{\theta} \qquad (1b)$$
$$\bar{\theta} \geq \theta_L + e^*\theta_H \qquad (2a)$$

Since $\bar{\theta} > \theta_L$ inequality (2a) can be satisfied if e^* is sufficiently close to 0. Furthermore, if $(\hat{e} - e^*)$ is sufficiently large then also (1b) is satisfied. Thus a pooling equilibrium where both types choose $e = e^*$ can exist. For example, if $\mu_H = \frac{2}{5}$, $\theta_L = 1$ and $\theta_H = 6$ so that $\bar{\theta} = \frac{2}{5}6 + \frac{3}{5}1 = 3$, then any pair $(e^*, \hat{e})$ such that $e^* \leq \frac{1}{3}$ and $\hat{e} \geq e^* + 3$.

Now let us look for a pooling equilibrium where both types choose *e* = 0. Then the incentive compatibility constraints are:

$$\theta_L \geq \bar{\theta} - e^*\theta_L \qquad (1a)$$
$$\theta_L \geq \theta_H - \hat{e}\theta_L \qquad (1b)$$
$$\theta_L \geq \bar{\theta} - e^*\theta_H \qquad (2a)$$
$$\theta_L \geq \theta_H - \hat{e}\theta_H \qquad (2b)$$

Since $\bar{\theta} > \theta_L$, (1a) implies (2a) and (1b) implies (2b). Thus a necessary and sufficient condition is $\theta_L \geq Max\left\{\bar{\theta} - e^*\theta_L,\ \theta_H - \hat{e}\theta_L)\right\}$, that is, $e^* \geq \frac{\bar{\theta}}{\theta_L} - 1$ and $\hat{e} \geq \frac{\theta_H}{\theta_L} - 1$.

One could also look for necessary and sufficient conditions for a pooling equilibrium where both types choose $e = \hat{e}$. The logic is the same.

(c.2) When $\mu_H = \frac{2}{5}$, $\theta_L = 1$ and $\theta_H = 6$ we get that $\bar{\theta} = \frac{2}{5}6 + \frac{3}{5}1 = 3$ and thus a sufficient condition for a pooling equilibrium where both types choose *e* = 0 is $e^* \geq 2$ and $\hat{e} \geq 5$.