AFFINE TRANSFORMATION

Theorem 2. Let $\succsim$ be a von Neumann-Morgenstern ranking of the set of basic lotteries $\mathcal{L}$. Then the following are true.

(A) If $U: Z \to \mathbb{R}$ is a von Neumann-Morgenstern utility function that represents $\succsim$, then, for any two real numbers a and b with $a > 0$, the function $V: Z \to \mathbb{R}$ defined by $V(z_i) = aU(z_i) + b$ ($i = 1,2,\ldots,m$) is also a von Neumann-Morgenstern utility function that represents $\succsim$.

(B) If $U: Z \to \mathbb{R}$ and $V: Z \to \mathbb{R}$ are two von Neumann-Morgenstern utility functions that represent $\succsim$, then there exist two real numbers a and b with $a > 0$ such that $V(z_i) = aU(z_i) + b$ ($i = 1,2,\ldots,m$).

$$U = \begin{cases} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ 10 & 6 & 16 & 8 & 6 & 14 \end{cases}$$

subtract 6 ($a = 1$, $b = -6$)

$W \quad 4 \quad 0 \quad 10 \quad 2 \quad 0 \quad 8$

multiply by $\frac{1}{10}$ ($a = \frac{1}{10}$, $b = 0$)

$\boxed{V \quad \frac{4}{10} \quad 0 \quad 1 \quad \frac{2}{10} \quad 0 \quad \frac{8}{10}}$

normalized utility function

$Z = \{z_1, z_2, z_3\}$

Q.1: how do you rank the basic outcomes?

best	z_2	1
	z_1	$\frac{3}{5}$
worst	z_3	0

$U(z_1) = ?$

$\boxed{\text{Continuity axiom:}}$

$\begin{pmatrix} z_1 \\ 1 \end{pmatrix}$ compare to $L(p) = \begin{array}{cc} \text{best} & \text{worst} \\ \end{array} \begin{pmatrix} z_2 & z_3 \\ p & 1-p \end{pmatrix}$

if $p = 1$ $\quad L(1) \succ z_1$

if $p = 0$ $\quad L(0) \prec z_1$

For every intermediate outcome z_i, $\exists\, p_i$ such that $z_i \sim \begin{pmatrix} z_{best} & z_{worst} \\ p_i & 1-p_i \end{pmatrix}$

Q.2: what value of p makes you indifferent between z_1 and $L(p)$?

Suppose answer is: $p = \frac{3}{5}$ $\qquad z_1 \sim \begin{pmatrix} z_2 & z_3 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$

$$z_1 \sim \begin{pmatrix} z_2 & z_3 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$$

By theorem 1

$$\underbrace{E\left[U\begin{pmatrix} z_1 \\ 1 \end{pmatrix}\right]}_{1 \cdot U(z_1) = U(z_1)} = \underbrace{E\left[U\begin{pmatrix} z_2 & z_3 \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}\right]}_{\frac{3}{5}\underbrace{U(z_2)}_{=1} + \frac{2}{5}\underbrace{U(z_3)}_{=0}}$$

$$= \frac{3}{5}$$

best	z_2	120
	z_1	$\frac{440}{5}$
worst	z_3	40

$$U(z_1) = \frac{3}{5}\,120 + \frac{2}{5}\,40$$

$$= \frac{440}{5}$$

$$Z = \{z_1, \dots, z_{12}\}$$

how many question?

$$1 + 10 = 11$$

$$Z = \{z_1, \dots, z_m\}$$

$$m - 1$$

EXAMPLE. Consider the following game frame:

$\frac{1}{2}\frac{5}{6}+\frac{1}{2}0=\frac{5}{12}$

Player 2

		C	D
Player 1	A	$\begin{pmatrix} z_1 & z_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	z_3
	B	$\begin{pmatrix} z_1 & z_3 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$	z_4

$\frac{5}{18}+\frac{12}{18}=\frac{17}{18}$

"

$\frac{1}{3}\frac{5}{6}+\frac{2}{3}1$

$\frac{5}{12},\frac{1}{2}$	$1, 0$
$\frac{17}{18}, 0$	$\frac{5}{6}, \frac{1}{3}$

Suppose that Player 1 has the following ranking of the set of basic outcomes: $\begin{pmatrix} \text{best} & z_3 \\ & z_1, z_4 \\ \text{worst} & z_2 \end{pmatrix}$

Construct the normalized utility function for Player 1:

best	z_3	1
	z_1, z_4	$\frac{5}{6}$
worst	z_2	0

Suppose he says:

$z_1 \sim \begin{pmatrix} z_3 & z_2 \\ \frac{5}{6} & \frac{1}{6} \end{pmatrix}$

$\frac{1}{3}0+\frac{2}{3}0=0$

$\frac{1}{2}0+\frac{1}{2}1=\frac{1}{2}$

Player 2 has the following ranking of the set of basic outcomes: $\begin{pmatrix} \text{best} & z_2 \\ & z_4 \\ \text{worst} & z_1, z_3 \end{pmatrix}$

Construct the normalized utility function for Player 2:

best	z_2	1
	z_4	$\frac{1}{3}$
worst	z_1, z_3	0

$z_4 \sim \begin{pmatrix} z_2 & z_1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

Suppose she says: $p=\frac{1}{3}$

Player 2

		C	D
Player 1	A	$\begin{pmatrix} z_1 & z_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	z_3
	B	$\begin{pmatrix} z_1 & z_3 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$	z_4

Multiply 1's utilities by 36

Multiply 2's utilities by 6

Player 2

		C		D	
Player 1	A	$\boxed{\frac{5}{12}}$	$\boxed{\frac{1}{2}}$	**1**	**0**
	B	$\boxed{\frac{17}{18}}$	**0**	$\boxed{\frac{5}{6}}$	$\boxed{\frac{1}{3}}$

15, 3	36, 0
34, 0	30, 2

If we don't want to deal with fractional numbers, we can multiply the payoffs of Player 1 by 36 and the payoffs of Player 2 by 6 to get the following game, which is the same game as the one given above:

$S_1 = \{A, B\}$ set of pure strategies of Player 1

Player 2

		C		D	
Player 1	A	**15**	**3**	**36**	**0**
	B	**34**	**0**	**30**	**2**

There is no N.E. in pure strategies

This game has no Nash equilibria in pure strategies, but – as we will see – it has a Nash equilibrium in mixed strategies.

$\begin{pmatrix} A & B \\ 1 & 0 \end{pmatrix} = A$

$\begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix} = B$

$\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ mixed strategy for Player 1

Σ_1 set of mixed strategie for Player 1 : $\begin{pmatrix} A & B \\ p & 1-p \end{pmatrix}$

$\sigma_1 = \begin{pmatrix} A & B \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} C & D \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

$\pi_1(\sigma_1, \sigma_2)$

AC	AD	BC	BD
$\frac{1}{5}\cdot\frac{2}{3}=\frac{2}{15}$	$\frac{1}{5}\frac{1}{3}=\frac{1}{15}$	$\frac{4}{5}\frac{2}{3}=\frac{8}{15}$	$\frac{4}{5}\frac{1}{3}=\frac{4}{15}$
15	36	34	30

$$\pi_1(\sigma_1,\sigma_2)= 15\cdot\frac{2}{15} + 36\cdot\frac{1}{15} + 34\cdot\frac{8}{15} + 30\cdot\frac{4}{15} =$$

NASH: every finite game (S_i is finite, finite # of players) has a Nash equilibrium in mixed strategies

B)

Player 2

		C	D
Player 1	A	15, 3	36, 0
	B	34, 0	30, 2

$$\sigma_1 = \begin{pmatrix} A & B \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} C & D \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

is this a N.E.?

AC	AD	BC	BD
$\frac{1}{20}$	$\frac{3}{20}$	$\frac{4}{20}$	$\frac{12}{20}$
15	36	34	30

$$\pi_1 = 15\frac{1}{20} + 36\frac{3}{20} + 34\frac{4}{20} + 30\frac{12}{20} = 30.95$$

if player 1 switches from $\sigma_1 = \begin{pmatrix} A & B \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$ to $\begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix}$

$$\pi_1' = 34\frac{1}{4} + 30\frac{3}{4} = 31$$

So (σ_1, σ_2) is not a N.E.

$$\pi_1\left(A, \begin{pmatrix} C & D \\ \frac{6}{25} & \frac{19}{25} \end{pmatrix}\right) = 15\frac{6}{25} + 36\frac{19}{25} = 30.96$$

$$\pi_1(B, \sigma_2^*) = 34\frac{6}{25} + 30\frac{19}{25} \text{ ''}$$

Claim: $\left(\begin{pmatrix} A & B \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{6}{25} & \frac{19}{25} \end{pmatrix} \right)$ is a N.E.

$$\pi_1 : 15 \cdot \frac{12}{125} + 36 \cdot \frac{38}{125} + 34 \cdot \frac{18}{125} + 30 \cdot \frac{57}{125} = \boxed{30.96}$$

What if Player 1 switched to $\begin{pmatrix} A & B \\ p & 1-p \end{pmatrix}$

$$\pi_1 : \underbrace{15 \cdot p \frac{6}{25}} + \underbrace{36 \cdot p \frac{19}{25}} + \underbrace{34 (1-p) \frac{6}{25}} + \underbrace{30 \cdot (1-p) \cdot \frac{19}{25}}$$

$$= p \underbrace{\left(\frac{6}{25} 15 + \frac{19}{25} 36 \right)}_{30.96} + (1-p) \underbrace{\left(\frac{6}{25} 34 + \frac{19}{25} 30 \right)}_{30.96}$$

$$= \boxed{30.96}$$

Indifference

Theorem. S_1, S_2 finite sets of pure strategies

if (σ_1^*, σ_2^*) is a N.E. then

$$\pi_1(\sigma_1^*, \sigma_2^*) = \pi_1(s_1, \sigma_2^*)$$

For every $s_1 \in S_1$ such that $\sigma_1^*(s_1) > 0$