Misrepresentation of Preferences

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1 Social choice functions

Arrow's theorem says that it is not possible to extract from a profile of individual preferences a preference ranking for society with a procedure that satisfies five desirable properties: Unrestricted Domain, Rationality, Unanimity, Non-dictatorship and Independence of Irrelevant Alternatives. Perhaps Arrow's approach is too demanding, in that it requires that a ranking of the entire set of alternatives be obtained for society. After all, if the purpose of voting procedures is to arrive at some choice among the alternatives, then we can dispense with a complete ranking and just focus on the final choice. Thus, we could look for a simpler object that extracts from a profile of individual preferences one alternative, to be thought of as *society's choice*. Such an object is called a Social Choice Function (SCF).

Definition 1.1. Let $X = \{x_1, x_2, \ldots, x_m\}$ ($m \geq 2$) be a finite set of alternatives, $N = \{1, 2, \ldots, n\}$ ($n \geq 2$) a finite set of individuals, $\mathcal{R}$ the set of complete and transitive binary relations on X, $\mathcal{R}^n$ the cartesian product $\underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{n \text{ times}}$ (thus an element of $\mathcal{R}^n$ is a list of complete and transitive preference relations on the set of alternatives X, one for each individual; we call an element of $\mathcal{R}^n$ a *profile of preferences*) and let S be a subset of $\mathcal{R}^n$. A *social choice function* is a function $f : S \to X$ that takes as input a profile of preferences for the individuals

$(\succsim_1, \succsim_2, \ldots, \succsim_n) \in S$ and produces as output an alternative $f(\succsim_1, \succsim_2, \ldots, \succsim_n) \in X$ to be thought of as "society's choice".

For example, suppose that there are only two alternatives, a and b (thus $X = \{a, b\}$), only strict rankings can be reported (that is, $S = \{a \succ b, b \succ a\} \times \{a \succ b, b \succ a\}$), and two voters ($N = \{1, 2\}$). Then, in order to construct a SCF we need to replace each $\square$ in the following table with either an a or a b:

		Individual 2's ranking	
		$a \succ_2 b$	$b \succ_2 a$
Individual 1's ranking	$a \succ_1 b$	$\square$	$\square$
	$b \succ_1 a$	$\square$	$\square$

Thus, there are $2^4 = 16$ possible SCFs, which are listed below:

(SCF-1)

1 \ 2	$a \succ_2 b$	$b \succ_2 a$
$a \succ_1 b$	a	a
$b \succ_1 a$	a	a

(SCF-2)

1 \ 2	$a \succ_2 b$	$b \succ_2 a$
$a \succ_1 b$	a	a
$b \succ_1 a$	a	b

(SCF-3)

1 \ 2	$a \succ_2 b$	$b \succ_2 a$
$a \succ_1 b$	a	a
$b \succ_1 a$	b	a

(SCF-4)

1 \ 2	$a \succ_2 b$	$b \succ_2 a$
$a \succ_1 b$	a	b
$b \succ_1 a$	a	a

(SCF-5)

1 \ 2	$a \succ_2 b$	$b \succ_2 a$
$a \succ_1 b$	b	a
$b \succ_1 a$	a	a

(SCF-6)

1 \ 2	$a \succ_2 b$	$b \succ_2 a$
$a \succ_1 b$	a	a
$b \succ_1 a$	b	b

(SCF-7)

1 \ 2	$a \succ_2 b$	$b \succ_2 a$
$a \succ_1 b$	a	b
$b \succ_1 a$	a	b

(SCF-8)

1 \ 2	$a \succ_2 b$	$b \succ_2 a$
$a \succ_1 b$	a	b
$b \succ_1 a$	b	a

(SCF-9)

1 \ 2	$a \succ_2 b$	$b \succ_2 a$
$a \succ_1 b$	b	a
$b \succ_1 a$	a	b

(SCF-10)

1 \ 2	$a \succ_2 b$	$b \succ_2 a$
$a \succ_1 b$	b	a
$b \succ_1 a$	b	a

(SCF-11)

		2	
		$a \succ_2 b$	$b \succ_2 a$
1	$a \succ_1 b$	b	b
	$b \succ_1 a$	a	a

(SCF-12)

		2	
		$a \succ_2 b$	$b \succ_2 a$
1	$a \succ_1 b$	a	b
	$b \succ_1 a$	b	b

(SCF-13)

		2	
		$a \succ_2 b$	$b \succ_2 a$
1	$a \succ_1 b$	b	a
	$b \succ_1 a$	b	b

(SCF-14)

		2	
		$a \succ_2 b$	$b \succ_2 a$
1	$a \succ_1 b$	b	b
	$b \succ_1 a$	a	b

(SCF-15)

		2	
		$a \succ_2 b$	$b \succ_2 a$
1	$a \succ_1 b$	b	b
	$b \succ_1 a$	b	a

(SCF-16)

		2	
		$a \succ_2 b$	$b \succ_2 a$
1	$a \succ_1 b$	b	b
	$b \succ_1 a$	b	b

Which of these SCFs should one reject on the basis of some general "reasonable" requirements?

First requirement: **Unanimity**. If all the individuals list alternative x at the top of their reported rankings then x should be chosen. In the above example this requirement amounts to insisting that the main diagonal be as follows: $\begin{array}{cc} a & \\ & b \end{array}$.

By appealing to Unanimity we can thus reject SCF-1, SCF-3, SCF-4, SCF-5, SCF-8, SCF-9, SCF-10, SCF-11, SCF-13, SCF-14, SCF-15 and SCF-16. Thus, we are left with the following four SCFs:

(SCF-2)

		2	
		$a \succ_2 b$	$b \succ_2 a$
1	$a \succ_1 b$	a	a
	$b \succ_1 a$	a	b

(SCF-6)

		2	
		$a \succ_2 b$	$b \succ_2 a$
1	$a \succ_1 b$	a	a
	$b \succ_1 a$	b	b

(SCF-7)

		2	
		$a \succ_2 b$	$b \succ_2 a$
1	$a \succ_1 b$	a	b
	$b \succ_1 a$	a	b

(SCF-12)

		2	
		$a \succ_2 b$	$b \succ_2 a$
1	$a \succ_1 b$	a	b
	$b \succ_1 a$	b	b

Second requirement: **Non-dictatorship**. There should not be a "dictator", that is, an individual whose top alternative is always chosen. In the above example there should not be an individual who is such that if he reports $a \succ b$ then a is chosen and if he reports $b \succ a$ then b is chosen.

On the basis of Non-dictatorship we must thus reject SCF-6 (where Individual 1 is a dictator) and SCF-7 (where Individual 2 is a dictator).

Hence, we are left two SCFs:

(SCF-2)

		2	
		$a \succ_2 b$	$b \succ_2 a$
1	$a \succ_1 b$	a	a
	$b \succ_1 a$	a	b

(SCF-12)

		2	
		$a \succ_2 b$	$b \succ_2 a$
1	$a \succ_1 b$	a	b
	$b \succ_1 a$	b	b

Can these two remaining SCFs be considered "reasonable"or "good"? Are there any other requirements that one should impose?

One issue that we have not addressed so far is the issue of misrepresentation of preferences. We have implicitly assumed up to now that each individual, when asked to report her ranking of the alternatives, will do so sincerely, that is, she will not report a ranking that is different from her true ranking. Is this an issue one should worry about? In the next section we will go through a number of popular SCFs and show that they all provide incentives for individuals to lie in reporting their preferences.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 5.1 at the end of this chapter.

2 Strategic voting

We shall illustrate the issue of strategic voting, or misrepresentation of preferences, in several popular voting methods which can be viewed as social choice functions.

Plurality voting with a default alternative. We illustrate this procedure for the case of three alternatives: $X = \{a, b, c\}$ and three voters: $N = \{1, 2, 3\}$. We assume that each voter can only report a strict ranking of the alternatives (that is, indifference is not allowed). Thus – as we saw in the previous chapter – there are six possible rankings that an individual can choose from when deciding what to report: $a \succ b \succ c$, $a \succ c \succ b$, $b \succ a \succ c$, $b \succ c \succ a$, $c \succ a \succ b$, $c \succ b \succ a$. To simplify the notation, we shall write them as abc, acb, bac, bca, cab, cba, that is, we read xyz as $x \succ y \succ z$. We take a to be the designated default alternative and the voting procedure is as follows:

- If two or more individuals list alternative b at the top of their ranking, then b is chosen,
- if two or more individuals list alternative c at the top of their ranking, then c is chosen,
- otherwise, the default alternative a is chosen (thus, a is chosen when two or more individuals list it at the top of their ranking or when there is complete disagreement, in the sense that one individual lists a at the top, another lists b at the top and the third lists c at the top).

How can we represent this voting procedure or SCF? We need six tables: each table labeled with one possible reported ranking of Individual 3; each table has six rows: each row labeled with one possible reported ranking of Individual 1, and six columns: each column labeled with one possible reported ranking of Individual 2. Inside each cell of each table we write the alternative chosen by the procedure described above. This is shown in Figure 1.

Let us first check if this SCF satisfies Unanimity and Non-dictatorship. Unanimity requires that when an alternative is listed at the top of each reported ranking then it should be chosen, that is, it requires the following, which is highlighted in Figure 2:

1. in the two tables at the top (corresponding to the cases where Voter 3 reports abc or acb) there should be an a in the following cells: (row 1, column 1), (row 1, column 2), (row 2, column 1) and (row 2, column 2) [these are the cases where every voter ranks a at the top],
2. in the two tables in the middle (corresponding to the cases where Voter 3 reports bac or bca) there should be a b in the following cells: (row 3, column 3), (row 3, column 4), (row 4, column 3) and (row 4, column 4) [these are the cases where every voter ranks b at the top],
3. in the two tables at the bottom (corresponding to the cases where Voter 3 reports cab or cba) there should be a c in the following cells: (row 5, column 5), (row 5, column 6), (row 6, column 5) and (row 6, column 6) [these are the cases where every voter ranks c at the top].

Thus, Unanimity only restricts the values in four cells in each table as shown in Figure 2.

Non-dictatorship is also satisfied, since for each individual there is at least one situation where she lists an alternative, say x, at the top and yet that alternative is not chosen because the other two individuals list a different alternative, say y, at the top.

2's → / 1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*a*	*a*
acb	*a*	*a*	*a*	*a*	*a*	*a*
bac	*a*	*a*	*b*	*b*	*a*	*a*
bca	*a*	*a*	*b*	*b*	*a*	*a*
cab	*a*	*a*	*a*	*a*	*c*	*c*
cba	*a*	*a*	*a*	*a*	*c*	*c*

3 reports abc

2's → / 1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*a*	*a*
acb	*a*	*a*	*a*	*a*	*a*	*a*
bac	*a*	*a*	*b*	*b*	*a*	*a*
bca	*a*	*a*	*b*	*b*	*a*	*a*
cab	*a*	*a*	*a*	*a*	*c*	*c*
cba	*a*	*a*	*a*	*a*	*c*	*c*

3 reports acb

2's → / 1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*b*	*b*	*a*	*a*
acb	*a*	*a*	*b*	*b*	*a*	*a*
bac	*b*	*b*	*b*	*b*	*b*	*b*
bca	*b*	*b*	*b*	*b*	*b*	*b*
cab	*a*	*a*	*b*	*b*	*c*	*c*
cba	*a*	*a*	*b*	*b*	*c*	*c*

3 reports bac

2's → / 1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*b*	*b*	*a*	*a*
acb	*a*	*a*	*b*	*b*	*a*	*a*
bac	*b*	*b*	*b*	*b*	*b*	*b*
bca	*b*	*b*	*b*	*b*	*b*	*b*
cab	*a*	*a*	*b*	*b*	*c*	*c*
cba	*a*	*a*	*b*	*b*	*c*	*c*

3 reports bca

` / 1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*c*	*c*
acb	*a*	*a*	*a*	*a*	*c*	*c*
bac	*a*	*a*	*b*	*b*	*c*	*c*
bca	*a*	*a*	*b*	*b*	*c*	*c*
cab	*c*	*c*	*c*	*c*	*c*	*c*
cba	*c*	*c*	*c*	*c*	*c*	*c*

3 reports cab

` / 1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*c*	*c*
acb	*a*	*a*	*a*	*a*	*c*	*c*
bac	*a*	*a*	*b*	*b*	*c*	*c*
bca	*a*	*a*	*b*	*b*	*c*	*c*
cab	*c*	*c*	*c*	*c*	*c*	*c*
cba	*c*	*c*	*c*	*c*	*c*	*c*

3 reports cba

Figure 1: Plurality voting with *a* as the default alternative

2's → / 1's ↓	abc	acb	bac	bca	cab	cba
abc	*a*	*a*	*a*	*a*	*a*	*a*
acb	*a*	*a*	*a*	*a*	*a*	*a*
bac	*a*	*a*	*b*	*b*	*a*	*a*
bca	*a*	*a*	*b*	*b*	*a*	*a*
cab	*a*	*a*	*a*	*a*	*c*	*c*
cba	*a*	*a*	*a*	*a*	*c*	*c*

3 reports abc

2's → / 1's ↓	abc	acb	bac	bca	cab	cba
abc	*a*	*a*	*a*	*a*	*a*	*a*
acb	*a*	*a*	*a*	*a*	*a*	*a*
bac	*a*	*a*	*b*	*b*	*a*	*a*
bca	*a*	*a*	*b*	*b*	*a*	*a*
cab	*a*	*a*	*a*	*a*	*c*	*c*
cba	*a*	*a*	*a*	*a*	*c*	*c*

3 reports acb

2's → / 1's ↓	abc	acb	bac	bca	cab	cba
abc	*a*	*a*	*b*	*b*	*a*	*a*
acb	*a*	*a*	*b*	*b*	*a*	*a*
bac	*b*	*b*	*b*	*b*	*b*	*b*
bca	*b*	*b*	*b*	*b*	*b*	*b*
cab	*a*	*a*	*b*	*b*	*c*	*c*
cba	*a*	*a*	*b*	*b*	*c*	*c*

3 reports bac

2's → / 1's ↓	abc	acb	bac	bca	cab	cba
abc	*a*	*a*	*b*	*b*	*a*	*a*
acb	*a*	*a*	*b*	*b*	*a*	*a*
bac	*b*	*b*	*b*	*b*	*b*	*b*
bca	*b*	*b*	*b*	*b*	*b*	*b*
cab	*a*	*a*	*b*	*b*	*c*	*c*
cba	*a*	*a*	*b*	*b*	*c*	*c*

3 reports bca

1's ↓	abc	acb	bac	bca	cab	cba
abc	*a*	*a*	*a*	*a*	*c*	*c*
acb	*a*	*a*	*a*	*a*	*c*	*c*
bac	*a*	*a*	*b*	*b*	*c*	*c*
bca	*a*	*a*	*b*	*b*	*c*	*c*
cab	*c*	*c*	*c*	*c*	*c*	*c*
cba	*c*	*c*	*c*	*c*	*c*	*c*

3 reports cab

1's ↓	abc	acb	bac	bca	cab	cba
abc	*a*	*a*	*a*	*a*	*c*	*c*
acb	*a*	*a*	*a*	*a*	*c*	*c*
bac	*a*	*a*	*b*	*b*	*c*	*c*
bca	*a*	*a*	*b*	*b*	*c*	*c*
cab	*c*	*c*	*c*	*c*	*c*	*c*
cba	*c*	*c*	*c*	*c*	*c*	*c*

3 reports cba

Figure 2: The highlights show the restrictions imposed by Unanimity

It remains to verify if it is the case that no individual can ever gain by lying about her preferences, that is, by reporting a ranking that is not her true ranking. We call this requirement *Non-manipulability* or *Strategy-proofness*. Unfortunately, this requirement is violated in this voting procedure. To see this, focus on the first table in Figure 1, corresponding to the case where Individual 3 reports the ranking *abc*. This table is reproduced in Figure 3. Consider the sixth column, corresponding to the case where Individual 2 reports the ranking *cba*. Suppose that the true ranking of Individual 1 is *bca* (4^{th} row); if she reports her preferences truthfully, that is, if she reports *bca* (recall that this means $b \succ_1 c \succ_1 a$) then the chosen alternative is a,[1] which is the worst, according to her true preferences; if, on the other hand, she lies and reports the false ranking *cab* then the chosen alternative is c, which – according to her true ranking – is better than a (in her true ranking, namely *bca*, c is the middle-ranked alternative while a is the worst).

	2's → 1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
	abc	*a*	*a*	*a*	*a*	*a*	*a*
	acb	*a*	*a*	*a*	*a*	*a*	*a*
	bac	*a*	*a*	*b*	*b*	*a*	*a*
true ranking	$\boxed{bca}$	*a*	*a*	*b*	*b*	*a*	*a*
reported ranking	*cab*	*a*	*a*	*a*	*a*	*c*	*c*
	cba	*a*	*a*	*a*	*a*	*c*	*c*

3 reports abc

Figure 3: The top-left table in Figure 2

The Condorcet method with a default alternative. The Condorcet method selects that alternative – called the *Condorcet winner* – that would win a majority vote in all the pairwise comparisons with each of the other alternatives; if such an alternative does not exist, then a pre-determined default alternative is

[1]Because there is complete disagreement: Voter 1 lists b at the top, Voter 2 lists c at the top and Voter 3 lists a at the top; hence, the default alternative, namely a, is chosen.

selected. As we did with plurality voting, we illustrate this procedure for the case of three alternatives: $X = \{a, b, c\}$ and three voters: $N = \{1, 2, 3\}$, assuming that each voter can only report a strict ranking of the alternatives. As before, we denote the ranking $x \succ y \succ z$ by xyz. We take a to be the designated default alternative. Let us first see what alternative the Condorcet method would select in a couple of situations. If the reported rankings are as follows:

	Voter 1	Voter 2	Voter 3
best	c	b	a
	b	a	b
worst	a	c	c

then b is the Condorcet winner: a majority (consisting of Voters 1 and 2) prefers b ro a and a majority (consisting of Voters 2 and 3) prefers b to c. Thus, b is selected. On the other hand, if the reported rankings are:

	Voter 1	Voter 2	Voter 3
best	a	c	b
	b	a	c
worst	c	b	a

then there is no Condorcet winner: a beats b but is beaten by c, b beats c but is beaten by a and c beats a but is beaten by b (indeed, the majority rule yields a cycle: a majority prefers a to b, a majority prefers b to c and a majority prefers c to a). Thus, since there is no Condorcet winner, the default alternative a is chosen.

As we did with plurality voting, we can represent this SCF by means of six tables, each with six rows and six columns, as shown in Figure 4. The reader might want to try to construct the tables before looking at Figure 4.
Note that this is a different SCF from the one shown in Figure 1. For example, the entry in the first table, row 6 and column 3 (corresponding to reported rankings cba for Voter 1, bac for Voter 2 and abc for Voter 3) is a for plurality voting (the default alternative since no two voters rank the same alternative at the top), but b for the Condorcet method (b is the Condorcet winner).

It is straightforward to verify that the SCF shown in Figure 4 satisfies Unanimity and Non-dictatorship (see Exercise 6). On the other hand, it fails to satisfy Non-manipulability. To see this, suppose that Voter 2's true ranking is bca and consider the first table, row 5 and column 4, corresponding to the case where Voter 1 reports cab, Voter 2 reports bca (thus, a truthful report) and Voter 3 reports abc. Then the chosen alternative is a, which is the worst in Voter 2's

1's ↓ / 2's →	abc	acb	bac	bca	cab	cba
abc	a	a	a	a	a	a
acb	a	a	a	a	a	a
bac	a	a	b	b	a	b
bca	a	a	b	b	a	b
cab	a	a	a	a	c	c
cba	a	a	b	b	c	c

3 reports abc

1's ↓ / 2's →	abc	acb	bac	bca	cab	cba
abc	a	a	a	a	a	a
acb	a	a	a	a	a	a
bac	a	a	b	b	a	a
bca	a	a	b	b	c	c
cab	a	a	a	c	c	c
cba	a	a	a	c	c	c

3 reports acb

1's ↓ / 2's →	abc	acb	bac	bca	cab	cba
abc	a	a	b	b	a	b
acb	a	a	b	b	a	a
bac	b	b	b	b	b	b
bca	b	b	b	b	b	b
cab	a	a	b	b	c	c
cba	b	a	b	b	c	c

3 reports bac

1's ↓ / 2's →	abc	acb	bac	bca	cab	cba
abc	a	a	b	b	a	b
acb	a	a	b	b	c	c
bac	b	b	b	b	b	b
bca	b	b	b	b	b	b
cab	a	c	b	b	c	c
cba	b	c	b	b	c	c

3 reports bca

1's ↓	abc	acb	bac	bca	cab	cba
abc	a	a	a	a	c	c
acb	a	a	a	c	c	c
bac	a	a	b	b	c	c
bca	a	c	b	b	c	c
cab	c	c	c	c	c	c
cba	c	c	c	c	c	c

3 reports cab

1's ↓	abc	acb	bac	bca	cab	cba
abc	a	a	b	b	c	c
acb	a	a	a	c	c	c
bac	b	a	b	b	c	c
bca	b	c	b	b	c	c
cab	c	c	c	c	c	c
cba	c	c	c	c	c	c

3 reports cba

Figure 4: The Condorcet method with *a* as the default alternative

true ranking. If Voter 2 were to misrepresent his preferences by reporting *cab*, then the chosen alternative would be *c*, which – according to his true ranking *bca* – is better than *a*.

The Borda count. The Borda count is the following SCF. Each voter states a strict ranking (that is, no indifference is allowed) of the m alternatives. For each voter's ranking, m points are assigned to the alternative ranked first, $m-1$ points to the alternative ranked second, and so on, up to 1 point for the worst alternative. Then, for each alternative, all the points are added up and the alternative with the largest score is chosen. A tie-breaking rule must be specified in case two or more alternatives receive the largest score.

Like the previous two SCFs, the Borda count satisfies Unanimity and Non-dictatorship but fails to satisfy Non-manipulability. For example, suppose that there are five alternatives: $X = \{a, b, c, d, e\}$ and five voters: $N = \{1, 2, 3, 4, 5\}$. Suppose that Voter 1's true ranking is:

	Voter 1's true ranking
best	a
	c
	d
	b
worst	e

(1)

Suppose also that Voter 1 expects the other voters to report the following rankings:

	Voter 2	Voter 3	Voter 4	Voter 5
best	b	b	c	a
	c	c	d	b
	e	a	a	e
	d	e	e	d
worst	a	d	b	c

If Voter 1 reports her true ranking, then we get the following profile of rankings:

	Voter 1	Voter 2	Voter 3	Voter 4	Voter 5	score
best	a	b	b	c	a	5
	c	c	c	d	b	4
	d	e	a	a	e	3
	b	d	e	e	d	2
worst	e	a	d	b	c	1

Applying the Borda count we get the following scores, so that **alternative c is chosen**.

$$
\begin{aligned}
a &: 5+1+3+3+5 = 17 \\
b &: 2+5+5+1+4 = 17 \\
c &: 4+4+4+5+1 = \boxed{18} \\
d &: 3+2+1+4+2 = 12 \\
e &: 1+3+2+2+3 = 11
\end{aligned}
$$

If, instead of her true ranking (1), Voter 1 were to report the following ranking:

best	a
	e
	d
	c
worst	b

then we would get the following profile of rankings:

	Voter 1	Voter 2	Voter 3	Voter 4	Voter 5	score
best	a	b	b	c	a	5
	e	c	c	d	b	4
	d	e	a	a	e	3
	c	d	e	e	d	2
worst	b	a	d	b	c	1

Applying the Borda count we get the following scores:

$$
\begin{aligned}
a &: 5+1+3+3+5 = \boxed{17} \\
b &: 1+5+5+1+4 = 16 \\
c &: 2+4+4+5+1 = 16 \\
d &: 3+2+1+4+2 = 12 \\
e &: 4+3+2+2+3 = 14
\end{aligned}
$$

so that **alternative a would be chosen**, which – according to her true ranking (1) – Voter 1 prefers to c. Hence, Voter 1 has an incentive to misrepresent her preferences.

Note that manipulability of a SCF does not mean that for *every* individual there is a situation where that individual can bring about a better outcome by misrepresenting her preferences. A SCF is manipulable as long as there is *at least one* individual who can bring about a better outcome by reporting a ranking which is different from her true ranking in *at least one* situation.

For example, consider the following SCF. There are three alternatives: $X = \{a, b, c\}$ and three voters: $N = \{1, 2, 3\}$. Each voter reports a strict ranking of the alternatives. Voter 1 is given privileged status in that her top-ranked alternative is assigned 1.5 points (and the other two alternatives 0 points), while for each of the other two voters his top-ranked alternative is assigned 1 point (and the other two alternatives 0 points). The alternative with the largest number of points is selected. This SCF is shown in Figure 5. The privileged status of Voter

2's → / 1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*a*	*a*
acb	*a*	*a*	*a*	*a*	*a*	*a*
bac	*a*	*a*	*b*	*b*	*b*	*b*
bca	*a*	*a*	*b*	*b*	*b*	*b*
cab	*a*	*a*	*c*	*c*	*c*	*c*
cba	*a*	*a*	*c*	*c*	*c*	*c*

3 reports abc

2's → / 1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*a*	*a*
acb	*a*	*a*	*a*	*a*	*a*	*a*
bac	*a*	*a*	*b*	*b*	*b*	*b*
bca	*a*	*a*	*b*	*b*	*b*	*b*
cab	*a*	*a*	*c*	*c*	*c*	*c*
cba	*a*	*a*	*c*	*c*	*c*	*c*

3 reports acb

2's → / 1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*b*	*b*	*a*	*a*
acb	*a*	*a*	*b*	*b*	*a*	*a*
bac	*b*	*b*	*b*	*b*	*b*	*b*
bca	*b*	*b*	*b*	*b*	*b*	*b*
cab	*c*	*c*	*b*	*b*	*c*	*c*
cba	*c*	*c*	*b*	*b*	*c*	*c*

3 reports bac

2's → / 1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*b*	*b*	*a*	*a*
acb	*a*	*a*	*b*	*b*	*a*	*a*
bac	*b*	*b*	*b*	*b*	*b*	*b*
bca	*b*	*b*	*b*	*b*	*b*	*b*
cab	*c*	*c*	*b*	*b*	*c*	*c*
cba	*c*	*c*	*b*	*b*	*c*	*c*

3 reports bca

1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*c*	*c*
acb	*a*	*a*	*a*	*a*	*c*	*c*
bac	*b*	*b*	*b*	*b*	*c*	*c*
bca	*b*	*b*	*b*	*b*	*c*	*c*
cab	*c*	*c*	*c*	*c*	*c*	*c*
cba	*c*	*c*	*c*	*c*	*c*	*c*

3 reports cab

1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*c*	*c*
acb	*a*	*a*	*a*	*a*	*c*	*c*
bac	*b*	*b*	*b*	*b*	*c*	*c*
bca	*b*	*b*	*b*	*b*	*c*	*c*
cab	*c*	*c*	*c*	*c*	*c*	*c*
cba	*c*	*c*	*c*	*c*	*c*	*c*

3 reports cba

Figure 5: Majority voting with a slight advantage given to Voter 1

1 has an impact only when there is complete disagreement, as is the case, for example, in the first table, row 3, column 5, corresponding to the case where Voter 1 reports *bac*, Voter 2 reports *cab* and Voter 3 reports *abc*. In this case b gets 1.5 points, c gets 1 point and a gets 1 point, so that b – the top alternative in Voter 1's reported ranking – is selected. In this SCF *there is no situation where Voter 1 can benefit from misreporting her preferences*. To see this, suppose that the

top-ranked alternative in Voter 1's *true* ranking is x. One of two scenarios must occur:

1. Voters 2 and 3 report the same alternative, call it y, at the top of their ranking (it might be that $y = x$ or might it be be that $y \neq x$). In this case alternative y is chosen and it will be chosen no matter what ranking Voter 1 reports (y already gets 2 points and Voter 1's report either adds 1.5 points to y or assigns 1.5 points to an alternative different from y). Thus, in this scenario, telling the truth and lying produce the same outcome; in particular, lying cannot be better than truthful reporting.

2. Voters 2 and 3 report different alternatives at the top of their rankings. In this case if Voter 1 reports truthfully, the chosen alternative will be x (either there is complete disagreement and x gets 1.5 points, while the other two alternatives get 1 point each, or one of Voters 2 and 3 has x at the top, in which case x gets 2.5 points). If Voter 1 lies then the alternative at the top of her reported ranking is chosen (same reasoning: either it is chosen because there is complete disagreement or it is chosen because Voter 1 forms a majority with one of the other two voters). Thus, if the alternative at the top of her reported ranking is not x, then Voter 1 is worse off by lying.

On the other hand, for both Voter 2 and Voter 3 there are situations where they gain by misrepresenting their preferences. We shall show this for Voter 2 and let the reader show that Voter 3 can gain by misrepresentation (Exercise 7). For Voter 2, consider the situation represented by the first table (Voter 3 reports *abc*) and row 6 (Voter 1 reports *cba*) and suppose that Voter 2's true ranking is *bac* (column 3). If Voter 2 reports truthfully, then the selected alternative is c, which is the worst from his point of view; if, on the other hand, Voter 3 reports *abc*, then the selected alternative is a, which – according to Voter 2's true ranking *bac* – is better than c.

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 5.2 at the end of this chapter.

3 The Gibbard-Satterthwaite theorem

The Gibbard-Satterthwaite theorem is a result published independently by philosopher Allan Gibbard in 1973[2] and economist Mark Satterthwaite in 1975.[3]

As for the case of Arrow's theorem, the objective is to determine if there are Social Choice Functions (see Definition 1.1) that satisfy some "reasonable" properties, which we will call *axioms*, as we did in the previous chapter.

We assume that the domain of the Social Choice Function (SCF) is the set of profiles of **strict rankings** of the set of alternatives X (that is, indifference is ruled out). Let $\mathcal{P}$ denote the set of strict rankings of the elements of X. Then the domain of the SCF is taken to be $\mathcal{P}^n = \underbrace{\mathcal{P} \times \cdots \times \mathcal{P}}_{n \text{ times}}$. Thus, individuals are not allowed to report indifference between any two alternatives, but – subject to this restriction – any strict ranking can be reported. Hence, this is a limited form of the property of Unrestricted Domain considered in the previous chapter. The axioms that we consider are the following:

- **Axiom 1: Unanimity**. If alternative x is the top-ranked alternative in the reported ranking of every individual, then it should be chosen by society: if, for every individual i, $x \succ_i y$ for every alternative $y \neq x$ then $f(\succ_1, \ldots, \succ_n) = x$.

- **Axiom 2: Non-dictatorship**. There is no individual i whose top alternative – in her reported ranking – is always chosen. Formally, this can be stated as follows: for every individual $i \in N$, there is a profile of reported preferences $(\succ_1, \ldots, \succ_n)$ such that if $f(\succ_1, \ldots, \succ_n) = x \in X$ then x is not at the top of $\succ_i$ (that is, there exists a $y \in X$ such that $y \neq x$ and $y \succ_i x$).

- **Axiom 3: Non-manipulability or Strategy-proofness**. There is no situation where some individual can gain by reporting a ranking different from her true ranking. Formally, this can be stated as follows. Fix an arbitrary individual $i \in N$ and an arbitrary profile $(\succ_1, \ldots, \succ_{i-1}, \succ_i, \succ_{i+1}, \ldots, \succ_n) \in \mathcal{P}^n$

[2] Allan Gibbard, "Manipulation of voting schemes: a general result", *Econometrica*, 1973, Vol. 41 (4), pages 587-601.

[3] Mark Satterthwaite, "Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions", *Journal of Economic Theory*, 1975, Vol. 10 (2), pages 187-217.

and let $f(\succ_1, \ldots, \succ_{i-1}, \succ_i, \succ_{i+1}, \ldots, \succ_n) = x \in X$. Then there is no $\succ_i' \in \mathcal{P}$ such that $f(\succ_1, \ldots, \succ_{i-1}, \succ_i', \succ_{i+1}, \ldots, \succ_n) \succ_i x$ (think of $\succ_i$ as the true ranking of individual i and $\succ_i'$ as a possible lie).

The following theorem provides an "impossibility result" similar to Arrow's impossibility theorem.

Theorem 1. *[Gibbard-Satterthwaite theorem] If the set of alternatives X contains at least three elements, there is no Social Choice Function $f : \mathcal{P}^n \to X$ that satisfies Unanimity, Non-dictatorship and Non-manipulability.*

An alternative way of stating the above theorem is as follows: if a SCF satisfies Unanimity and one of the other two axioms then it fails to satisfy the third axiom (for example, if a SCF satisfies Unanimity and Non-dictatorship then it violates Non-manipulability).[4]

In the next section we illustrate the logic of the proof of Theorem 1 by focusing on the simple case of three alternatives and two voters.[5]

Test your understanding of the concepts introduced in this section, by going through the exercises in Section 5.3 at the end of this chapter.

4 Illustration of the proof of the Gibbard-Satterthwaite theorem

In this section we prove the Gibbard-Satterthwaite theorem (Theorem 1) for the case where there are three alternatives, called x, y and z, and two individuals ($N = \{1, 2\}$). There are six possible strict rankings of the set $X = \{x, y, z\}$ and thus any SCF can be represented as a table with six rows and six columns. We will show that any SCF that satisfies Unanimity and Non-manipulability must violate Non-dictatorship. Fix a SCF that satisfies Unanimity. Then the blocks on the main diagonal must be filled as shown in Figure 6 (Unanimity forces the values of 12 out of 36 entries; as usual, xyz means $x \succ y \succ z$ and similarly for the other rankings).

[4]Sometimes the Gibbard-Satterthwaite theorem is stated with the premise 'if the range of the SCF contains at least three alternatives ...', but this clause is implied by the assumptions that the set X contains at least three elements and that the SCF satisfies Unanimity.

[5]A relatively simple proof for the general case can be found in Jean-Pierre Benoit, "The Gibbard-Satterthwaite theorem: a simple proof", *Economics Letters*, Vol. 69, 2000, pages 319-322. See also the references therein for alternative proofs.

Voter 1		1 *xyz*	2 *xzy*	3 *yxz*	4 *yzx*	5 *zxy*	6 *zyx*
A	*xyz*	x	x				
B	*xzy*	x	x				
C	*yxz*			y	y		
D	*yzx*			y	y		
E	*zxy*					z	z
F	*zyx*					z	z

Voter 2 (columns)

Figure 6: The requirement of Unanimity

Now consider the highlighted cell A4 in Figure 6. By Non-manipulability there cannot be a z there otherwise Voter 1, with true preferences xyz (row A), would gain by lying and reporting yxz (row C) when Voter 2 reports yzx (column 4). Thus, in cell A4 there must be either an x or a y. *The strategy of the proof is to show that if there is an x in cell A4 then Voter 1 must be a dictator, while if there is a y in cell A4 then Voter 2 must be a dictator.* We will only prove the first part, that is, that if there is an x in cell A4 then Voter 1 must be a dictator.

Suppose that there is an x in cell A4. Then there must be an x also in cell B4 (the cell marked with a 1 in Figure 7) otherwise Voter 1, with true preferences xzy (row B), would gain by reporting xyz (row A) when Voter 2 reports yzx (column 4). Now, from Voter 2's point of view, there must be an x in all the boxes marked with a 2 otherwise Voter 2, with true preferences yzx (column 4), would gain by "moving" either left or right to get the "non-x" which she prefers to x. Thus, the top two rows are entirely made of x's.

Now consider the highlighted cell C6 in Figure 7. There cannot be a z there because Voter 1, with true preferences yxz (row C), would gain by reporting xzy (row B) when Voter 2 reports zyx (column 6); furthermore, there cannot be an x in cell C6 because Voter 2, with true preferences zyx (column 6), would gain by reporting yzx (column 4) when Voter 1 reports yxz (row C). Thus, **there**

Voter 2

Voter 1	1 *xyz*	2 *xzy*	3 *yxz*	4 *yzx*	5 *zxy*	6 *zyx*
A *xyz*	***x***	***x***	② *x*	*(not z)* ***x***	② *x*	② *x*
B *xzy*	***x***	***x***	② *x*	① *x*	② *x*	② *x*
C *yxz*			***y***	***y***		
D *yzx*			***y***	***y***		
E *zxy*					***z***	***z***
F *zyx*					***z***	***z***

Figure 7: Inferences from the presence of x in cell A4

must be a y in cell C6. It follows that there must be a y also in cell D6 below, otherwise Voter 1, with true preferences yzx (row D), would gain by reporting yxz (row C) when Voter 2 reports zyx (column 6). Thus, we have reached the configuration shown in Figure 8.

			Voter 2				
		1 xyz	2 xzy	3 yxz	4 yzx	5 zxy	6 zyx
Voter 1	A xyz	x	x	x	x	x	x
	B xzy	x	x	x	x	x	x
	C yxz			y	y	①	y
	D yzx			y	y	*not x* ① *not z* ②	y
	E zxy					z	z
	F zyx					z	z

Figure 8: Further inferences

Now consider the highlighted cell D5: there cannot be a z there otherwise Voter 2, with true preferences zyx (column 6), would gain by reporting zxy (column 5) when Voter 1 reports yzx (row D) and there cannot be an x, otherwise Voter 1, with true preferences yzx (row D), would gain by reporting zxy (row E) when Voter 2 reports zxy (column 5). Hence, **there must be a y in cell D5**. Then there must be a y also in cell C5 otherwise Voter 1 with true preferences yxz (row C) would gain by reporting yzx (row D) when Voter 2 reports zxy (column 5). Thus, we have reached the configuration shown in Figure 9.

Now there must be a y in the remaining cells of rows C and D (marked with a ② in Figure 9) because otherwise Voter 2 with true preferences zxy (column 5) would gain by reporting either xyz (column 1) or xzy (column 2) when Voter 1 reports a ranking corresponding to either row C or row D. Thus, we have shown that rows C and D consist entirely of y's.

Now consider cell E4 in Figure 10: there cannot be a y there because Voter 1, with true preferences zxy (row E), would gain by reporting xyz (row A) in

				Voter 2			
		1	2	3	4	5	6
		xyz	*xzy*	*yxz*	*yzx*	*zxy*	*zyx*
Voter 1	A *xyz*	***x***	***x***	***x***	***x***	***x***	***x***
	B *xzy*	***x***	***x***	***x***	***x***	***x***	***x***
	C *yxz*	2	2	***y***	***y***	***y***	***y***
	D *yzx*	2	2	***y***	***y***	***y***	***y***
	E *zxy*					***z***	***z***
	F *zyx*					***z***	***z***

Figure 9: Updated configuration

the situation represented by column 4 and there cannot be an x because Voter 2, with true preferences yzx (column 4), would gain by reporting zxy (column 5) in the situation represented by row E. Thus, **there must be a z in cell E4**. Then there must be a z also in cell F4 below otherwise Voter 1, with true preferences zyx (row F), would gain by reporting zxy (row E) in the situation represented by column 4.

Now in the highlighted cells F1, F2 and F3 there cannot be an x because Voter 1, with true preferences zyx (row F), would gain by reporting yzx (row D) and there cannot be a y because Voter 2, with true preferences yzx (column 4), would gain by reporting the ranking corresponding to either column 1 or column 2 or column 3 in the situation represented by row F. Thus, **there must be a z in F1, F2 and F3**. This implies that there must be a z in the remaining cells too because Voter 1, with true preferences zxy (row E) would gain by reporting zyx (row F). Hence, we have shown that the SCF must have all x's in rows A and B, all y's in rows C and D and all z's in rows E and F, making Voter 1 a Dictator (in rows A and B her reported top alternative is x and it is chosen no matter what Voter 2 reports, in rows C and D her reported top alternative is y and it is chosen no matter what Voter 2 reports and in rows E and F her reported top alternative is z and it is chosen no matter what Voter 2 reports).

			Voter 2					
			1	2	3	4	5	6
			xyz	*xzy*	*yxz*	*yzx*	*zxy*	*zyx*
Voter 1	A	*xyz*	***x***	***x***	***x***	***x***	***x***	***x***
	B	*xzy*	***x***	***x***	***x***	***x***	***x***	***x***
	C	*yxz*	***y***	***y***	***y***	***y***	***y***	***y***
	D	*yzx*	***y***	***y***	***y***	***y***	***y***	***y***
	E	*zxy*				***not x*** ***not y***	***z***	***z***
	F	*zyx*	*not x* ① *not y* ②	*not x* ① *not y* ②	*not x* ① *not y* ②	***(z)***	***z***	***z***

Figure 10: The last steps

The proof that if there had been a y in cell A4, then Voter 2 would have been a Dictator is similar and we will omit it.

5 Exercises

The solutions to the following exercises are given in Section 6 at the end of this chapter.

5.1 Exercises for Section 1: Social choice functions

Exercise 1. *Suppose that there are three alternatives: $X = \{a, b, c\}$ and two voters: $N = \{1, 2\}$ and consider SCFs that only allow the reporting of strict rankings so that each individual must report one of the following: $a \succ b \succ c$, $a \succ c \succ b$, $b \succ a \succ c$, $b \succ c \succ a$, $c \succ a \succ b$, $c \succ b \succ a$. To simplify the notation, write them as abc, acb, bac, bca, cab, cba. In this case we can represent a SCF by means of a table with six rows (each row labeled with one ranking for Individual 1) and six columns (each column labeled with one ranking for Individual 2).*

(a) *How many SCFs are there?*

(b) *Fill in the table as much as you can by using only the Unanimity principle.*

(c) *How many SCFs that satisfy the Unanimity principle are there?*

(d) *Show the SCF that corresponds to the case where Individual 2 is a dictator.*

Exercise 2. *Consider again the case where there are three alternatives: $X = \{a, b, c\}$ and two voters: $N = \{1, 2\}$ and only strict rankings can be reported. Consider the SCF shown in Figure 11.*

(a) *Does this SCF satisfy Unanimity?*

(b) *Show that this SCF satisfies Non-dictatorship.*

1's ranking ↓ \ 2's ranking →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*c*	*a*
acb	*a*	*a*	*b*	*a*	*a*	*c*
bac	*b*	*a*	*b*	*b*	*b*	*c*
bca	*a*	*b*	*b*	*b*	*c*	*b*
cab	*a*	*c*	*c*	*b*	*c*	*c*
cba	*c*	*a*	*b*	*c*	*c*	*c*

Figure 11: An SCF when $X = \{a, b, c\}$ and $N = \{1, 2\}$

5.2 Exercises for Section 2: Strategic voting

Exercise 3. *In Section 1 we considered the case of two alternatives and two voters, with only strict rankings being allowed. We saw that in this case there are 16 possible SCFs, but by appealing to Unanimity and Non-dictatorship, one can reduce the number to the two SCFs shown below:*

(SCF-2)

		2: $a \succ_2 b$	2: $b \succ_2 a$
1	$a \succ_1 b$	a	a
1	$b \succ_1 a$	a	b

(SCF-12)

		2: $a \succ_2 b$	2: $b \succ_2 a$
1	$a \succ_1 b$	a	b
1	$b \succ_1 a$	b	b

(a) *For SCF-2 show that neither individual can ever gain by misrepresenting his/her preferences. Give enough details in your argument.*

(b) *For SCF-12 show that neither individual can ever gain by misrepresenting his/her preferences. Give enough details in your argument.*

Exercise 4. *Consider the SCF of Exercise 2, which is reproduced in Figure 12.*

(a) *Show that there is at least one situation where Individual 1 can gain by misrepresenting her preferences.*

(b) *Show that there is at least one situation where Individual 2 can gain by misrepresenting his preferences.*

1's ranking ↓ / 2's ranking →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*c*	*a*
acb	*a*	*a*	*b*	*a*	*a*	*c*
bac	*b*	*a*	*b*	*b*	*b*	*c*
bca	*a*	*b*	*b*	*b*	*c*	*b*
cab	*a*	*c*	*c*	*b*	*c*	*c*
cba	*c*	*a*	*b*	*c*	*c*	*c*

Figure 12: An SCF when $X = \{a, b, c\}$ and $N = \{1, 2\}$

Exercise 5. *Consider the Borda count explained in Section 2, with the following tie-breaking rule: if two or more alternatives get the highest score, then the alternative that comes first in alphabetical order is chosen. Suppose that there are three alternatives:* $X = \{a, b, c\}$ *and three voters:* $N = \{1, 2, 3\}$*. Voter 1's true ranking is:*

	Voter 1's true ranking
best	*a*
	b
worst	*c*

(2)

Suppose that Voter 1 expects the other two voters to report the following rankings:

	Voter 2	*Voter 3*
best	*c*	*c*
	b	*b*
worst	*a*	*a*

(a) *What alternative will be chosen if Voter 1 reports her true ranking (2)?*

(b) *Show that, by misrepresenting her preferences, Voter 1 can obtain a better alternative than the one found in Part (a).*

Exercise 6. *Consider the SCF shown in Figure 13 (the Condorcet method with a default alternative, previously shown in Figure 4), where there are three voters and three alternatives.*

(a) *Show that it satisfies the Unanimity principle.*

(b) *Show it satisfies the Non-dictatorship principle.*

1's ↓ / 2's →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*a*	*a*
acb	*a*	*a*	*a*	*a*	*a*	*a*
bac	*a*	*a*	*b*	*b*	*a*	*b*
bca	*a*	*a*	*b*	*b*	*a*	*b*
cab	*a*	*a*	*a*	*a*	*c*	*c*
cba	*a*	*a*	*b*	*b*	*c*	*c*

3 reports abc

1's ↓ / 2's →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*a*	*a*
acb	*a*	*a*	*a*	*a*	*a*	*a*
bac	*a*	*a*	*b*	*b*	*a*	*a*
bca	*a*	*a*	*b*	*b*	*c*	*c*
cab	*a*	*a*	*a*	*c*	*c*	*c*
cba	*a*	*a*	*a*	*c*	*c*	*c*

3 reports acb

1's ↓ / 2's →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*b*	*b*	*a*	*b*
acb	*a*	*a*	*b*	*b*	*a*	*a*
bac	*b*	*b*	*b*	*b*	*b*	*b*
bca	*b*	*b*	*b*	*b*	*b*	*b*
cab	*a*	*a*	*b*	*b*	*c*	*c*
cba	*b*	*a*	*b*	*b*	*c*	*c*

3 reports bac

1's ↓ / 2's →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*b*	*b*	*a*	*b*
acb	*a*	*a*	*b*	*b*	*c*	*c*
bac	*b*	*b*	*b*	*b*	*b*	*b*
bca	*b*	*b*	*b*	*b*	*b*	*b*
cab	*a*	*c*	*b*	*b*	*c*	*c*
cba	*b*	*c*	*b*	*b*	*c*	*c*

3 reports bca

1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*c*	*c*
acb	*a*	*a*	*a*	*c*	*c*	*c*
bac	*a*	*a*	*b*	*b*	*c*	*c*
bca	*a*	*c*	*b*	*b*	*c*	*c*
cab	*c*	*c*	*c*	*c*	*c*	*c*
cba	*c*	*c*	*c*	*c*	*c*	*c*

3 reports cab

1's ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*b*	*b*	*c*	*c*
acb	*a*	*a*	*a*	*c*	*c*	*c*
bac	*b*	*a*	*b*	*b*	*c*	*c*
bca	*b*	*c*	*b*	*b*	*c*	*c*
cab	*c*	*c*	*c*	*c*	*c*	*c*
cba	*c*	*c*	*c*	*c*	*c*	*c*

3 reports cba

Figure 13: The Condorcet method with *a* as the default alternative

Exercise 7. *Consider the SCF shown in Figure 14 (majority voting with a slight advantage given to Voter 1), which reproduces Figure 5. Show that there is at least one situation where Voter 3 can benefit from misrepresenting his preferences.*

Exercise 8. *Consider again the SCF shown in Figure 14 (which reproduces Figure 5). At the end of Section 2 we showed that there is no situation where Voter 1 (the one*

2’s → / 1’s ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*a*	*a*
acb	*a*	*a*	*a*	*a*	*a*	*a*
bac	*a*	*a*	*b*	*b*	*b*	*b*
bca	*a*	*a*	*b*	*b*	*b*	*b*
cab	*a*	*a*	*c*	*c*	*c*	*c*
cba	*a*	*a*	*c*	*c*	*c*	*c*

3 reports abc

2’s → / 1’s ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*a*	*a*
acb	*a*	*a*	*a*	*a*	*a*	*a*
bac	*a*	*a*	*b*	*b*	*b*	*b*
bca	*a*	*a*	*b*	*b*	*b*	*b*
cab	*a*	*a*	*c*	*c*	*c*	*c*
cba	*a*	*a*	*c*	*c*	*c*	*c*

3 reports acb

2’s → / 1’s ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*b*	*b*	*a*	*a*
acb	*a*	*a*	*b*	*b*	*a*	*a*
bac	*b*	*b*	*b*	*b*	*b*	*b*
bca	*b*	*b*	*b*	*b*	*b*	*b*
cab	*c*	*c*	*b*	*b*	*c*	*c*
cba	*c*	*c*	*b*	*b*	*c*	*c*

3 reports bac

2’s → / 1’s ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*b*	*b*	*a*	*a*
acb	*a*	*a*	*b*	*b*	*a*	*a*
bac	*b*	*b*	*b*	*b*	*b*	*b*
bca	*b*	*b*	*b*	*b*	*b*	*b*
cab	*c*	*c*	*b*	*b*	*c*	*c*
cba	*c*	*c*	*b*	*b*	*c*	*c*

3 reports bca

` / 1’s ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*c*	*c*
acb	*a*	*a*	*a*	*a*	*c*	*c*
bac	*b*	*b*	*b*	*b*	*c*	*c*
bca	*b*	*b*	*b*	*b*	*c*	*c*
cab	*c*	*c*	*c*	*c*	*c*	*c*
cba	*c*	*c*	*c*	*c*	*c*	*c*

3 reports cab

` / 1’s ↓	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*c*	*c*
acb	*a*	*a*	*a*	*a*	*c*	*c*
bac	*b*	*b*	*b*	*b*	*c*	*c*
bca	*b*	*b*	*b*	*b*	*c*	*c*
cab	*c*	*c*	*c*	*c*	*c*	*c*
cba	*c*	*c*	*c*	*c*	*c*	*c*

3 reports cba

Figure 14: Majority voting with a slight advantage given to Voter 1

who has a slight advantage in that her top alternative is assigned 1.5 points instead of 1 point) can gain by misrepresenting her preferences. Suppose now that (perhaps as a result of previous discussions) it is common knowledge among the three voters that Voter 1's true ranking is acb (that is, $a \succ_1 c \succ_1 b$*). Hence, it is reasonable to assume that Voter 1 will report her ranking truthfully (she cannot gain by lying) and, indeed, it is common knowledge between Voters 2 and 3 that they expect Voter 1 to report acb. By postulating that Voter 1 reports acb, we can reduce the SCF of Figure 14 to an SCF with only two voters: Voter 2 and Voter 3.*

(a) *Draw a table that represents the reduced SCF. For example, this table should show that if Voter 2 reports bca and Voter 3 reports cba, then the chosen alternative is a (it gets 1.5 points from Voter 1's report, while each of b and c get only 1 point each).*

(b) *In the reduced SCF, can Voter 2 ever gain from misrepresenting his preferences?*

(c) *In the reduced SCF, can Voter 3 ever gain from misrepresenting his preferences?*

(d) *Suppose that Voter 3's true ranking is bac. Can Voter 3 gain by reporting a different ranking?*

(e) *Suppose that Voter 2 knows that Voter 3's true ranking is bac and expects her to report truthfully. Suppose also that Voter 2's true ranking is cba. What ranking should Voter 2 report?*

5.3 Exercises for Section 3: The Gibbard-Satterthwaite theorem

Exercise 9. *Consider the two SCFs of Exercise 3 (SCF-2 and SCF-12), reproduced below:*

(SCF-2)

		2: $a \succ_2 b$	2: $b \succ_2 a$
1	$a \succ_1 b$	a	a
1	$b \succ_1 a$	a	b

(SCF-12)

		2: $a \succ_2 b$	2: $b \succ_2 a$
1	$a \succ_1 b$	a	b
1	$b \succ_1 a$	b	b

In Section 1 they were shown to satisfy Unanimity and Non-dictatorship and in Exercise 3 they were shown to satisfy Non-manipulability. Explain why these two SCFs do not constitute a counterexample to the Gibbard-Satterthwaite theorem (Theorem 1).

Exercise 10. *There are five alternatives* ($X = \{a, b, c, d, e\}$) *and fourteen voters* ($N = \{1, 2, \ldots, 14\}$)*. Consider the following SCF: each voter submits a strict ranking of the alternatives and there are no restrictions on what strict ranking can be submitted. Then the procedure is as follows:*

1. *if Individuals 1-5 all rank the same alternative at the top, then that alternative is chosen, otherwise*
2. *if Individuals 6-10 all rank the same alternative at the top, then that alternative is chosen, otherwise*
3. *if there is a Condorcet winner in the reported rankings of Individuals 11-13 (the definition of Condorcet winner was explained in Section 2) then that alternative is chosen, otherwise*
4. *the top-ranked alternative of individual 14 is chosen.*

Does this SCF satisfy Non-manipulability?

6 Solutions to Exercises

Solution to Exercise 1.

(a) The table has 36 cells that need to be filled, each with one of a, b or c. Thus, there are $3^{36} = 1.5009 \times 10^{17}$, that is, more than 150,000 trillions (recall that a trillion is 10^{12} or a million million) SCFs!

(b) The Unanimity principle restricts only the values in 12 of the 36 cells, as shown in Figure 15 below.

(c) In Figure 15 there are 24 remaining cells to be filled in (each with one of a, b or c) and thus there are $3^{24} = 282.43 \times 10^{9}$ (that is, more than 282 billions) SCFs that satisfy the Unanimity principle!

(d) The case where Individual 2 is a dictator corresponds to the SCF shown in Figure 16 below. □

1's ↓ \ 2's →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*				
acb	*a*	*a*				
bac			*b*	*b*		
bca			*b*	*b*		
cab					*c*	*c*
cba					*c*	*c*

Figure 15: The restrictions imposed by the Unanimity principle

1's ↓ \ 2's →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*b*	*b*	*c*	*c*
acb	*a*	*a*	*b*	*b*	*c*	*c*
bac	*a*	*a*	*b*	*b*	*c*	*c*
bca	*a*	*a*	*b*	*b*	*c*	*c*
cab	*a*	*a*	*b*	*b*	*c*	*c*
cba	*a*	*a*	*b*	*b*	*c*	*c*

Figure 16: Individual 2 is a dictator

Solution to Exercise 2.

(a) Yes, this SCF satisfies Unanimity: when both individuals rank alternative x at the top, x is chosen by society (the main diagonal consists of a block of four a's, a block of four b's and a block of four c's, fulfilling the requirement shown in Figure 15).

(b) This SCF also satisfies Non-dictatorship. In Figure 17 we have highlighted two cells to show this. For Individual 1, consider the cell in row 4 and column 1: her ranking is *bca*, thus her top-ranked alternative is *b*, and yet the chosen alternative (when Individual 2 reports the ranking *abc*) is *a*, not *b* . For Individual 2, consider the cell in row 1 and column 6: his ranking is *cba*, thus his top-ranked alternative is *c*, and yet the chosen alternative (when Individual 1 reports the ranking *abc*) is *a*, not *c*. (Of course, other cells could have been used to make the same point.) □

1's ranking ↓ / 2's ranking →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*c*	*a*
acb	*a*	*a*	*b*	*a*	*a*	*c*
bac	*b*	*a*	*b*	*b*	*b*	*c*
bca	*a*	*b*	*b*	*b*	*c*	*b*
cab	*a*	*c*	*c*	*b*	*c*	*c*
cba	*c*	*a*	*b*	*c*	*c*	*c*

Figure 17: Neither individual is a dictator

Solution to Exercise 3.

(a) SCF-2. Individual 1 cannot gain by misrepresenting her preferences: *if her true ranking is* $a \succ_1 b$ then by reporting truthfully she gets her top alternative a and by misrepresenting she might get a or b; *if her true ranking is* $b \succ_1 a$ and she reports truthfully, then there are two possibilities: (1) Individual 2 reports $a \succ_2 b$, in which case the outcome is a, and would still be a if Individual 1 lied, and (2) Individual 2 reports $b \succ_2 a$, in which case if Individual 1 reports truthfully then she gets her top alternative b, while if she lies then she get her worst alternative, namely a.
Individual 2 cannot gain by misrepresenting his preferences: *if his true*

ranking is $a \succ_2 b$ then by reporting truthfully he gets his top alternative a and by misrepresenting he might get a or b; *if his true ranking is* $b \succ_2 a$ and he reports truthfully, then there are two possibilities: (1) Individual 1 reports $a \succ_1 b$, in which case the outcome is a, and would still be a if Individual 2 lied, and (2) Individual 1 reports $b \succ_1 a$, in which case if Individual 2 reports truthfully then he gets his top alternative b, while if he lies then he gets his worst alternative, namely a.

(b) SCF-12. Individual 1 cannot gain by misrepresenting her preferences: *if her true ranking is* $b \succ_1 a$ then by reporting truthfully she gets her top alternative b and by misrepresenting she might get a or b; *if her true ranking is* $a \succ_1 b$ and she reports truthfully, then there are two possibilities: (1) Individual 2 reports $b \succ_2 a$, in which case the outcome is b, and would still be b if Individual 1 lied, and (2) Individual 2 reports $a \succ_2 b$, in which case if Individual 1 reports truthfully then she gets her top alternative a, while if she lies she get her worst alternative, namely b.
Individual 2 cannot gain by misrepresenting his preferences: *if his true ranking is* $b \succ_2 a$ then by reporting truthfully he gets his top alternative b and by misrepresenting he might get a or b; *if his true ranking is* $a \succ_2 b$ and he reports truthfully, then there are two possibilities: (1) Individual 1 reports $b \succ_1 a$, in which case the outcome is b, and would still be b if Individual 2 lied, and (2) Individual 1 reports $a \succ_1 b$, in which case if Individual 2 reports truthfully then he gets his top alternative a, while if he lies he gets his worst alternative, namely b. □

Solution to Exercise 4.

(a) There are several situations where Individual 1 can gain by lying. For example, suppose that her true ranking is *bca* (row 4) and Individual 2 reports *abc* (column 1). Then, by reporting truthfully, Individual 1 brings about outcome a, which is her worst outcome, but by lying and reporting *bac* she obtains her most preferred outcome, namely b.

(b) There are several situations where Individual 2 can gain by lying. For example, suppose that his true ranking is *abc* (column 1) and Individual 1 reports *bac* (row 3). Then, by reporting truthfully, Individual 2 brings about outcome b, which is his middle-ranked outcome, but by lying and reporting *acb* he obtains his most preferred outcome, namely a. □

Solution to Exercise 5.

(a) If Voter 1 reports her true ranking then we get the following profile:

	Voter 1	Voter 2	Voter 3	score
best	a	c	c	3
	b	b	b	2
worst	c	a	a	1

The scores computed according to the Borda rule are:

$$\begin{aligned} a &: 3+1+1=5 \\ b &: 2+2+2=6 \\ c &: 1+3+3=\boxed{7} \end{aligned}$$

so that alternative c is chosen (Voter 1's worst).

(b) If instead of her true ranking Voter 1 reports the following ranking:

best	b
	a
worst	c

then the profile of reported rankings is

	Voter 1	Voter 2	Voter 3	score
best	b	c	c	3
	a	b	b	2
worst	c	a	a	1

The scores computed according to the Borda rule are:

$$\begin{aligned} a &: 2+1+1=4 \\ b &: 3+2+2=7 \\ c &: 1+3+3=7 \end{aligned}$$

The largest score is 7 and is shared by both b and c. According to the tie-breaking rule, in case of ties the alternative that comes first in alphabetical order is chosen. Thus, the chosen alternative b, which Voter 1 (according to her true ranking $a \succ_1 b \succ_1 c$) prefers to c. Thus, Voter 1 gains by lying and reporting a ranking which is different from her true ranking. $\square$

Solution to Exercise 6.

(a) Unanimity requires the following:
(1) in the two tables at the top (corresponding to the cases where Voter 3 reports *abc* or *acb*) there should be an *a* in the following cells: (row 1, column 1), (row 1, column 2), (row 2, column 1) and (row 2, column 2) [these are the cases where every voter ranks *a* at the top],
(2) in the two tables in the middle (corresponding to the cases where Voter 3 reports *bac* or *bca*) there should be a *b* in the following cells: (row 3, column 3), (row 3, column 4), (row 4, column 3) and (row 4, column 4) [these are the cases where every voter ranks *b* at the top],
(3) in the two tables at the bottom (corresponding to the cases where Voter 3 reports *cab* or *cba*) there should be a *c* in the following cells: (row 5, column 5), (row 5, column 6), (row 6, column 5) and (row 6, column 6) [these are the cases where every voter ranks *c* at the top].
The SCF shown in Figure 13 indeed satisfies these constraints.

(b) To see that Voter 1 is not a dictator, consider the first table (Voter 3 reports *abc*), row 4 (Voter 1 reports *bca*) and column 1 (Voter 2 reports *abc*): *b* is at the top of Voter 1's reported ranking and yet the chosen alternative is *a*.
To see that Voter 2 is not a dictator, consider the first table (Voter 3 reports *abc*), row 1 (Voter 1 reports *abc*) and column 4 (Voter 2 reports *bca*): *b* is at the top of Voter 2's reported ranking and yet the chosen alternative is *a*.
Finally, to see that Voter 3 is not a dictator, consider the first table (Voter 3 reports *abc*), row 6 (Voter 1 reports *cba*) and column 4 (Voter 2 reports *bca*): *a* is at the top of Voter 3's reported ranking and yet the chosen alternative is *b*. □

Solution to Exercise 7. Consider the situation where Voter 1 reports the ranking *cab* (row 5 of any table) and Voter 2 reports *bca* (column 4 of any table). Suppose that Voter 3's true ranking is *abc* (first table). If Voter 3 reports truthfully, then the selected alternative is *c*, which is the worst from his point of view; if, on the other hand, Voter 3 reports *bac* (the second table in the first column of tables), then the selected alternative is *b*, which – according to Voter 3's true ranking *abc* – is better than *c*. □

Solution to Exercise 8.

(a) The reduced SCF is shown in Figure 18 below.

(b) Yes, there are situations where Voter 2 can gain by misrepresenting his preferences. For example, if his true ranking is *cba* (row 6) and he expects Voter 3 to report *bac* (column 3), then by reporting truthfully he brings about his worst outcome, namely *a*, while by lying and reporting *bca* (row 4) he brings about outcome *b* which he prefers to *a*.

(c) Yes, there are situations where Voter 3 can gain by misrepresenting her preferences. For example, if her true ranking is *cba* (column 6) and she expects Voter 2 to report *bac* (row 3), then by reporting truthfully she brings about her worst outcome, namely *a*, while by lying and reporting *bca* (column 4) she brings about outcome *b* which she prefers to *a*.

(d) If Voter 3's true ranking is *bac*, then there is no situation where she can gain by misrepresenting her preferences: (1) if Voter 2 reports *abc* or *acb*, then the outcome is *a* no matter what Voter 3 reports, (2) if Voter 2 reports *bac* or *bca*, then, by reporting truthfully, Voter 3 gets her best outcome, namely *b*, (3) if Voter 2 reports *cab* or *cba*, then, by reporting truthfully, Voter 3 gets outcome *a* and by lying she gets either *a* or her worst outcome, namely *c*.

(e) If Voter 2's true ranking is *cba* and he expects Voter 3 to report *bac*, then he should lie and report either *bac* or *bca* (and thus bring about outcome *b* which he prefers to *a*, which is the outcome he would get if he reported truthfully). □

2's ↓ / 3's →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*a*	*a*	*a*
acb	*a*	*a*	*a*	*a*	*a*	*a*
bac	*a*	*a*	*b*	*b*	*a*	*a*
bca	*a*	*a*	*b*	*b*	*a*	*a*
cab	*a*	*a*	*a*	*a*	*c*	*c*
cba	*a*	*a*	*a*	*a*	*c*	*c*

Assuming that Voter 1 reports acb

Figure 18: The reduced SCF from Figure 5

Solution to Exercise 9. They do not constitute a counterexample to the Gibbard-Satterthwaite theorem because the set of alternatives contains only two elements. The Gibbard-Satterthwaite theorem is based on the premise that there are at least three alternatives. □

Solution to Exercise 10. This SCF fails to satisfy Non-manipulability. One can try to show this by identifying a situation where some individual can gain by misrepresenting her preferences, but a quicker proof is by invoking the

Gibbard-Satterthwaite theorem. All we need to do is show that this SCF satisfies Unanimity and Non-dictatorship, so that – by the Gibbard-Satterthwaite theorem – it must violate Non-manipulability. That Unanimity is satisfied is obvious: if all the individuals list the same alternative x at the top, then – in particular – the first five individuals list x at the top and thus x is chosen. That the SCF satisfies Non-dictatorship is also straightforward: (1) for any of the first five individuals, if she reports x at the top, but at least one of the other first five does not and all of individuals 6-10 report $y \neq x$ at the top, then y is chosen ; (2) for any of Individuals 6-14, if he reports x at the top but the first five individuals all report $y \neq x$ at the top, then y is chosen. □