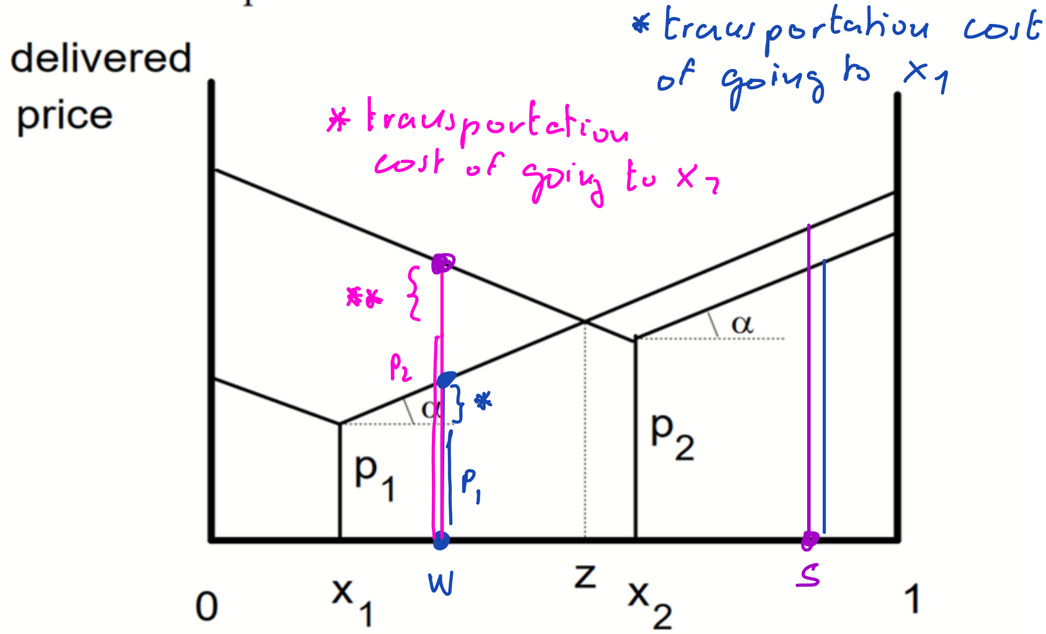
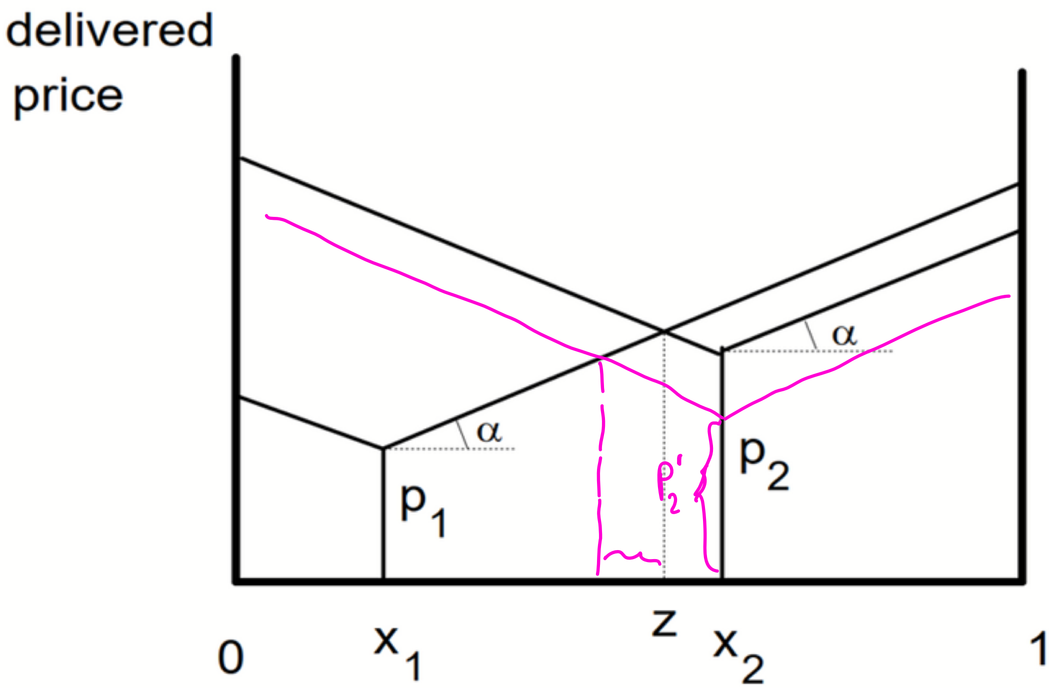
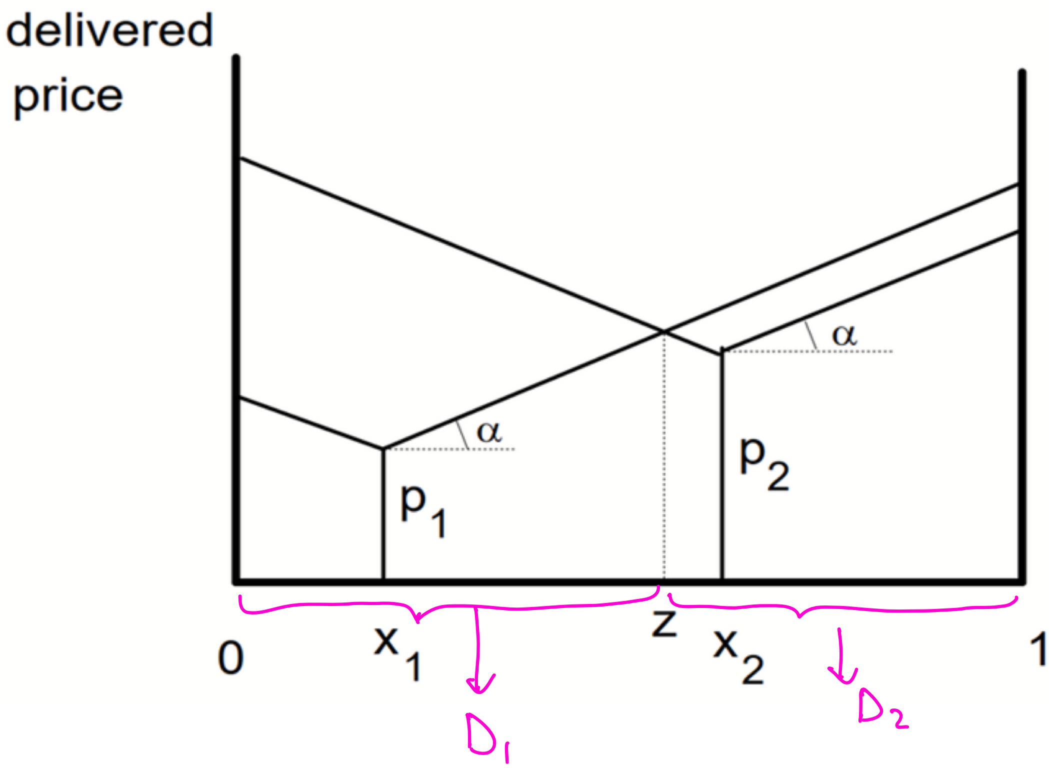
Differentiated Products: Hotelling's model (1929)

Imagine a town with a Main Street of length 1. There are N consumers living on this street and they are uniformly distributed along the street, that is, on a segment of length x there are xN consumers. Each consumer has an infinite reservation price and will buy exactly one unit (from the firm that offers the best deal). Two firms offer the same product and have zero production costs. Consumers have a roundtrip transportation cost of α per unit of distance.





delivered
price
α
α
p'2
p2
p1
0
x1
z
x2
1



$$p_1 + \alpha(z - x_1) = p_2 + \alpha(x_2 - z)$$

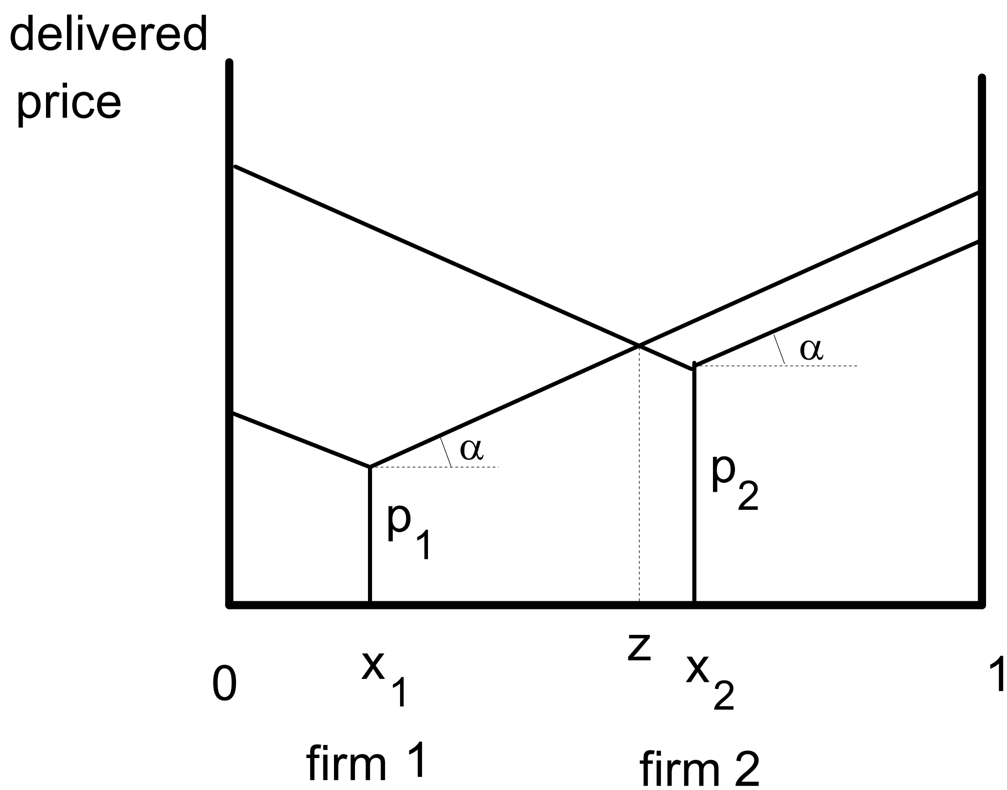
solve for $z = \dfrac{p_2 - p_1}{2\alpha} + \dfrac{x_1 + x_2}{2}$

$D_1(p_1, p_2) = zN$ $\qquad$ $D_2(p_1, p_2) = (1-z)N$

$\Pi_1(p_1, p_2) = zNp_1$ $\qquad$ $\Pi_2(p_1, p_2) = (1-z)N p_2$

strictly concave in p_1 $\qquad$ strictly concave in p_2

FOC $\quad \dfrac{\partial \Pi_1}{\partial p_1} = 0 \qquad \dfrac{\partial \Pi_2}{\partial p_2} = 0$



$$z = \frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2\alpha}$$

$$\Pi_1(p_1, p_2) = zN = \left(\frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2\alpha}\right)N \; p_1$$

$$\Pi_2(p_1, p_2) = (1 - z)N = \left(1 - \frac{x_1 + x_2}{2} - \frac{p_2 - p_1}{2\alpha}\right)N \; p_2$$

MC

The solution to $\begin{cases} \dfrac{\partial \Pi_1}{\partial p_1} = 0 \\ \dfrac{\partial \Pi_2}{\partial p_2} = 0 \end{cases}$ is $\boxed{\begin{cases} p_1^* = \frac{\alpha}{3}(x_1 + x_2 + 2) \\ p_2^* = \frac{\alpha}{3}(4 - x_1 - x_2) \end{cases}}$ $\begin{matrix} > 0 \\ > 0 \end{matrix}$

SUPPOSE $\alpha = 10$

Case 1: $x_1 = \frac{1}{6}$ $\quad x_2 = \frac{5}{6}$ $\quad \boxed{p_1^* = p_2^* = \alpha = 10}$

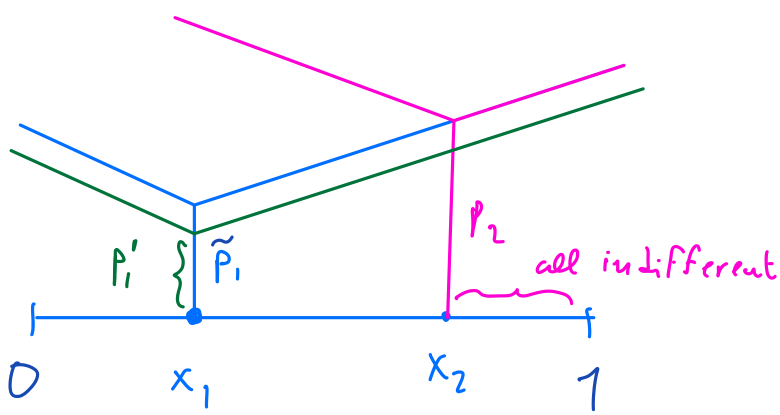
is a NE $\quad MC = 10$

Case 2: $x_1 = \frac{1}{8}$ $\quad x_2 = \frac{5}{6}$ $\quad \boxed{p_1^* = 9.86 \quad p_2^* = 10.14}$

is a NE

Case 3: $x_1 = \frac{1}{3}$ $\quad x_2 = \frac{2}{3}$ $\quad \boxed{p_1^* = p_2^* = \alpha = 10}$

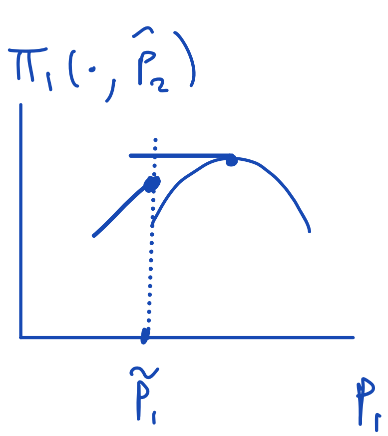
not a NE!

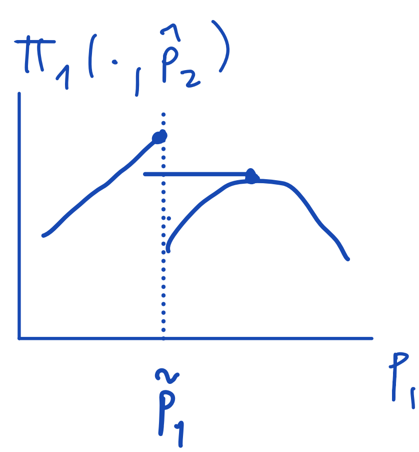


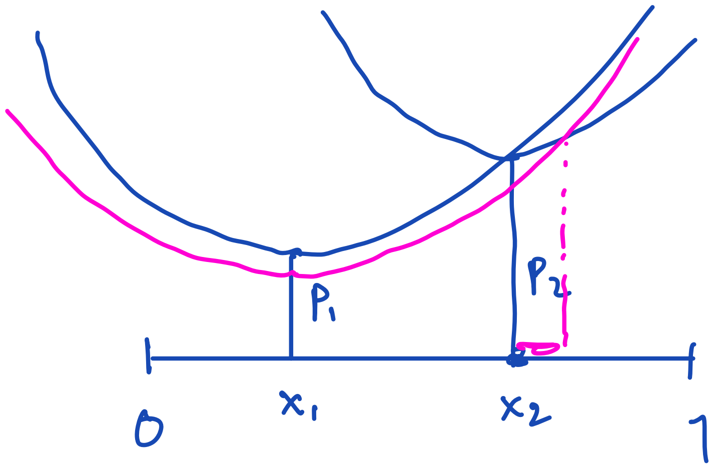
$C(d) = \alpha d$

$\downarrow$

αd^2







P_1
P_2
0
x_1
x_2
1

General remarks about Bertrand-Nash equilibria

- n single-product firms
- $D_i(p)$ is the demand function of firm i ($p = (p_1, p_2, \ldots p_n)$) is the price vector)
- $\frac{\partial D_i}{\partial p_i} < 0$ and, for $j \neq i$, $\frac{\partial D_i}{\partial p_j} > 0$ (products are substitutes)
- $C_i(q_i)$ is the cost function of firm i $\quad \frac{dC_i}{dq_i} \geq 0$

Theorem 1 (*anti-Bertrand theorem*). Let p^* be a Bertrand-Nash equilibrium with $p_i^* > 0$ and $D_i(p^*) > 0$ for all $i = 1,\ldots,n$. Then, for every firm $i = 1,\ldots,n$, $p_i^* > MC_i \equiv \frac{dC_i}{dq_i}(D_i(p^*))$.

$$\Pi_i = p_i D_i(p) - C(D_i(p))$$

$$\frac{\partial \Pi_i}{\partial p_i}(p^*) = 0 \qquad \frac{\partial \Pi_i}{\partial p_i} = D_i(p) + p_i \frac{\partial D_i}{\partial p_i} - \frac{dC}{dq_i}\frac{\partial D_i}{\partial p_i}$$

$$\frac{\partial \Pi_i}{\partial p_i}(p^*) = \underbrace{D_i(p^*)}_{+} + \underbrace{\underbrace{p_i^*}_{+}\underbrace{\frac{\partial D_i}{\partial p_i}(p^*)}_{-} - \underbrace{\frac{dC}{dq_i}(D_i(p^*))}_{\geq 0}\underbrace{\frac{\partial D_i}{\partial p_i}(p^*)}_{-}} \quad = 0$$

$$p_i^* \frac{\partial D_i}{\partial p_i}(p^*) - \overbrace{\frac{dC}{dq_i}(D_i(p^*))}^{MC_i^*}\frac{\partial D_i}{\partial p_i}(p^*) < 0$$

$$\underset{-}{\frac{\partial D_i}{\partial p_i}(p^*)}\underbrace{\left[p_i^* - MC_i^*\right]}_{\text{hence } +} < 0$$

$$p_i^* > MC_i^*$$

Theorem 2. Let p^* be a Bertrand-Nash equilibrium with $\underline{p_i^* > 0}$ and $\underline{D_i(p^*) > 0}$ for all $i = 1,\ldots,n$. Then there exists a $\hat{p}$ such that, for all $i = 1, \ldots, n$, (1) $\underbrace{\hat{p}_i > p_i^*}$ and (2) $\underbrace{\pi_i(\hat{p}) > \pi_i(p^*)}$.

Fix arbitrary $i = \{1, \ldots, n\}$

$$\Pi_i(\hat{p}) - \Pi_i(p^*) \simeq \sum_{j=1}^{n} \frac{\partial \Pi_i}{\partial p_j}(p^*) \cdot (\hat{p}_j - p_j^*)$$

for $j = i$ $\quad \frac{\partial \Pi_i}{\partial p_i}(p^*) = 0$ because p^* is a NE

$$= \sum_{\substack{j=1 \\ j \neq i}}^{n} \underbrace{\frac{\partial \Pi_i}{\partial p_j}}_{+}(p^*) \cdot \underbrace{(\hat{p}_j - p_j^*)}_{+} \quad \text{if } \hat{p}_j > p_j^*$$

$$\Pi_i = p_i D_i(p) - C(D_i(p))$$

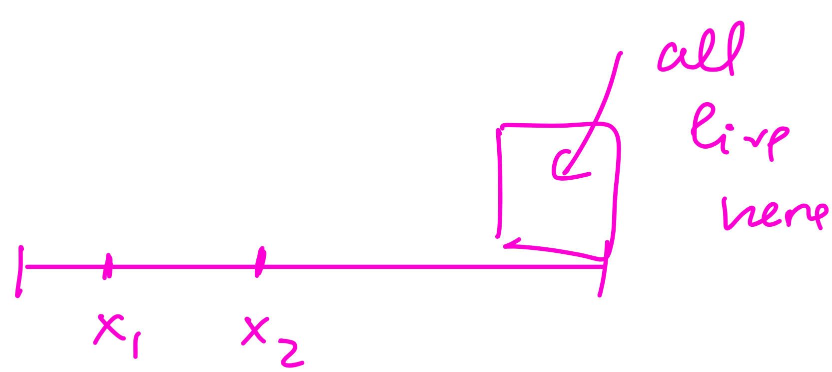
$$\underset{j \neq i}{\frac{\partial \Pi_i}{\partial p_j}} = p_i \frac{\partial D_i}{\partial p_j} - \frac{dC}{dq_i} \frac{\partial D_i}{\partial p_j} \quad \text{at } p^*$$

$$\underbrace{\frac{\partial D_i}{\partial p_j}(p^*)}_{+} \underbrace{\left[p_i^* - \frac{dC}{dq_i}(D_i(p^*)) \right]}_{+ \text{ by Theorem 1}}$$

Hotelling's model is an example of **horizontal product differentiation**, which is defined as a situation where, if prices are the same, some consumers will prefer one product and others will prefer the other product

- degree of sweetness of a drink
- color
- design
- etc.

An alternative type of product differentiation is **vertical differentiation** defined as a situation where, if prices are the same, then all the consumers choose the same product. Thus the source of differentiation can be called **quality** and all consumers agree on what constitutes higher quality.



$\Pi_1(x_1, x_2)$

$\Pi_2(x_1, x_2)$

COURNOT CASE:

$x_1 = q_1 \quad x_2 = q_2$

BERTRAND CASE:

$x_1 = p_1 \quad x_2 = p_2$

$$\frac{\partial^2 \Pi_1}{\partial x_1^2} < 0 \qquad \frac{\partial^2 \Pi_2}{\partial x_2^2} < 0$$

Solve

$$\frac{\partial \Pi_1}{\partial x_1} = 0 \quad \text{for } x_1$$

by strict concavity

for every x_2 $\exists$ unique x_1

such that $\frac{\partial \Pi_1}{\partial x_1}(x_1, x_2) = 0$

$x_2 \longmapsto$ unique solution to $\frac{\partial \Pi_1}{\partial x_1} = 0$

$x_1 = BR_1(x_2)$ best-reply function

or reaction function

of firm 1

By the implicit function theorem

$$\frac{dBR_1}{dx_2} = - \frac{\frac{\partial}{\partial x_1} \frac{\partial \pi_1}{\partial x_2}}{\frac{\partial^2 \pi_1}{\partial x_1^2}}$$

$$-$$

$$\text{Sign}\left(\frac{dBR_1}{dx_2}\right) = \text{Sign} \frac{\partial^2 \pi_1}{\partial x_1 \partial x_2}$$

if $+$ we say that x_1 and x_2 are strategic complements

if $-$ . . . — substitutes

with linear demand Cournot = strat. substitutes

Bertrand = strat. complements