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MIDTERM EXAM (total 100 points)

1. [56 points] Two agents, 1 and 2, can contribute to the provision of a public good. The good will be provided if and only if the sum of their contributions exceeds some critical level $X > 0$. In this case, each agent obtains a benefit $B > 0$. However, contributing is costly and contributions are non-refundable. The cost of contributing $x \geq 0$ is $c(x) = x$. The payoff of a player who contributes x is $B - x$ if the public good is provided and $-x$ if the public good is not provided. **Assume that $2B > X$** so that providing the good is efficient.

Suppose first that the agents simultaneously choose contribution levels x_1 and x_2, after which the good is provided if and only if $x_1 + x_2 \geq X$.

(a) [4 points] Find the pure strategy Nash equilibria for the case where $B > X$.

(b) [4 points] Find the pure strategy Nash equilibria for the case where $B = X$.

(c) **(c.1)** [4 points] Find the pure strategy Nash equilibria for the case where $B < X$.
(c.2) [4 points] Explain why there is at least one Nash equilibrium where both players contribute a positive amount.

Suppose now that the agents contribute sequentially, with agent 1 contributing in period 1 and agent 2 in period 2. Assume that contributions are observable and that the good is provided as soon as the total contribution level is at least X (that is, either in the first period, the second period, or never). Suppose that there is no discounting, so that if an agent contributes x her payoff is $B - x$, whether the public good is provided in period 1 or in period 2, and $-x$ if it isn't provided.

(d) Suppose that $B > X$.
(d.1) [4 points] Find the best reply function of Player 2.
(d.2) [4 points] Find the backward-induction solution(s).
(d.3) [4 points] Is the public good provided in the first period, the second period, or never?

(e) Suppose that $B = X$.
(e.1) [4 points] Find the best reply function of Player 2.
(e.2) [4 points] Find the backward-induction solution(s).
(e.3) [4 points] Is the public good provided in the first period, the second period, or never?
(e.4) [4 points] Find a Nash equilibrium that is not a backward-induction solution.

(f) Suppose that $B < X$ (recall that, by assumption, $2B > X$).
(f.1) [4 points] Find the best reply function of Player 2.
(f.2) [4 points] Find the backward-induction solution(s).
(f.3) [4 points] Is the public good provided in the first period, the second period, or never?

2. [32 points] It has been documented in the psychology literature that the probability that an individual offers assistance to someone in need is lower when he is in a group than when he alone (see Latané, B. and Nida, S., Ten years of research on group size and helping, *Psychological Bulletin*, 89, 1981, pp. 308-324). Let us show this phenomenon in a simple game. There are $N \geq 2$ potential helpers (call them players) of a person in need. Each player decides whether to Help or Ignore. Decisions are made simultaneously and independently. Each player cares about whether the victim is helped, whether she helps the victim and whether anyone else helps the victim. The von Neumann-Morgenstern payoffs are denoted as follows:

$a \equiv$ payoff to a player if she helps and no one else helps,

$b \equiv$ payoff to a player if she does **not** help and someone else helps,

$c \equiv$ payoff to a player if she helps and someone else helps,

$d \equiv$ payoff to a player if no one helps.

All the players are identical and the game and payoffs are common knowledge.

(a) [4 points] Assume that each player is concerned with the welfare of the victim (wants the victim to be helped) but, at the same time, help involves effort, so that each player would rather free ride on someone else's help. Express these two assumptions in terms of restrictions on the parameters *a, b, c, d.*

(b) [4 points] Find all the pure-strategy Nash equilibria of this game.

(c)[4 points] Find the symmetric mixed-strategy equilibrium (symmetric means that all the players adopt the same strategy).

In what follows consider the special case where $a = 10, b = 16, c = 8, d = 2$.

(d) **(d.1)** [4 points] Show that, for each player, at the mixed-strategy equilibrium the probability that she helps is decreasing in N.

(d.2) [4 points] Calculate the probability, call it *h*, that the victim is helped at the mixed-strategy equilibrium.

(d.3) [4 points] Show that *h* is decreasing in *N*.

(d.4) [4 points] Verify this by computing the value of *h* when $N = 2$ and when $N = 8$.

(d.5) [4 points] Calculate the limit of *h* as *N* goes to infinity.

3. [12 points] Consider the following two-player game-frame. Rudy is a fugitive and it is common knowledge between him and the Chief of Police that (1) there are only two locations, A and B, where he could be hiding and (2) the police will be able to pinpoint the exact location of Rudy's cellphone, but only if it is on, that is, if it communicates with the cell towers. Rudy makes three consecutive decisions: first whether to hide at location A or at location B, second whether to turn off his cell phone or leave it on and third - in case he leaves his cellphone on - whether to keep it with himself or give it to an accomplice to take it to the other location and leave it there (turned on). The next day, the police chief -- not having observed any of the above decisions by Rudy -- after consulting the tracking device and checking whether or not there is a signal from the cell-phone and, if there is, where it is coming from, decides whether to storm location A or location B.

(a) [6 points] Draw an extensive-form game-frame to represent this situation.

(b) [3 points] How many strategies does the **police chief** have?

(c) [3 points] How many strategies does **Rudy** have?

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SPRING 2023 **ECN/ARE 200C: MICROECONOMIC THEORY** Giacomo Bonanno

Midterm Exam ANSWER

1. **(a)** When $B > X$ the set of Nash equilibria is the set of pairs (x_1, x_2) such that (1) $0 \leq x_1 \leq X$, (2) $0 \leq x_2 \leq X$ and (3) $x_1 + x_2 = X$.

(b) When $B = X$ the set of Nash equilibria is the set of pairs (x_1, x_2) of part (a) as well as the pair (0,0).

(c) **(c.1)** When $B < X$ the set of Nash equilibria is the set of pairs (x_1, x_2) such that (1) ~~$0 \leq x_1 \leq B$~~ $X - B \leq x_1 \leq B$, (2) ~~$0 \leq x_2 \leq B$~~ $X - B \leq x_2 \leq B$ and (3) $x_1 + x_2 = X$, as well as the pair (0,0).

(c.2) Since $2B > X$, $X - B < \frac{X}{2} < B$ and thus the pair (x_1, x_2) with $x_1 = x_2 = \frac{X}{2}$ belongs to the set detailed in part (c.1).

(d) Assume that $B > X$.

(d.1) Player 2's best reply function is $x_2 = \begin{cases} X - x_1 & \text{if } x_1 \leq X \\ 0 & \text{if } x_1 > X \end{cases}$.

(d.2) The unique backward-induction solution is $x_1 = 0$ together with Player 2's strategy given in part (d.1).

(d.3) The backward-induction outcome is $(x_1 = 0,\ x_2 = X)$ and thus the public good is provided in period 2.

(e) Assume that $B = X$.

(e.1) Player 2's best reply function is $x_2 = \begin{cases} \{0, X\} & \text{if } x_1 = 0 \\ X - x_1 & \text{if } 0 < x_1 \leq X \\ 0 & \text{if } x_1 > X \end{cases}$.

(e.2) There is only one backward-induction solution: $x_1 = 0$ together with Player 2's strategy given in part (d.1) with $x_2 = X$. If Player 2's responds to $x_1 = 0$ with $x_2 = 0$ then Player 1 would prefer to contribute a positive amount $\hat{x}_1$ to which Player 2 would respond with $X - \hat{x}_1$; the problem is that Player 1 would want to choose the smallest such positive amount and it does not exist if the strategy sets are continuous. If the strategy sets were discrete then Player 1 would choose the smallest positive contribution and we would have a second backward-induction solution.

(e.3) The backward-induction outcome is $(x_1 = 0,\ x_2 = X)$ with the public good being provided in period 2.

(e.4) There are several. One of them is: $x_1 = \frac{X}{2}$ and Player 2's strategy is to choose $x_2 = \frac{X}{2}$ if $x_1 = \frac{X}{2}$ and $x_2 = 0$ otherwise.

(f) Assume that $B < X$.

(f.1) Player 2's best reply function is $x_2 = \begin{cases} 0 & \text{if } 0 \leq x_1 < X - B \text{ (i.e. if } X - x_1 > B) \\ \{0, B\} & \text{if } x_1 = X - B \\ X - x_1 & \text{if } x_1 > X - B \text{ (i.e. if } X - x_1 < B) \end{cases}$.

(f.2) Note that, since, by hypothesis, $2B > X$, $B > X - B$. The backward-induction solution is $x_1 = X - B$ together with Player 2's strategy given in part (f.1) with $x_2 = B$ (the same logic applies here as in part (e.2) about Player 1's best reply to Player 2's strategy given in part (f.1) with $x_2 = 0$.).

(f.3) The backward-induction outcome is $(x_1 = X - B,\ x_2 = B)$, so that the public good is provided in period 2.

2. **NOTE: this was Question 6.15 in the textbook.**

(a) $b > a > d$ and $b > c$; not enough information to decide whether $a > c$, $c > a$ or $a = c$.

(b) There are N pure-strategy equilibria. At each of these there is exactly one player who helps.

(c) Let p be the probability that a player **helps** (the same for all players, because we are looking for a symmetric equilibrium). Then p must be such that each player is indifferent between helping and not helping. The probability that *none* of the other players help is $(1-p)^{N-1}$. Thus if a player does *not* help her expected payoff is

$$d\,(1-p)^{N-1} + b\,[1-(1-p)^{N-1}].$$

On the other hand, her expected payoff if she does help is

$$a\,(1-p)^{N-1} + c\,[1-(1-p)^{N-1}].$$

Thus p is the solution to

$$a\,(1-p)^{N-1} + c\,[1-(1-p)^{N-1}] = d\,(1-p)^{N-1} + b\,[1-(1-p)^{N-1}].$$

Hence the equilibrium probability of helping is

$$p = 1 - \left(\frac{b-c}{a-d+b-c}\right)^{\left(\frac{1}{N-1}\right)}$$

(d.1) When $a = 10$, $b = 16$, $c = 8$, $d = 2$, the above expression becomes $p = 1-\left(\frac{1}{2}\right)^{\frac{1}{N-1}}$ which is decreasing in N. In fact, $\frac{dp}{dN} = -\frac{\ln(2)}{(N-1)^2}\left(\frac{1}{2}\right)^{\frac{1}{N-1}} < 0$.

(d.2) The probability that *nobody* helps is $(1-p)^N$ where p is as given above. Thus the probability that the victim is helped is

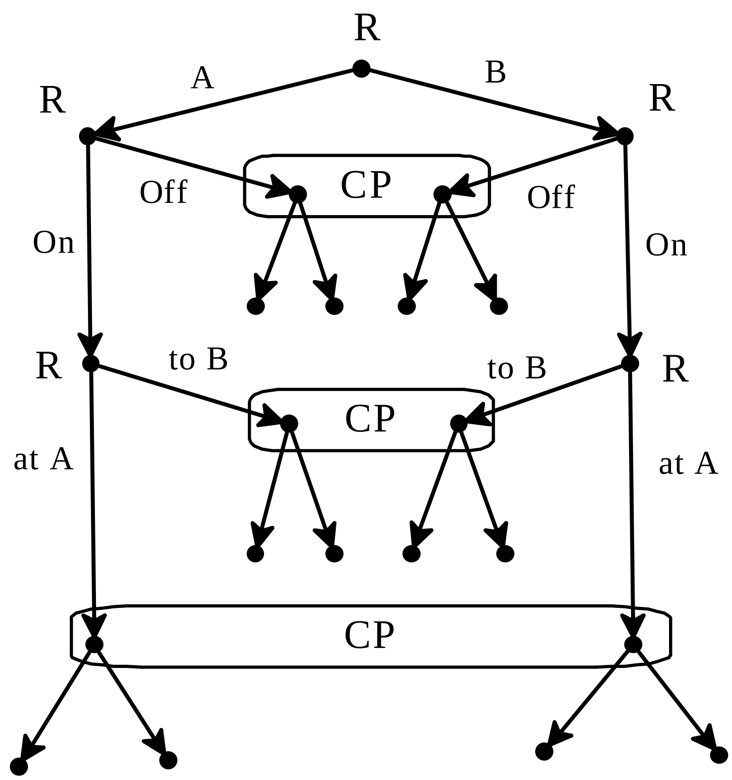
$$h(N) = 1-(1-p)^N = 1-\left(\frac{1}{2}\right)^{\frac{N}{N-1}}$$

(d.3) This is decreasing in N. In fact $\frac{dh}{dN} = -\frac{\ln(2)}{(N-1)^2}\left(\frac{1}{2}\right)^{\frac{N}{N-1}} < 0$

(d.4) When $N = 2$ the probability that the victim is helped is $h(2) = 0.75$, while when $N = 8$ the probability is $h(8) = 0.547$.

(d.4) The limit of $h(N)$ as $N \to \infty$ is 0.5.

3. **(a)** The game-frame is as follows:



(b) $2^3 = 8$

(c) $2^5 = 32$

FINAL EXAM

ANSWER ALL QUESTIONS (total 100 points)

1. [50 points] There are two groups of individuals. All the individuals in Group 1 have the same utility function which is as follows:

$$\begin{cases} m+q & \text{if owns a car of quality } q \text{ and } \$m \\ \text{m} & \text{if owns } \$m \text{ but no car} \end{cases}$$

All the individuals in Group 2 have the same utility function which is as follows:

$$\begin{cases} m+\gamma q & \text{if owns a car of quality } q \text{ and } \$m \\ m & \text{if owns \$m but no car} \end{cases} \quad \text{with } \gamma > 1.$$

All cars are owned by individuals in Group 1. Each car is of one of the qualities in the set $Q = \{50, 100, \ldots, 50n\}$, where $n \geq 2$. There is an equal number of cars of each quality (i.e. as many cars of quality $50i$ as cars of quality $50j$, for every $i,j \in \{1,2,\ldots,n\}$). The quality of each car is known to the owner but cannot be determined by the buyer. Thus there can be only one price for second-hand cars. Call this common price P. All agents are **risk neutral**. Assume that, if indifferent between selling or not selling, Group 1 persons will decide **not** to sell and if indifferent between buying and not buying a Group 2 person will decide **not** to buy. Assume also that the number of individuals in Group 2 is sufficiently large for there to be a potential buyer for every car and that each member of this group has a sufficiently large endowment of money.

(a) **(a.1)** [20 points] Assume that $1 < \gamma < 2$. For every integer $i \in \{1, 2, \ldots, n\}$, give a necessary and sufficient condition on the value of γ for the existence of an equilibrium where **all and only** the cars of quality up to (and including) $50i$ are traded at price P.

(a.2) [8 points] Assume now that $\gamma \geq 2$. For what values of $i \in \{1, 2, \ldots, n\}$ is there an equilibrium where **all and only** the cars of quality up to (and including) $50i$ are traded at price P ?

(b) [10 points] Assume only that $\gamma > 1$. For what values of γ is there an equilibrium at which the final allocation of cars (and money) in the population is Pareto efficient?

(c) Suppose that there are 9 quality levels ($n = 9$) and $\gamma = 1.58$. Furthermore, suppose that there are 100 cars of each quality (thus a total of 900 cars).

(c.1) [6 points] What is the largest number of cars that can be traded at an equilibrium?

(c.2) [6 points] Give an equilibrium price at which the number of cars traded is the largest possible.

2. [30 points] There are N types of workers. The marginal productivity of type i = 1,2,…,N is given by $MP(i) = 1 + \frac{i-1}{N-1}$. A worker of type i can obtain a level of education $y \geq 0$ at cost $c_i(y) = \left(1 - 0.5\frac{i-1}{N-1}\right)y$. Note that education has no effect on a worker's marginal productivity. The utility of type i is given by $u_i(w, y) = w_i - c_i(y)$ where w_i is the wage received. The payoff to a firm hiring a type i worker is equal to $MP(i) - w_i$. The employers observe workers' education levels but not their types. A signaling equilibrium is a separating equilibrium where (different types make different education choices and) each type ends up with a wage equal to that type's marginal productivity.

(a) Consider first the case where N = 2. Suppose that employers offer to pay a wage equal to $MP(1)$ to those whose education level is less than y^* and a wage equal to $MP(2)$ to those whose education level is at least y^*.

(a.1) [6 points] Find all the values of y^* that give rise to a signaling equilibrium.

(a.2) [2 points] From an efficiency point of view, what is the best value of y^*, that is, what is the most efficient signaling equilibrium?

(a.3) [6 points] Let $q_1 \in (0,1)$ be the fraction of type 1 in the population. Suppose that the economy starts at a signaling equilibrium and then the Supreme Court in a 6-3 vote declares that education is not a constitutionally protected right. Some Southern states immediately pass laws that make it illegal to acquire education and require employers to pay every worker a wage equal to the average productivity in the population. Under what conditions do all the workers in those states benefit from this new legislation?

(b) Consider now the case where N = 3. Suppose that employers offer to pay a wage equal to $MP(1)$ to those whose education level is less than $\bar{y}$, a wage equal to $MP(2)$ to those whose education level is at least $\bar{y}$ but less than $\bar{\bar{y}}$ and a wage equal to $MP(3)$ to those whose education level is at least $\bar{\bar{y}}$.

(b.1) [12 points] Find all the values of $\bar{y}$ and $\bar{\bar{y}}$ that give rise to a signaling equilibrium.

(b.2) [4 points] From an efficiency point of view, what are the best values of $\bar{y}$ and $\bar{\bar{y}}$, that is, what is the most efficient signaling equilibrium?

3. [20 points] In the insurance industry there are two types of potential customers, *L* and *H*.

Both types have the same initial wealth of \$16,000, face a potential loss of \$7,000 and have the von Neumann-Morgenstern utility-of-wealth function $U(m) = \sqrt{m}$. There are a total of 4,200 potential customers, of which 700 are of type *H*. The probability of loss for the *H* type is 20%, while the probability of loss for the *L* type is 10%. Assume that, if indifferent between not insuring and insuring, the consumer decides to insure. Assume also that the insurance industry is a monopoly.

(a) [4 points] Calculate the monopolist's profits if it decides to offer only one contract and chooses the contract that extracts the maximum surplus from the *H* type.

(b) **(b.1)** [6 points] If the monopolist decided to offer only one contract that would attract both types, what would be the best such contract (best in the sense that it maximizes profits)? You don't need to calculate the premium and deductible: just write the relevant equations.

(b.2) [4 points] Prove that such a contract exists.

(b.3) [6 points] Find a two-contract menu that yields higher profits than the contract of part (b.1). Again, no need to calculate: just write the relevant equations.

Final Exam ANSWERS

1. **(a.1)** For any $j \le n$, a necessary and sufficient condition for cars of quality 50*j* to be *offered* for sale at price *P* is $P > 50j$. Fix $i \in \{1,2,\ldots,n\}$. Then *all and only* cars of quality $j \in \{1,\ldots,i\}$ will be *offered* for sale at price *P* if and only if $50i < P \le 50(i+1)$. Let *m* be the number of cars of each quality level (thus the total number of cars is $n\,m$); then the fraction of cars of each quality is $\frac{m}{mn} = \frac{1}{n}$ (uniform distribution). A buyer who purchases a car at price *P* with $50i < P \le 50(i+1)$ faces a lottery where he gets a car of quality $j \in \{1,\ldots,i\}$ with probability $\frac{m}{im} = \frac{1}{i}$ (uniform distribution). Thus her expected utility if she buys at price *P* is (where ω is her initial endowment of money):

$$\omega - P + \frac{1}{i}\sum_{j=1}^{i}\left[\gamma(50j)\right] = \omega - P + \frac{50\gamma}{i}\sum_{j=1}^{i} j = \omega - P + \frac{50\gamma}{i}\left(\frac{i(i+1)}{2}\right) = \omega - P + 25\gamma(i+1)$$

whereas her utility if she does not buy is ω. Thus she will buy if and only if $P < 25\gamma(i+1)$. Hence necessary and sufficient conditions for *all and only* cars of quality $j \in \{1,\ldots,i\}$ to be traded are: (1) $50i < P < 50(i+1)$ and (2) $P < 25\gamma(i+1)$. Assuming that $\gamma < 2$ (so that $25\gamma(i+1) < 50(i+1)$), there is a *P* that satisfies both inequalities if and only if $50i < 25\gamma(i+1)$ i.e. if and only if $\boxed{\gamma > \frac{2i}{i+1}}$.

(a.2) If $\gamma \ge 2$ then $25\gamma(i+1) \ge 50(i+1)$ and thus condition (2) above is implied by condition (1) and the latter can always be satisfied, since $50i < 50(i+1)$. Thus, for every $i \in \{1, 2, \ldots, n\}$ there is an equilibrium where **all and only** the cars of quality up to (and including) 50*i* are traded.

(b) Since $\gamma > 1$, Group 2 individuals value cars more than Group 1 individuals and thus Pareto efficiency requires that all cars be traded. If $\gamma \ge 2$ it follows from part (a.2) that there exists a price at which this happens, no matter what *n* is. If $1 < \gamma < 2$, by the previous analysis, all cars can be traded if and only if $\gamma > \frac{2n}{n+1}$.

(c) Let $n = 9$ (so that the quality levels are 50, 100, …, 450).

(c.1) When $\gamma = 1.58$, the condition $\gamma > \frac{2i}{i+1}$ of part (a.1) is satisfied if and only if $i \le 3$. Thus there is an equilibrium where all cars of quality up to 150 are traded, but there is no equilibrium where cars of quality greater than 150 are traded. Thus the maximum number of cars that can be traded in equilibrium is 100(3) = 300.

(c.2) Such an equilibrium occurs when *P* is such that $50(3) = 150 < P < 25(4)(1.58) = 158$.

2. **(a.1)** Clearly type 1 will choose $y = 0$ and type 2 will choose $y = y^*$. For a signaling equilibrium the following inequalities need to be satisfied: (1) $1 \geq 2 - y^*$, that is, $y^* \geq 1$ and (2) $2 - 0.5y^* \geq 1$, that is, $y^* \leq 2$. Thus there is a signaling equilibrium for every value of y^* between 1 and 2.

(a.2) Since education does not increase productivity and only performs the role of a signal, the most efficient value of y^* is the smallest, namely 1, thus requiring the least amount spent on education.

(a.3) In the new situation everybody chooses $y = 0$ and is paid the average productivity which is $q_1 + (1 - q_1)2 = 2 - q_1$. This is greater than 1 so that type 1 individuals are better off than before. Let $y^* \in [1, 2]$ be the level of education at the previous signaling equilibrium. Then the utility of type 2 individuals was $2 - 0.5y^*$ while now it is $2 - q_1$. Hence type 2 individuals are better off if and only if $2 - 0.5y^* < 2 - q_1$, that is, if and only if $q_1 < 0.5y^*$.

(b.1) When N = 3 we have that $MP(1) = 1$, $MP(2)$=1.5 and $MP(3) = 2$, $c_1 = y$, $c_2 = 0.75y$ and $c_3 = 0.5y$. Each type will chose the lowest level of education in each range, thus either 0 or $\bar{y}$ or $\bar{\bar{y}}$. For a signaling equilibrium the following inequalities need to be satisfied:

(1a) $1 \geq 1.5 - \bar{y}$	(1b) $1 \geq 2 - \bar{\bar{y}}$
(2a) $1.5 - 0.75\bar{y} \geq 1$	(2b) $1.5 - 0.75\bar{y} \geq 2 - 0.75\bar{\bar{y}}$
(3a) $2 - 0.5\bar{\bar{y}} \geq 1$	(3b) $2 - 0.5\bar{\bar{y}} \geq 1.5 - 0.5\bar{y}$

From (1a) and (2a) we get that $\boxed{0.5 \leq \bar{y} \leq \tfrac{2}{3} = 0.667}$. From (1b) and (3a) we get that $\boxed{1 \leq \bar{\bar{y}} \leq 2}$. From (2b) and (3b) we get that $\boxed{\tfrac{2}{3} + \bar{y} = 0.677 + \bar{y} \leq \bar{\bar{y}} \leq 1 + \bar{y}}$

(b.2) Again, we require the least amount to be spent on education. Thus the most efficient equilibrium is given by $\bar{y} = 0.5$ and $\bar{\bar{y}} = \tfrac{1}{2} + \tfrac{2}{3} = \tfrac{7}{6} = 1.167$.

3. **(a)** The monopolist would offer full insurance at the maximum premium that the H type are willing to pay, given by the solution to $\sqrt{16,000-h}=\frac{1}{5}\sqrt{9,000}+\frac{4}{5}\sqrt{16,000}$, which is $h_H^*=1,560$. The corresponding profits are $\pi_a=700\left(1,560-\frac{1}{5}7,000\right)=112,000$.

(b.1) The average probability of loss is $\bar{p}=\frac{700}{4,200}\cdot\frac{1}{5}+\frac{3,500}{4,200}\cdot\frac{1}{10}=\frac{7}{60}=0.1167$. The profit-maximizing contract is the point on the L-reservation indifference curve at which the slope is equal to $-\frac{\bar{p}}{1-\bar{p}}$.

Thus the contract must satisfy the following equations:

(1) $$\underbrace{\frac{\sqrt{16,000-h}}{9\sqrt{16,000-h-D}}}_{\frac{p_L}{1-p_L}\cdot\frac{U'(16,000-h-D)}{U'(16,000-h)}}=\underbrace{\frac{7}{53}}_{\frac{\bar{p}}{1-\bar{p}}}$$ and

(2) $$\frac{1}{10}\sqrt{9,000}+\frac{9}{10}\sqrt{16,000}=\frac{1}{10}\sqrt{16,000-h-D}+\frac{9}{10}\sqrt{16,000-h}.$$

(b.2) Denote the solution to the above two equations by $(\hat{h},\hat{D})$. Such solution exists if and only if the slope of the L-reservation indifference curve at the no insurance (NI) point is greater than $\frac{\bar{p}}{1-\bar{p}}=\frac{7}{53}$ in absolute value (since it is less than that amount at the point where it intersects the 45^o line). And indeed the slope at the NI point is $\frac{\sqrt{16,000}}{9\sqrt{9,000}}=\frac{4}{27}>\frac{7}{53}$.

(b.3) If the monopolist keeps the contract of part (b.1) but adds a second contract, given by the intersection of the 45^o line and the H-indifference curve through the contract of part (b.1) then its profits will increase. The additional contract solves the equation

$$\sqrt{16,000-h}=\frac{1}{5}\sqrt{16,000-\hat{h}-\hat{D}}+\frac{4}{5}\sqrt{16,000-\hat{h}}.$$