n-player game

S_i = finite set of pure strategies of player i

Σ_i = " mixed "

$(\sigma_1^*, \sigma_2^*, \dots, \sigma_n^*)$ Nash equilibrium

Indifference theorem: $\forall i$, $\forall s_i \in S_i$

if $\sigma_i^*(s_i) > 0$ then $\pi_i(s_i, \sigma_{-i}^*) = \pi_i(\sigma_i^*, \sigma_{-i}^*)$

Hence if $s_i^1, s_i^2 \in S_i$ are such that

$\sigma_i^*(s_i^1) > 0$ and $\sigma_i^*(s_i^2) > 0$ then

$$\pi_i(s_i^1, \sigma_{-i}^*) = \pi_i(s_i^2, \sigma_{-i}^*)$$

Computing the mixed-strategy Nash equilibria

Theorem. At a Nash equilibrium in mixed strategies, a player must be indifferent between any two PURE strategies that she plays with positive probability.

$0 < q < 1$

$0 < p < 1$

		Player 2: C (q)	Player 2: D ($1-q$)
Player 1	A (p)	15, 3	36, 0
	B ($1-p$)	34, 0	30, 2

NE:

$$\left(\begin{pmatrix} A & B \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{6}{25} & \frac{19}{25} \end{pmatrix} \right)$$

$$\underbrace{\pi_1\left(A, \begin{pmatrix} C & D \\ q & 1-q \end{pmatrix}\right)}_{15q + 36(1-q)} = \underbrace{\pi_1\left(B, \begin{pmatrix} C & D \\ q & 1-q \end{pmatrix}\right)}_{34q + 30(1-q)}$$

$$q = \frac{6}{25}$$

$$\underbrace{\pi_2\left(\begin{pmatrix} A & B \\ p & 1-p \end{pmatrix}, C\right)}_{3p + 0(1-p)} = \pi_2\left(\begin{pmatrix} A & B \\ p & 1-p \end{pmatrix}, D\right)$$

$$3p + 0(1-p) = 0 \cdot p + 2(1-p)$$

$$p = \frac{2}{5}$$

Theorem. At a Nash equilibrium in mixed strategies, a player must be indifferent between any two strategies that she plays with positive probability.

Provides a necessary, *but not sufficient*, condition for a mixed-strategy profile to be a Nash equilibrium

		Player 2	
		D	E
Player 1	A	6, 0	0, 4
	B	0, 4	6, 0
	C	4, 0	4, 2

$$\left(\underbrace{\begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}}_{\sigma_1}, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right)$$

not a N.E.

because

$$\underbrace{\pi_1\left(A, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right)}_{6 \times \frac{1}{2} + 0 \times \frac{1}{2} = 3} = \underbrace{\pi_1\left(B, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right)}_{0 \times \frac{1}{2} + 6 \times \frac{1}{2} = 3}$$

$$\pi_1\left(C, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = 4 \times \frac{1}{2} + 4 \times \frac{1}{2} = 4$$

$$\underbrace{\pi_2(\sigma_1, D)}_{0 \times \frac{1}{2} + 4 \times \frac{1}{2} + 0 \times 0 = 2} = \underbrace{\pi_2(\sigma_1, E)}_{4 \times \frac{1}{2} + 0 \times \frac{1}{2} + 2 \times 0 = 2}$$

$$\left(\begin{pmatrix} A & B & C \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} D & E & F \\ 0 & \frac{1}{5} & \frac{4}{5} \end{pmatrix} \right)$$

Theorem: A strategy can be played with positive probability at a Nash equilibrium only if it survives the IDSDS procedure.

$$\left(\begin{pmatrix} B & C \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} E & F \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix} \right)$$

		D	E	F
	~~A~~	~~1, 4~~	~~4, 2~~	~~0, 8~~
1	B	~~4, 0~~	2, 1	2, 0
	C	~~2, 3~~	6, 4	1, 6

step 1

step 2

		q	2	1-q
		E		F
p	B	2, 1		2, 0
1				
1-p	C	6, 4		1, 6

no pure-strategy NE.

Player 1: $B \rightarrow 2 \cdot q + 2 \cdot (1-q) = 2$

$C \rightarrow 6 \cdot q + 1 \cdot (1-q) = 5q + 1$

must be equal $\quad 2 = 5q+1 \quad 5q = 1 \quad q = \frac{1}{5}$

Player 2 : $\quad E : \quad 1 \cdot p + 4(1-p) = 4 - 3p$

$\qquad\qquad\quad F : \quad 0 \cdot p + 6(1-p) = 6 - 6p$

must be equal $\qquad 4 - 3p = 6 - 6p$

$$3p = 2 \qquad p = \frac{2}{3}$$

Theorem: a pure strategy can be played with positive probability at a N.E. only if it survives the Cardinal IDSDS

a pure strategy can be strictly dominated by a mixed strategy

		Player 2		
		D	*E*	*F*
Player 1	*A*	0, 1	3, 0	5, 0
	B	1, 2	1, 4	2, 3
	C	3, 0	0, 2	0, 1

B is strictly dominated by $\begin{pmatrix} A & C \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

	D	E	F
π_1 from B	1	1	2
π_1 from $\begin{pmatrix} A & C \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	$0 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 1.5$	$3 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 1.5$	$5 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 2.5$

$$\begin{pmatrix} A & B & C \\ p_1 & p_2 & 1-p_1-p_2 \end{pmatrix}$$

	D	E
A	0, 1	3, 0
C	3, 0	0, 2

Player 2

Player 1	D		E		~~F~~	
A	0	1	3	0	~~5~~	~~0~~
~~B~~	~~1~~	~~2~~	~~1~~	~~4~~	~~2~~	~~3~~
C	3	0	0	2	~~0~~	~~1~~

Step 1

Now F is strictly dominated by $\begin{pmatrix} D & E \\ \frac{1}{2}-\varepsilon & \frac{1}{2}+\varepsilon \end{pmatrix}$

Player 2		A	C
	with F	0	1
	with $\begin{pmatrix} D & E \\ \frac{1}{2}-\varepsilon & \frac{1}{2}+\varepsilon \end{pmatrix}$	$\frac{1}{2} 1 + \frac{1}{2} 0 = 0.5$	$\frac{1}{2} 0 + \frac{1}{2} 2 = 1$
		little less than 0.5	little more than 1

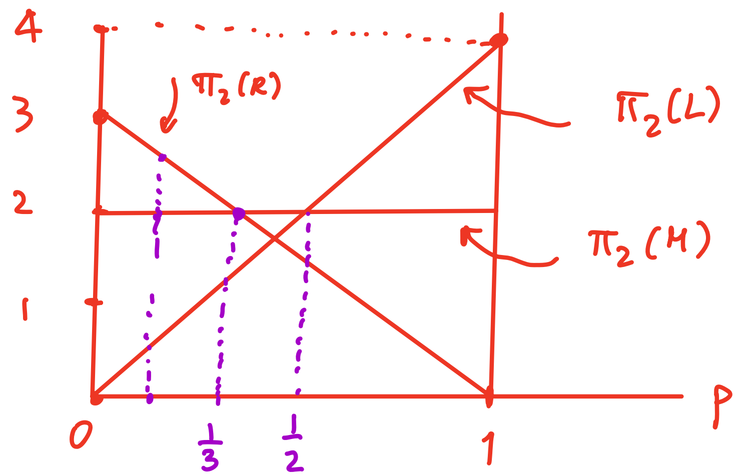
			2	
		L	M	R
1	p T	0,4	1,2	1,0
	$1-p$ B	2,0	3,2	0,3

$$\underbrace{\begin{pmatrix} T & B \\ p & 1-p \end{pmatrix}}$$

$\pi_2(L, p) = 4p$

$\pi_2(M, p) = 2$

$\pi_2(R, p) = 3 - 3p$



Best reply of Player 2 to $\begin{pmatrix} T & B \\ p & 1-p \end{pmatrix}$

$$BR_2(p) = \begin{cases} R & \text{if } p < \frac{1}{3} \\ R \text{ or } M \text{ or } \begin{pmatrix} R & M \\ q & 1-q \end{pmatrix} \text{ for any } q & \text{if } p = \frac{1}{3} \\ M & \text{if } \frac{1}{3} < p < \frac{1}{2} \\ L \text{ or } M \text{ or } \begin{pmatrix} L & M \\ q & 1-q \end{pmatrix} \text{ for any } q & \text{if } p = \frac{1}{2} \\ L & \text{if } p > \frac{1}{2} \end{cases}$$

NE has to be

either $\begin{pmatrix} T & B \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \begin{pmatrix} R & M \\ q & 1-q \end{pmatrix}$ $(*)$

or $\begin{pmatrix} T & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} L & M \\ q & 1-q \end{pmatrix}$ $(**)$

$$\begin{array}{ccc|c|c|c|}
 & & & & 2 & \\
 & & & L & M & R \\
\hline
 & p & T & 0,4 & 1,2 & 1,0 \\
\hline
1 & 1-p & B & 2,0 & 3,2 & 0,3 \\
\hline
\end{array}$$

(*) only need to check if $\begin{pmatrix} T & B \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ is a best reply to $\begin{pmatrix} R & M \\ q & 1-q \end{pmatrix}$

$$T \to 1(1-q) + 1 \cdot q = 1$$

$$B \to 3(1-q) + 0 \cdot q = 3 - 3q$$

need $3 - 3q = 1 \qquad 2 = 3q$

$$q = \frac{2}{3}$$

N.E. : $\left(\begin{pmatrix} T & B \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \begin{pmatrix} L & M & R \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right)$