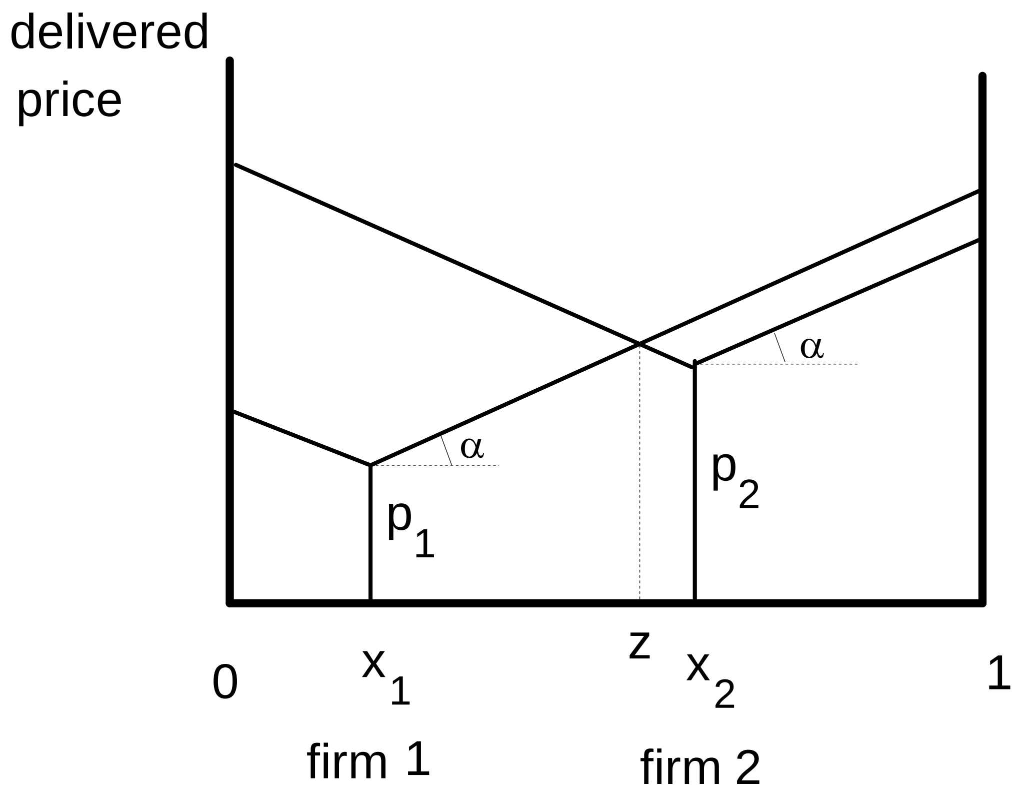
Differentiated Products: Hotelling's model (1929)

Imagine a town with a Main Street of length 1. There are N consumers living on this street and they are uniformly distributed along the street, that is, on a segment of length x there are xN consumers. Each consumer has an infinite reservation price and will buy exactly one unit (from the firm that offers the best deal). Two firms offer the same product and have zero production costs. Consumers have a roundtrip transportation cost of α per unit of distance.



$$z = \frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2\alpha}$$

$$\Pi_1(p_1, p_2) = zN = \left(\frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2\alpha}\right)N$$

$$\Pi_2(p_1, p_2) = (1 - z)N = \left(1 - \frac{x_1 + x_2}{2} - \frac{p_2 - p_1}{2\alpha}\right)N$$

The solution to $\begin{cases} \dfrac{\partial \Pi_1}{\partial p_1} = 0 \\ \dfrac{\partial \Pi_2}{\partial p_2} = 0 \end{cases}$ is $\begin{cases} p_1^* = \frac{\alpha}{3}(x_1 + x_2 + 2) \\ p_2^* = \frac{\alpha}{3}(4 - x_1 - x_2) \end{cases}$

General remarks about Bertrand-Nash equilibria

- n single-product firms
- $D_i(p)$ is the demand function of firm i ($p = (p_1, p_2, \ldots p_n)$) is the price vector)
- $\frac{\partial D_i}{\partial p_i} < 0$ and, for $j \neq i$, $\frac{\partial D_i}{\partial p_j} > 0$ (products are substitutes)
- $C_i(q_i)$ is the cost function of firm i

Theorem 1 (*anti-Bertrand theorem*). Let p^* be a Bertrand-Nash equilibrium with $p_i^* > 0$ and $D_i(p^*) > 0$ for all $i = 1,\ldots,n$. Then, for every firm $i = 1,\ldots,n$,

$$p_i^* > MC_i \equiv \frac{dC_i}{dq_i}(D_i(p^*)) .$$

Theorem 2. Let p^* be a Bertrand-Nash equilibrium with $p_i^* > 0$ and $D_i(p^*) > 0$ for all $i = 1,\ldots,n$. Then there exists a $\hat{p}$ such that, for all $i = 1, \ldots, n$, (1) $\hat{p}_i > p_i^*$ and (2) $\pi_i(\hat{p}) > \pi_i(p^*)$.

Hotelling’s model is an example of **horizontal product differentiation**, which is defined as a situation where, if prices are the same, some consumers will prefer one product and others will prefer the other product

- degree of sweetness of a drink
- color
- design
- etc.

An alternative type of product differentiation is **vertical differentiation** defined as a situation where, if prices are the same, then all the consumers choose the same product. Thus the source of differentiation can be called **quality** and all consumers agree on what constitutes higher quality.

STRATEGIC COMPLEMENTS AND SUBSTITUTES

Two-player game. Let x_j be the decision variable of firm j (e.g. output or price).

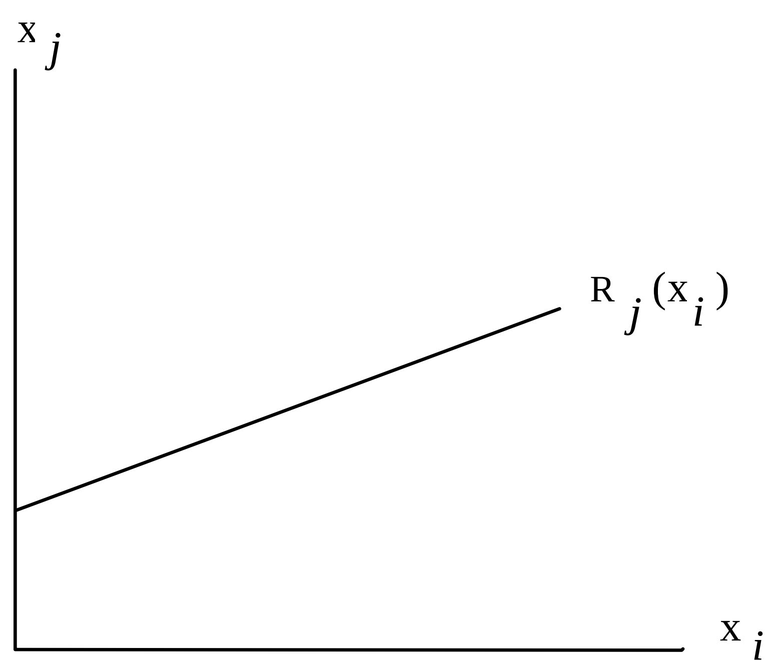
Let $\Pi_j(x_1,x_2)$ be the payoff function of firm *j*. Consider the set of points (x_1,x_2)

that satisfy the FOC $\frac{\partial \Pi_j}{\partial x_j}(x_1, x_2) = 0$. Assume that $\Pi_j(x_1,x_2)$ is strictly concave in

x_j; then, for every x_i there is a unique x_j that satisfies the FOC $\frac{\partial \Pi_j}{\partial x_j}(x_1, x_2) = 0$.

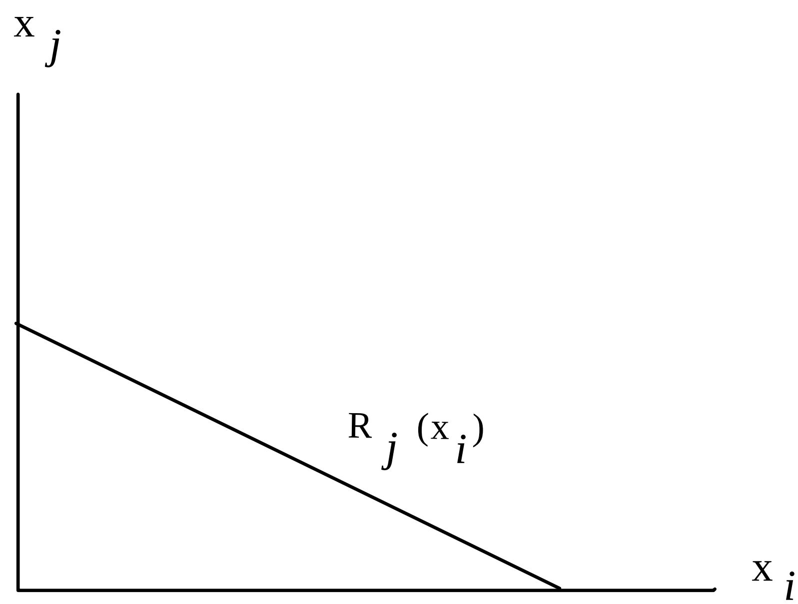
The reaction curve of firm *j* be written as a function $x_j = R_j(x_i)$. It can be shown using the implicit function theorem that

- $R_j(x_i)$ is a strictly increasing function of x_i if and only if $\frac{\partial}{\partial x_i}\left(\frac{\partial \Pi_j}{\partial x_j}\right) > 0$. In this case we say that x_i and x_j are *strategic complements.* An example of this is Hotelling's model.



- $R_j(x_i)$ is a strictly decreasing function of x_i if and only if $\frac{\partial}{\partial x_i}\left(\frac{\partial \Pi_j}{\partial x_j}\right) < 0$.

  In this case we say that x_i and x_j are *strategic substitutes.* An example is Cournot's model.

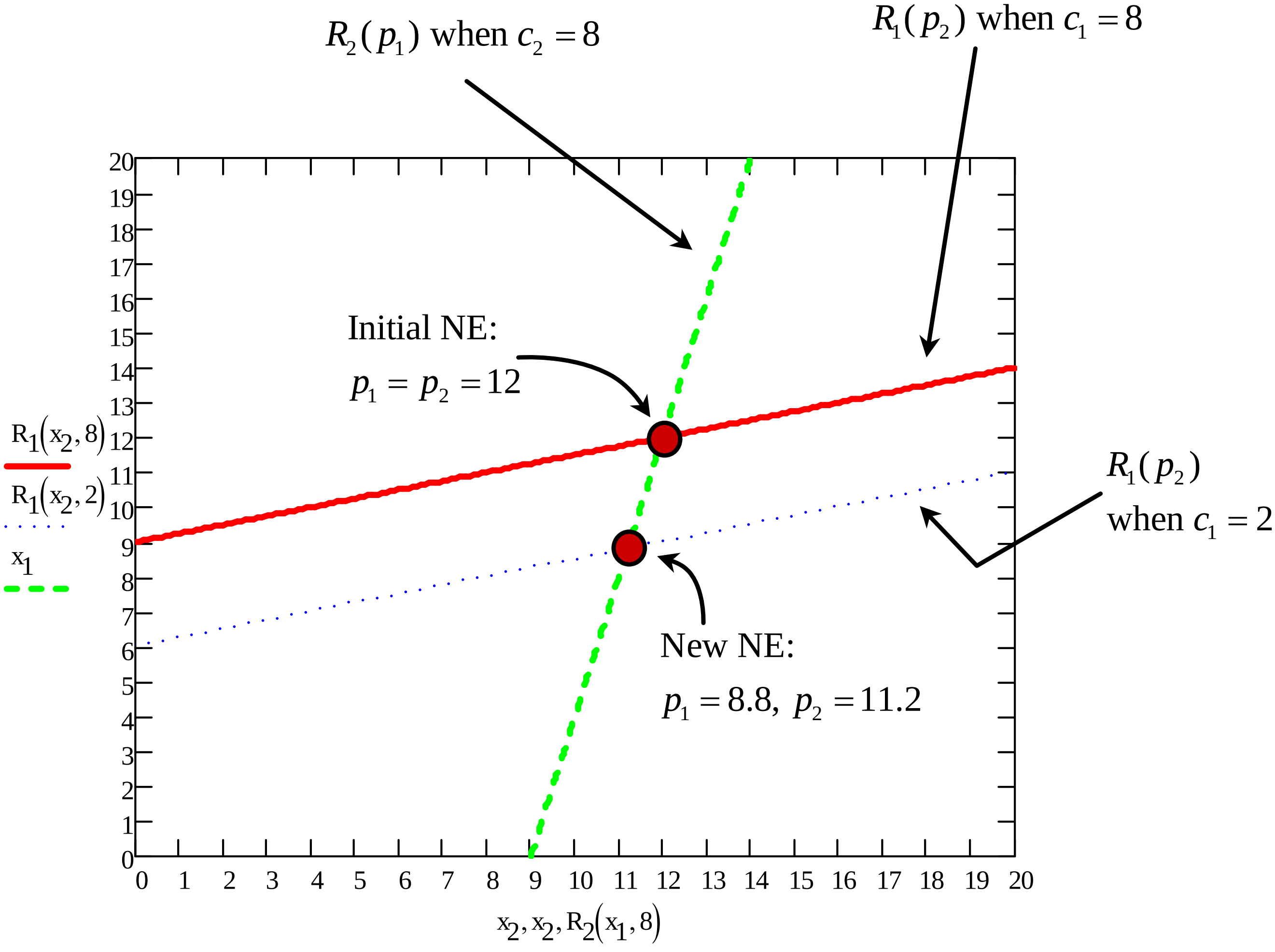


What happens to the Nash equilibrium in these situations if one of the reaction curves shifts due to a change in a parameter?

The strategic complements case.

$$\begin{cases} D_1 = 40 - 4p_1 + 2p_2 \\ D_2 = 40 + 2p_1 - 4p_2 \end{cases} \qquad \begin{cases} C_1 = c_1 q_1 \\ C_2 = c_2 q_2 \end{cases} \quad \text{then} \quad \begin{cases} R_1(p_2) = 5 + \frac{1}{4}p_2 + \frac{1}{2}c_1 \\ R_2(p_1) = 5 + \frac{1}{4}p_1 + \frac{1}{2}c_2 \end{cases}$$

The reaction curves are upward-sloping. **If the cost of firm 1 goes down, firm 1 becomes more aggressive** (for any p_2 the profit-maximizing price for firm 1 is lower), that is, firm 1's **reaction curve shifts down.** Firm 2 will react by lowering its price too (i.e. will react aggressively) and the result is a new Nash equilibrium with lower prices.



The strategic substitutes case.

$$P = 24 - 4Q \qquad C_1 = c_1 q_1 \qquad C_2 = c_2 q_2 . \quad \text{Then} \quad \begin{cases} R_1(q_2) = 3 - \frac{1}{2} q_2 - \frac{1}{8} c_1 \\ R_2(q_1) = 3 - \frac{1}{2} q_1 - \frac{1}{8} c_2 \end{cases}$$

If the cost of firm 1 goes down, firm 1 becomes more aggressive (for any q_2 the profit-maximizing output for firm 1 is higher), that is, firm 1's **reaction curve shifts up.** Firm 2 will react by lowering its output and the result is a new Nash equilibrium with higher output for firm 1 and lower output for firm 2.