Cournot oligopoly

- Fixed number of firms n (no entry, no exit)
- Homogeneous product
- Inverse demand function $P = P(Q)$ $\quad Q =$ industry output
- Firm i's cost function $C_i = C_i(q_i)$
- Objective of each firm is to maximize its own profits, $\pi_i = q_i P(q_1 + q_2 + \cdots + q_n) - C_i(q_i)$
- $q = (q_1, q_2, \ldots, q_n)$ $\qquad q_{-i} = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots q_n)$
- $q^* = (q_1^*, \ldots, q_n^*)$ is an equilibrium if

$$\pi_i(q^*) \geq \pi_i(q_i, q_{-i}^*) \quad \forall q_i \geq 0, \quad \forall i$$

Theorem 1. At a Cournot-Nash equilibrium q^*, with $q_i^* > 0 \ \forall i$, $p^* > MC_i^* = \frac{d}{dq_i} C(q_i^*) \quad \forall i$

$p^* = P(Q^*)$, $Q^* = q_1^* + q_2^* + \dots + q_n^*$

[Cournot: $P(Q) = a - bQ$

$C_i(q_i) = c q_i$]

$S_i = [0, \infty)$

At a CNE $\pi_i(q_i^*, q_{-i}^*) \geq \pi_i(q_i, q_{-i}^*) \ \forall q_i \geq 0$

q_i^* maximizes $\pi_i(\cdot, q_{-i}^*)$

it must be that $\frac{\partial \pi_i}{\partial q_i}(q_i^*, q_{-i}^*) = 0$

$$\pi_i = q_i P(Q) - C_i(q_i)$$

$$\frac{\partial \pi_i}{\partial q_i} = P(Q) + q_i \frac{dP}{dQ} \underbrace{\frac{\partial Q}{\partial q_i}}_{= 1} - \frac{dC_i}{dq_i}$$

$$\frac{\partial \pi_i}{\partial q_i}(q^*) = \underbrace{P(Q^*)}_{P^*} + \underbrace{q_i^*}_{+} \underbrace{\frac{dP}{dQ}(Q^*)}_{-} - \underbrace{\frac{dC_i}{dq_i}(q_i^*)}_{\substack{\geq 0 \\ MC_i^*}} = 0$$

$$P^* = MC_i^* \underbrace{- \underbrace{q_i^*}_{+} \underbrace{\frac{dP}{dQ}(Q^*)}_{-}}_{+} > MC_i^*$$

Properties of CNE continued

Example. Two firms, each can produce either 1 or 2 units at zero cost. The demand function is:

P	**Q**
5	0
4	1
3	2
2	3
1.1	4

		Firm 2's output	
		1	2
Firm 1's output	1	3, 3	2, 4
	2	4, 2	2.2, 2.2

CNE

Theorem 2. Let $q^* = (q^*_1, \ldots, q^*_n)$ be a CNE with $q_i^* > 0 \;\; \forall i = 1, \ldots, n$. Then there exists a $\hat{q} \neq q^*$ such that, $\hat{q}_i < q_i^*$ and $\pi_i(\hat{q}) > \pi_i(q^*)$, $\forall i = 1, \ldots, n$.

$$\pi_i(\hat{q}) - \pi_i(q^*) = \boxed{\sum_{j=1}^{n} \frac{\partial \pi_i}{\partial q_j}(q^*) \cdot (\hat{q}_j - q_j^*)}$$

Since q^* is a CNE

$$\frac{\partial \pi_i}{\partial q_i}(q^*) = 0$$

if choose $\hat{q}_j < q_i^*$

$$\pi_i(\hat{q}) > \pi_i(q^*) \quad = \sum_{\substack{j=1 \\ j \neq i}}^{n} \overset{-}{\frac{\partial \pi_i}{\partial q_j}(q^*)} \cdot \underbrace{(\hat{q}_j - q_j^*)}_{-} > 0$$

$$\pi_i = q_i \, P(q_1 + \dots + q_n) - C_i(q_i)$$

$$j \neq i \qquad \frac{\partial \pi_i}{\partial q_j}(q^*) = \underbrace{q_i^*}_{+} \; \underbrace{\frac{dP}{dQ}(Q^*)}_{-} < 0$$

Linear demand and identical firms:

$P(Q) = a - bQ \qquad C_i(q_i) = c\, q_i \qquad (0 < c < a,\ b > 0)$

$$q_i^* = \frac{a-c}{(n+1)b} \qquad Q^* = n\,\frac{a-c}{(n+1)b}$$

$$P^* = a - bQ^* = \frac{a+nc}{n+1} = \frac{\frac{a}{n}+c}{1+\frac{1}{n}} \xrightarrow[n\to\infty]{} \frac{c}{1} = c$$

$$\pi_i^* = \frac{(a-c)^2}{(n+1)^2 b} \longrightarrow 0$$

as $n \to \infty$

$$q_i^*(n) = \frac{a-c}{(n+1)b}$$ (output of each firm)

$$Q^*(n) = \frac{n(a-c)}{(n+1)b} = \frac{a-c}{\left(1+\frac{1}{n}\right)b}$$ (industry output)

$$P^*(n) = \frac{a+nc}{n+1} =$$ (price) $$\frac{dP^*}{dn} =$$ (since $a > c$), as $n \to \infty$, $P^* \to c$

$$\pi_i^*(n) = \frac{(a-c)^2}{(n+1)^2 b}$$ (profit per firm).

Existence of CNE

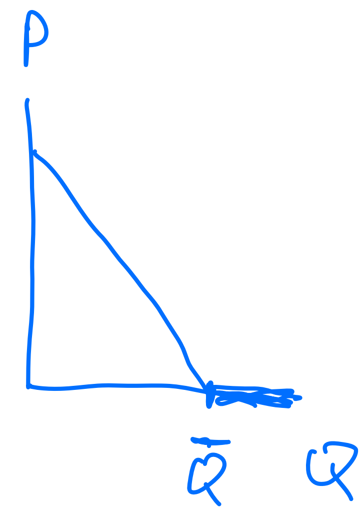
Existence theorem (sufficient conditions) for general games: if

S_i is convex and compact

π_i is continuous and concave in s_i

then a NE exists

$S_i = [0, \cancel{\infty}) \; \bar{Q}]$



$\pi_i = q_i P(Q) - C_i(q_i)$

$$\frac{\partial \pi_i}{\partial q_i} = P(Q) + q_i \frac{dP}{dQ} - \frac{dC_i}{dq_i}$$

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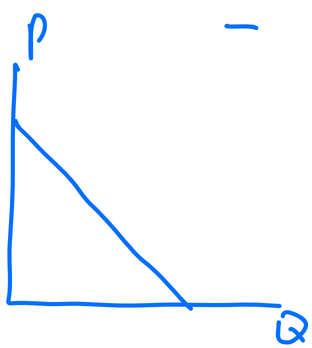
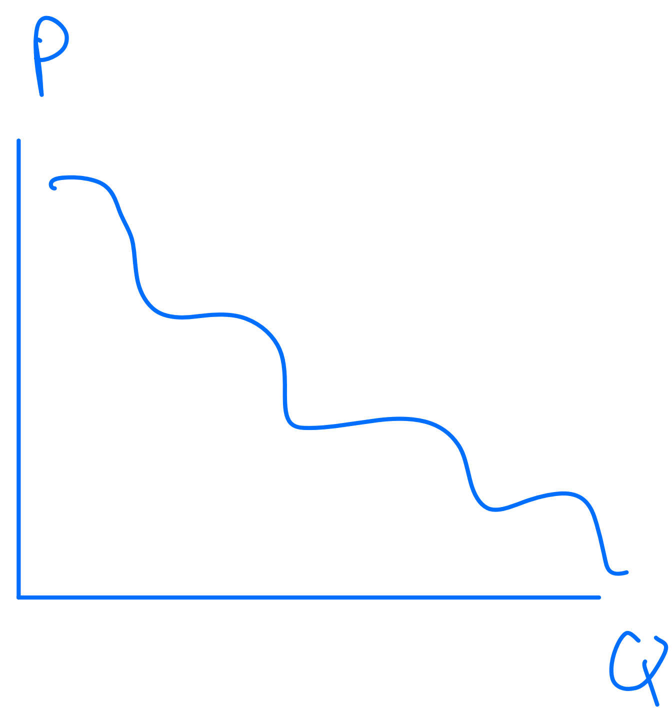
$$\frac{\partial^2 \pi_i}{\partial q_i^2} = \underbrace{\frac{dP}{dQ}}_{-} + \underbrace{\frac{dP}{dQ}}_{-} + \underbrace{q_i \underbrace{\frac{d^2P}{dQ^2}}_{+}}_{\leq 0} - \underbrace{\frac{d^2C_i}{dq_i^2}}_{\text{if } \geq 0} \leq 0$$

if ≥ 0 (MC_i either constant or increasing)




P
Q

Joseph Bertrand (1883): what if we maintain the assumptions of Cournot's model but replace quantity competition with price competition? Assume that

- If all firms choose the same price, then consumers pick a firm at random so that each firm expects to get $\frac{1}{n}$ of the total demand (where n is the number of firms);
- If prices are different, then all consumers buy from the cheapest firm (if there is more than one cheapest firm, then consumers pick randomly among them).
- All firms have the same cost function given by $C_i(q_i) = c\,q_i$

Bertrand's theorem. Let $p^* = (p_1^*, \ldots, p_n^*)$ be a Bertrand-Nash equilibrium. Then,

(1) for all $i = 1, \ldots n$, $p_i^* \geq c$, and

(2) for at least two firms j and k ($j \neq k$), $p_j^* = p_k^* = c$.

p_m^* = lowest price $\quad$ p_s^* = second lowest price

Step 1: $\quad p_m^* \geq c \quad$ imagine $p_m^* < c \quad \pi_m^* < 0$

Step 2: $\quad p_s^* = p_m^* \quad$ imagine $p_s^* > p_m^*$

$$\Pi_s = 0$$

Step 3: rule out $p_s^* = p_m^* > c$

$\Pi_s^* > 0 \quad \Pi_m^* > 0$

$$\Pi_m^* - \varepsilon$$

1929 Hotelling

N consumers uniformly distributed along this street

transportation cost = αd distance traveled

$\alpha > 0$