MIDTERM EXAM

ANSWER ALL QUESTIONS (total 100 points)

1. [30 points] Consider the following strategic-form game. There are n ($n \geq 3$) scientific labs that have been invited to apply for funds. Each lab can submit one of three requests: \$1 million (M), \$100,000 (H) or \$1,000 (T). Each lab is "selfish and greedy" in the sense that it only cares about how much funding it gets and prefers getting more funding to getting less funding. Let m be the number of labs that submit a request of M. It is common knowledge that the funding agency will apply the following rules to allocate funds:

- If $m \leq \frac{n}{3}$ then every lab receives the amount that it requested.
- If $m > \frac{n}{3}$ then every lab receives \$1,000.

(a) [4 points] Are there any strategies that are strictly dominated? Explain your answer.

(b) [6 points] Are there any strategies that are weakly dominated? Explain your answer.

(c) [10 points] Suppose that n = 54. Find all the pure-strategy Nash equilibria. Explain why what you propose are Nash equilibria and why there are no other Nash equilibria.

(d) [10 points] Suppose that n = 3. Find a symmetric mixed-strategy Nash equilibrium (symmetric means that each lab randomizes with the same probabilities), where only strategies that are not weakly dominated are played with positive probability. [If you don't have time, just write the relevant equation without attempting to solve it.]

2. [22 points] Consider an industry where there are 2 firms producing an identical product (that is, the products are perfect substitutes). Production costs are zero and the demand function is

$$Q = \max\{120 - p, 0\}.$$

The firms compete in prices. When one firm charges a lower price than the other then that firm serves the entire demand; if the two firms charge the same price then each firm caters for half of the demand. Prices can only be integers; thus the strategy set of each firm is the set of non-negative integers.

(a) [8 points] Write the best reply function of firm 2.

(b) [6 points] Find all the pure-strategy Nash equilibria.

(c) [8 points] Now suppose that firm 1 moves first and commits to a price, which is observed by firm 2. Firm 2 then announces its own price and consumers react as explained above (thus Firm 1 acts as a Stackelberg leader and Firm 2 as a Stackelberg follower). Find the backward-induction solution(s) of this perfect-information game.

3. [30 points] An industry is currently a monopoly. It is common knowledge that there is a potential entrant who is considering entering the industry. The monopolist can either be Passive or take an Action that costs $*K*. The potential entrant observes whether the incumbent was passive or took the action and decides whether or not to enter. If the potential entrant stays out then it earns $*A* in an alternative investment. If the potential entrant enters then there is a simultaneous duopoly game between the incumbent and the entrant. Let π_M be monopoly profits when the incumbent is passive and $\hat{\pi}_M - K$ be monopoly profit when the incumbent takes the costly action.

(a) [2 points] Sketch the extensive-form game.

Now analyze the following instance of the above game.

1. Inverse industry demand is given by $P(Q) = 120 - 2Q$.
2. Currently the incumbent monopolist's cost function is $C(q) = 12q$.
   This is also the cost function that the entrant would have if it entered.
3. For the potential entrant the return to the alternative investment is $600 (that is, A = 600).
4. If the incumbent monopolist is passive then it keeps its current cost function.
5. If the incumbent monopolist spends $*K* then its cost function changes to $C(q) = 6q$.
   (It will be a patented production technique that the entrant will not have access to: see Point 2 above).
6. If there is entry, then there will be a Cournot game between the two firms.

(b) [28 points] Find the subgame-perfect equilibria of this game for every possible value of *K*.

4. [18 points] First some notation: given a set of states, e.g. $\{a,b,c,d,e,f,g,h\}$, and an event, e.g. $F = \{a,d,e,f,h\}$, then $\neg F$ (read "not *F*") denotes the complement of *F*, that is, the set of states that are not in *F*; thus, in this example, $\neg F = \{b,c,g\}$.

The set of states is $\{a,b,c,d,e,f,g,h\}$,
individual 1's information partition is $\{\{a,d\},\{b\},\{c\}, \{e,f\}, \{g\}, \{h\}\}$,
individuals 2's information partition is $\{\{a,b\},\{c\},\{d\}, \{e\}, \{f,g\}, \{h\}\}$,
individual 3's information partition is $\{\{a,c\},\{b,d\}, \{e\}, \{f,g\}, \{h\}\}$.

(a) [6 points] Construct the common knowledge partition.

(b) Let E = {a,b,c,f,g,h}. Find the following events:
(b.1) [2 points] K_1E (1 knows E),
(b.2) [4 points] $\neg K_3K_2\neg K_1E$ (it is not the case that 3 knows that 2 knows that 1 does not know E),
(b.3) [3 points] CKE (it is common knowledge that E)
(b.4) [3 points] $\neg K_1CKE$ (it is not the case that 1 knows that E is commonly known).

Midterm Exam ANSWERS

1. **(a)** No. Consider player *i*. If more than $\frac{n}{3}$ of the other players choose *M* then player *i* gets \$1,000 no matter what she chooses.

(b) Yes: *T* is weakly dominated by both *H* and *M*. Choosing *T* guarantees that player *i* will get \$1,000, while choosing one of the other strategies will yield at least \$1,000 in every case and more than \$1,000 in at least one case. *H* in **not** weakly dominated by *M* because, if exactly $\frac{n}{3}$ of the other players choose *M*, then *H* yields \$100,000 while *M* yields \$1,000. Clearly, *M* is strictly better than *H* when nobody else chooses *M*.

(c) When $n = 54$, $m = 18$. Then

- Every strategy profile where exactly 18 labs choose *M* and the rest choose *H* is a Nash equilibrium. (Each of the labs that choose *M* gets \$1,000,000 and the others get \$100,000; if one of the ones that choose *M* reduces its request then it gets less and if one of the ones who choose *H* changes it strategy then it will get only \$1,000).
- Every strategy profile where 20 or more labs choose *M* and the rest choose either *H* or *T* is a Nash equilibrium. (They all get \$1,000 and there is no way that any lab can get more by unilaterally changing its strategy).

There are no other Nash equilibria:
- if $m < 18$ then one of the labs that is not choosing *M* can increase its funding by switching to *M,*
- if $m = 19$ then one of the labs that is choosing *M* can increase its funding from \$1,000 to \$100,000 by switching to *H.*

(d) When $n = 3$, $m = 1$. We are looking for a Nash equilibrium where everybody chooses *M* with probability *p* and *H* with probability $(1-p)$ with $0 < p < 1$. In order for this to be a Nash equilibrium it must be that choosing *M* for sure yields the same expected payoff as choosing *H* for sure (given that the other choose *M* with probability *p* and *H* with probability $(1-p)$). The probability that none of the other two players chooses *M* is $(1-p)^2$ and the probability that both of the other players chooses *M* is p^2.

Expected payoff from *M* : $1{,}000{,}000\ (1-p)^2 + 1{,}000\left[1-(1-p)^2\right]$

Expected payoff from choosing *H*: $100{,}000\left(1-p^2\right) + 1{,}000\ p^2$.

Thus we need to solve

$$1{,}000\ (1-p)^2 + \left[1-(1-p)^2\right] = 100\left(1-p^2\right) + p^2$$

The solution is $p = \frac{50}{61}$. The expected payoff (from either *M* or *H*) is \$33,485.62, which is greater than the payoff from choosing *T* (\$1,000); thus this is indeed a Nash equilibrium.

2. **(a)** The best reply of Firm 2 is as follows:

$$BR_2(p_1) = \begin{cases} 60 & \text{if } p_1 > 60 \\ p_1 - 1 & \text{if } 2 \le p_1 \le 60 \\ 1 & \text{if } p_1 = 1 \\ \mathbb{N} & \text{if } p_1 = 0 \end{cases}$$

In fact, monopoly price is 60 . Thus (1) the best reply to $p_1 > 60$ is to undercut Firm 1 by charging monopoly price; (2) if $p_1 = 1$ then Firm 2 gets zero profits if it charges more than 1 or less than 1, while by charging 1 it gets a positive profit equal to $\frac{1}{2}(120-1)$; (3) if $p_1 = 0$ then Firm 2 gets zero profits no matter what price it chooses. So we are left with the case $2 \le p_1 \le 60$. In this case if Firm 2 sets $p_2 = p_1$ it gets a profit equal to $s(p_1) = p_1 \frac{1}{2}(120 - p_1)$ while if it decides to undercut then it should do so by the smallest amount (since we are on the increasing portion of the profit function of a monopoly), thereby getting a profit of $u(p_1) = (p_1 - 1)(120 - p_1 + 1)$. The gain from undercutting Firm 1 by 1 unit is thus

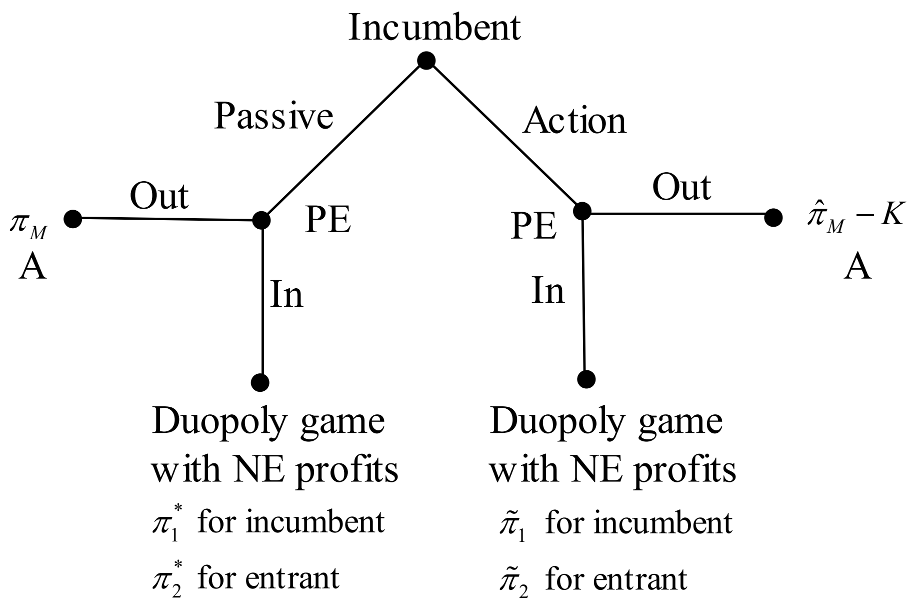
$$s(p_1) - u(p_1) = \frac{p_1^2}{2} - 62p_1 + 121$$

Thus expression is positive if $p_1 = 1$ (confirming that the best reply to $p_1 = 1$ is $p_2 = 1$) and negative for $p_2 \ge 2$ (in fact, the solution to $\frac{p_1^2}{2} - 62p_1 + 121 = 0$ is 1.9833).

(b) It follows from Part (a) that there are two pure-strategy Nash equilibria: (0,0) and (1,1).

(c) From Part (a) we have that if Firm 1 commits to a price $p_1 \ge 2$ Firm 2 will undercut Firm 1 and thus Firm 1 will get zero profits; if Firm 1 commits to $p_1 = 0$ then it will make zero profits; finally, if Firm 1 commits to $p_1 = 1$ then Firm 2 will react by choosing $p_2 = 1$ and both firms will make positive profits. Thus the backward-induction solution is $(1, BR_2(p_1))$ where $BR_2(p_1)$ is the function given in Part (a) (recall that a backward-induction solution is defined as a strategy profile: it is not enough to specify what price Firm 2 actually charges: one needs to specify what price Firm 2would charge in response to every possible price of Firm 1).

3. **(a)** The structure of the game is as follows:



(b) Referring to the above sketch, we have that $\pi_M = 1,458$ (with corresponding output of 27 units), $\hat{\pi}_M = 1,624.5$ (with corresponding output of 28.5 units), $\pi_1^* = \pi_2^* = 648$ (at the Cournot equilibrium each firm produces 18 units),
$\tilde{\pi}_1 = 800,\ \tilde{\pi}_2 = 578$ (at the Cournot equilibrium the incumbent produces 20 units and the entrant produces 17 unit.). Note that $1,624 - 648 = 976.5$. Thus

- If $0 \leq K < 976.5$ the incumbent will take the action and the entrant will react by staying out. The SPE strategies are:
  Incumbent: (Action, (q = 27 if no entry, q = 18 if entry), (q = 28.5 if no entry, q = 20 if entry)
  Potential Entrant: (In, Out, q = 18, q = 17).
- If $K > 976.5$ the incumbent will be passive and the entrant will react by entering. The SPE strategies are:
  Incumbent: (Passive, (q = 27 if no entry, q = 18 if entry), (q = 28.5 if no entry, q = 20 if entry)
  Potential Entrant: (In, Out, q = 18, q = 17).
- If $K = 976.5$ then both of the above are SPE

4. **(a)** $\{\{a,,b,c,d\}, \{e,f,g\}, \{h\}\}$.

(b) **(b.1)** $K_1E = \{b,c,g,h\}$

(b.2) compute from right to left: $\neg K_1E = \{a,d,e,f\}$, $K_2\neg K_1E = \{d,e\}$, $K_3K_2\neg K_1E = \{e\}$; thus $\neg K_3K_2\neg K_1E = \{a,b,c,d,f,g,h\}$.

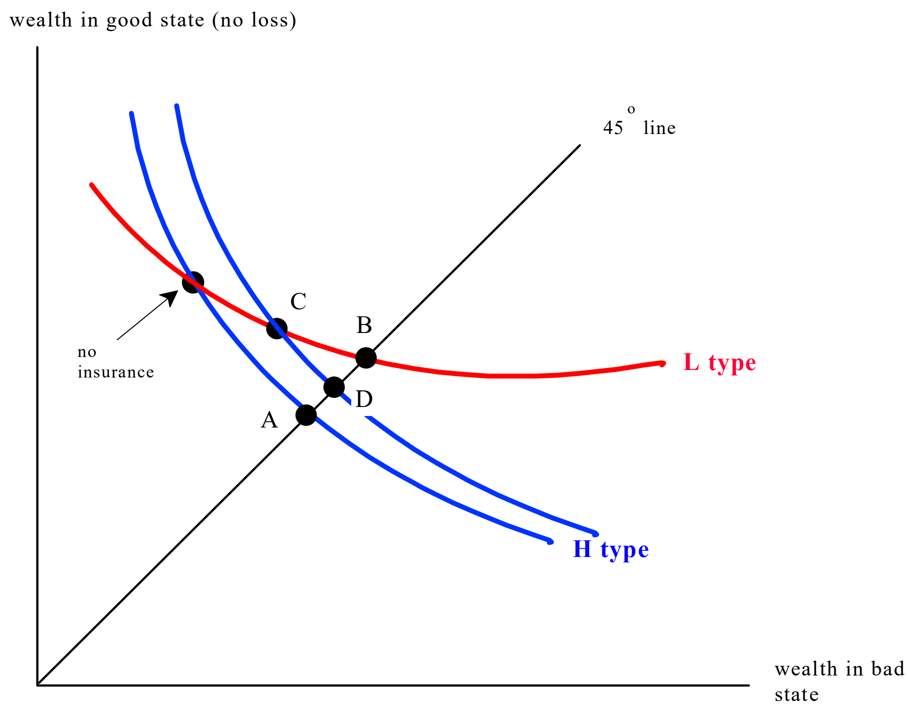
(b.3) $CKE = \{h\}$

(b.4) $K_1CKE = CKE = \{h\}$; thus $\neg K_1CKE = \{a,b,c,d,e,f,g\}$

FINAL EXAM

ANSWER ALL QUESTIONS (total 100 points)

5. [30 points] There are two types of individuals. They have identical initial wealth of \$2,500, they face a potential loss of \$900 and they have a utility-of-money function $U(m) = \sqrt{m}$. For individuals of type H the probability of loss is $p_H = 20\%$ while for individuals of type L the probability of loss is $p_L = 10\%$. Let $N_H \geq 1$ be the number of H types and $N_L \geq 1$ the number of L types. The insurance market is a monopoly. The monopolist knows all of the above data but cannot tell whether any particular customer is of type H or type L (while each potential customer knows her own type). The monopolist is considering several options (refer to the following figure). **Assume that (1) if indifferent between insuring and not insuring, a consumer would choose to insure and (2) if indifferent between two contracts, then the consumer would choose the one with lower deductible.**



(a) [8 points] Option 1: offer only contract A. Calculate the monopolist's profits in this case.

(b) [8 points] Option 2: offer only contract B. Calculate the monopolist's profits in this case.

(c) [14 points] Option 3: Offer contracts C and D and let consumer choose. The premium for contract C is \$6.24 and the premium for contract D is \$190. Calculate the monopolist's profits in this case.

6. [40 points] A monopolist faces 50 type-A consumers, 60 type-B consumers and 80 type-C consumers. Each type-A consumer has the demand function $D_A(P) = 60 - 2P$, each type-B consumer has the demand function $D_B(P) = 60 - 4P$ and each type-C consumer has the demand function $D_C(P) = 60 - 8P$. The monopolist's cost function is C(Q) = 2Q. The monopolist is considering selling the good in packages. The pair (Q, V) represents a package containing Q units at a total price of V (thus V is the price of the entire package, not the price per unit). For some reason unknown to us, the monopolist has decided that no consumer is allowed to buy more than one package. Furthermore, resale among consumers is prohibited and the ban is enforceable. The monopolist is considering the following options.

OPTION 1. Offer only one type of package ($Q_1 = 20$, $V_1 = 170$).

OPTION 2. Offer two types of packages: ($Q_{21} = 20$, $V_{21} = 124$) and ($Q_{22} = 30$, $V_{22} = 211$).

OPTION 3. Offer two types of packages: ($Q_{31} = 30$, $V_{31} = 168$) and ($Q_{32} = 56$, $V_{32} = 388$).

(a) [8 points] Calculate the monopolist's profits for Option 1.

(b) [8 points] Calculate the monopolist's profits for Option 2.

(c) [8 points] Calculate the monopolist's profits for Option 3.

(d) [8 points] What would the monopolist's profits be if it used linear pricing (that is, if it set a price per unit and sold whatever quantity was demanded at that price)?

(e) [8 points] What would the monopolist's profits be if it were able to determine the type of each consumer?

7. [12 points] Consider the following Spence-like model. There are three types of individuals: A, B and C. The cost of acquiring y units of education is given in (1) below; the productivity of each type is affected by education and is given in (2) below.

$$(1)\begin{cases}\text{For type A:} & C_A(y) = \dfrac{y}{5} \\ \text{For type B:} & C_B(y) = \dfrac{y}{4} \\ \text{For type C:} & C_C(y) = \dfrac{y}{2}\end{cases} \qquad (2)\begin{cases}\text{For type A:} & 3 + \dfrac{y}{6} \\ \text{For type B:} & 2 + \dfrac{y}{6} \\ \text{For type C:} & 1 + \dfrac{y}{6}\end{cases}$$

Suppose that the employer offers to pay each applicant a wage which is a function of the applicant's education, as follows: $\begin{cases} 1 + \dfrac{y}{6} & \text{if } 0 \le y < d \\ 2 + \dfrac{y}{6} & \text{if } d \le y < e \\ 3 + \dfrac{y}{6} & \text{if } y \ge e \end{cases}$.

Write a system of inequalities that need to be satisfied in order to have a signaling equilibrium (don't worry about the possibility of indifference: just write strict inequalities; **no need to simplify the inequalities**). [An equilibrium is a signaling equilibrium if it is fully separating and everybody is paid an amount that matches her true productivity.]

8. [18 points] Consider the market for second-hand cars. There are four possible qualities with the corresponding proportions, values to seller and buyer indicated in the table below, with $\alpha > 1$ and $s_A > s_B > s_C > s_D$.

Quality of car	A	B	C	D
proportion	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
value to seller	s_A	s_B	s_C	s_D
value to buyer	αs_A	αs_B	αs_C	αs_D

The utility-of-money function of a seller is $\begin{cases} m + s_i & \text{if he owns a car of quality } i \\ m & \text{if he does not own a car} \end{cases}$ and the utility of a buyer is $\begin{cases} m + \alpha s_i & \text{if he owns a car of quality } i \\ m & \text{if he does not own a car} \end{cases}$. In the initial situation each seller owns exactly one car and all sellers and buyers have the same initial endowment of money, which you can assume to be "sufficiently large".

All sellers and buyers are risk neutral. While each seller knows the quality of his car, each buyer is unable to determine the quality of any car offered for sale. Thus there can be only one price for used cars, call it *p*. If indifferent between selling and not selling (respectively, buying and not buying) a seller (resp. a buyer) would choose to sell (resp. buy).

When asked to give necessary and/or sufficient conditions, you should give them in terms of the parameters only (which are the s_i's and α).

(a) [9 points] Give necessary and sufficient conditions for there to exist a price *p* at which all and only cars of qualities B, C and D are traded.

(b) [9 points] Give necessary and sufficient conditions for there to exist a price *p* at which all and only cars of qualities C and D are traded.

Final Exam ANSWERS

1.

(a) Contract A is a full-insurance contract that will be bought only by the H types. To calculate the premium for contract A solve $\frac{4}{5}\sqrt{2500}+\frac{1}{5}\sqrt{1600}=\sqrt{2500-h}$. The solution is h = 196. Thus the monopolist's profits are $\pi_1=\left[196-\frac{1}{5}900\right]N_H=16N_H$.

(b) Contract B is a full-insurance contract that will be bought by all types. To calculate the premium for contract B solve $\frac{9}{10}\sqrt{2500}+\frac{1}{10}\sqrt{1600}=\sqrt{2500-h}$. The solution is h = 99. Thus the monopolist's profits are $\pi_2=\left[99-\frac{1}{5}900\right]N_H+\left[99-\frac{1}{10}900\right]N_L=9N_L-81N_H$.

(c) The H types would choose the full-insurance contract D while the L types would choose the partial-insurance contract C. To compute the deductible for contract C solve $\frac{9}{10}\sqrt{2,500-6.24}+\frac{1}{10}\sqrt{2,500-6.24-d}=\frac{9}{10}\sqrt{2500}+\frac{1}{10}\sqrt{1600}$. The solution is d = 848.48. Thus the monopolist's profits are
$\pi_3=\left[190-\frac{1}{5}900\right]N_H+\left[6.24-\frac{1}{10}\left(900-848.48\right)\right]N_L=10N_H+1.088\mathrm{N}_L$.
[Slightly different numbers are obtained if the premium for contract C is computed using the indifference curve of the H type by solving
$\sqrt{2,500-190}=\frac{1}{5}\sqrt{2,500-6.25-h}+\frac{4}{5}\sqrt{2,500-6.25}$; in this case the solution is 848.44 with corresponding profits of
$\pi_3=\left[190-\frac{1}{5}900\right]N_H+\left[6.25-\frac{1}{10}\left(900-848.44\right)\right]N_L=10N_H+1.094\mathrm{N}_L$.]

2.

Let $W_i(Q)$ be the willingness to pay of customer of type $i\in\{A,B,C\}$ for Q units. Then

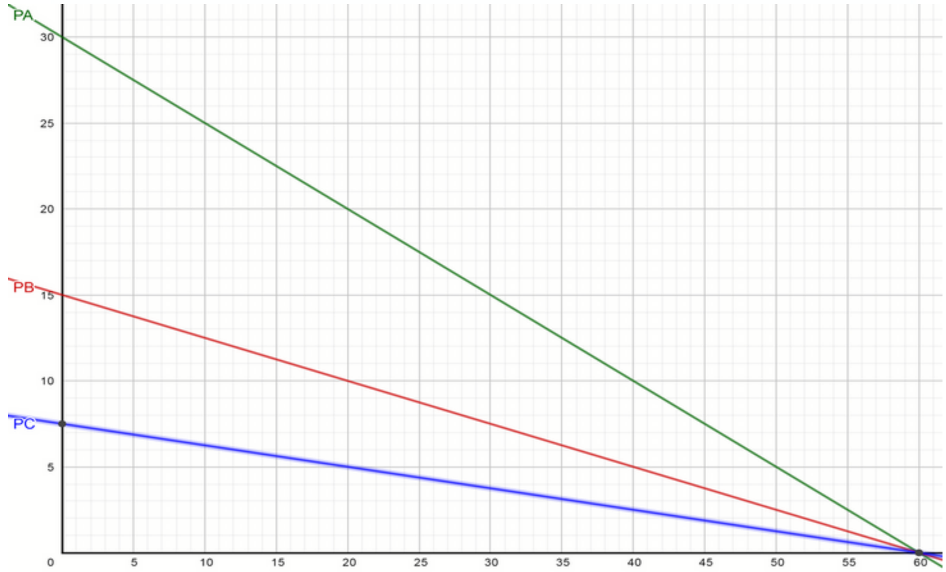
$$W_A(Q)=30Q-\frac{Q^2}{4},\ W_B(Q)=15Q-\frac{Q^2}{8} \text{ and } W_C(Q)=\frac{15}{2}Q-\frac{Q^2}{16}.$$

(a) Since $W_C(Q_1)-V_1=-45$, $W_B(Q_1)-V_1=80$ and $W_A(Q_1)-V_1=330$, only customers of type A and B buy. Thus the firm's profits will be $\Pi_1=(50+60)[170-2(20)]=\boxed{14,300}$

(b) Since $W_C(Q_{21})-V_{21}=1$, $W_C(Q_{22})-V_{22}=-42.25$, $W_B(Q_{21})-V_{21}=126$, $W_B(Q_{22})-V_{22}=126.5$, $W_A(Q_{21})-V_{21}=376$, $W_A(Q_{22})-V_{22}=464$, C-customers purchase the first package, while the others purchase the second package. Thus
$\Pi_2=80[124-2(20)]+110[211-2(30)]=\boxed{23,330}$

(c) Since $W_C(Q_{31})-V_{31}=0.75$, $W_C(Q_{32})-V_{32}=-164$, $W_B(Q_{31})-V_{31}=169.5$, $W_B(Q_{32})-V_{32}=60$, $W_A(Q_{31})-V_{31}=507$, $W_A(Q_{32})-V_{32}=508$, B and C customers purchase the first package, while A customers purchase the second package. Thus
$\Pi_3=140(168-60)+50(388-112)=\boxed{28,920}$

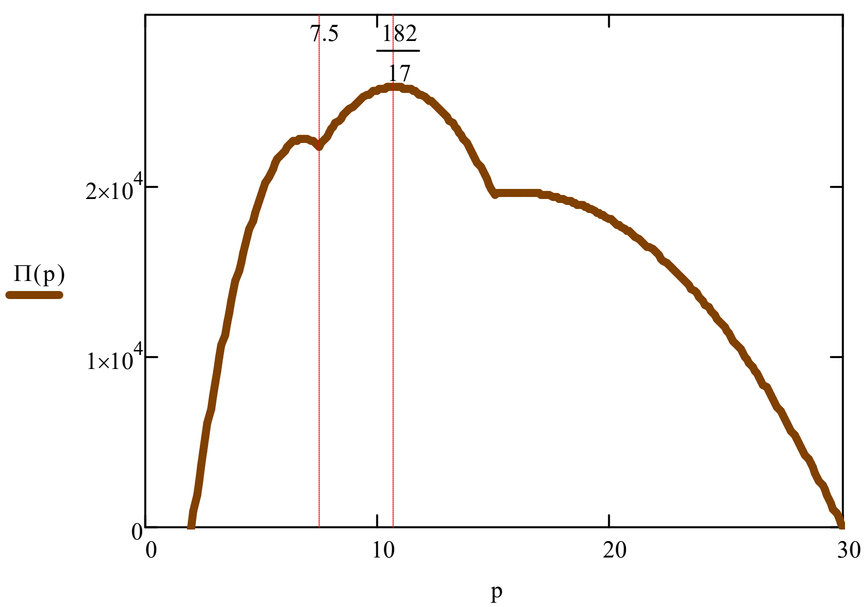
(d) For linear pricing we first need to calculate aggregate demand. The demand functions are:



Thus aggregate demand is

$$D(P)=\begin{cases}50D_A(P)+60D_B(P)+80D_C(P) & \text{if } 0\le P<7.5\\ 50D_A(P)+60D_B(P) & \text{if } 7.5\le P<15\\ 50D_A(P) & \text{if } P\ge 15\end{cases}$$

The profit function is $\pi(P)=(P-2)D(P)$. The maximum occurs at $P=\frac{182}{17}=10.706$ with corresponding profits of $\pi_{linear}=\Pi(10.706)=25,769.412$ (note that $\Pi(7.5)=22,275$ and $\Pi(15)=19,500$). Thus with linear pricing the monopolist serves only types A and B.



(e) The monopolist would sell to each consumer a bundle containing the quantity at which MC crosses the consumer's demand and charge a price for the bundle equal to the consumer's willingness to pay for that quantity. Bundle for Type A: (Q = 56, V = 896) (with a profit per bundle of 784), bundle for type B: (Q = 52, V = 442) (with a profit per bundle of 338), bundle for type C: (Q = 44, V = 209) (with a profit per bundle of 121). Its total profits would be 50(784)+60(338)+80(121) = 69,160.

3.

(a) Note that, for each type the cost of an extra unit of education exceeds the benefit (in terms of higher salary) of that extra unit. Thus each type will only consider three levels of education: 0, *d* and *e*. The inequalities are as follows:

$$\text{For type A: } \begin{cases} \text{(1) [e better than d]} & 3+\frac{e}{6}-\frac{e}{5}>2+\frac{d}{6}-\frac{d}{5} \\ \text{(2) [e better than 0]} & 3+\frac{e}{6}-\frac{e}{5}>1 \end{cases}$$

$$\text{For type B: } \begin{cases} \text{(3) [d better than e]} & 2+\frac{d}{6}-\frac{d}{4}>3+\frac{e}{6}-\frac{e}{4} \\ \text{(4) [d better than 0]} & 2+\frac{d}{6}-\frac{d}{4}>1 \end{cases}$$

$$\text{For type C: } \begin{cases} \text{(5) [0 better than e]} & 1>3+\frac{e}{6}-\frac{e}{2} \\ \text{(6) [0 better than d]} & 1>2+\frac{d}{6}-\frac{d}{2} \end{cases}$$

(b) Yes, when $d = 10$ and $e = 24$ all of the inequalities are satisfied.

(c) No, $d = 10$ and $e = 21$ inequality (3) is not satisfied (while the others are).

4.

(a) Let q_i denote the probability of quality *i* (thus $q_A = \frac{1}{8}$, etc.). We need to be able to find a *p* such that $s_B \le p < s_A$ (this is always possible since, by hypothesis, $s_B < s_A$) and $p \le \sum_{i \in \{B,C,D\}} \left(\frac{q_i}{q_B + q_C + q_D} \right) \alpha s_i$, that is, $p \le \frac{\alpha}{7}(3s_B + 3s_C + s_D)$. Thus the necessary and sufficient condition is $\boxed{s_B \le \frac{\alpha}{7}(3s_B + 3s_C + s_D)}$.

(b) We need to be able to find a *p* such that $s_C \le p < s_B$ (this is always possible since, by hypothesis, $s_C < s_B$) and $p \le \sum_{i \in \{C,D\}} \left(\frac{q_i}{q_C + q_D} \right) \alpha s_i$, that is, $p \le \frac{\alpha}{4}(3s_C + s_D)$ Thus the necessary and sufficient condition is $\boxed{s_C \le \frac{\alpha}{4}(3s_C + s_D)}$.