Theorem 2. Let $\succsim$ be a von Neumann-Morgenstern ranking of the set of basic lotteries $\mathcal{L}$. Then the following are true.

(A) If $U : Z \to \mathbb{R}$ is a von Neumann-Morgenstern utility function that represents $\succsim$, then, for any two real numbers a and b with $a > 0$, the function $V : Z \to \mathbb{R}$ defined by $V(z_i) = aU(z_i) + b$ ($i = 1,2,\ldots,m$) is also a von Neumann-Morgenstern utility function that represents $\succsim$.

(B) If $U : Z \to \mathbb{R}$ and $V : Z \to \mathbb{R}$ are two von Neumann-Morgenstern utility functions that represent $\succsim$, then there exist two real numbers a and b with $a > 0$ such that $V(z_i) = aU(z_i) + b$ ($i = 1,2,\ldots,m$).

$$U = \begin{cases} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ 10 & 6 & 16 & 8 & 6 & 14 \end{cases}$$

EXAMPLE. Consider the following game frame:

		Player 2: C	Player 2: D
Player 1	A	$\begin{pmatrix} z_1 & z_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	z_3
	B	$\begin{pmatrix} z_1 & z_3 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$	z_4

Suppose that Player 1 has the following ranking of the set of basic outcomes: $\begin{pmatrix} \text{best} & z_3 \\ & z_1, z_4 \\ \text{worst} & z_2 \end{pmatrix}$

Construct the normalized utility function for Player 1:

best $\quad z_3$

$\quad z_1, z_4$

worst $\quad z_2$

Player 2 has the following ranking of the set of basic outcomes: $\begin{pmatrix} \text{best} & z_2 \\ & z_4 \\ \text{worst} & z_1, z_3 \end{pmatrix}$

Construct the normalized utility function for Player 2:

best $\quad z_2$

$\quad z_4$

worst $\quad z_1, z_3$

Player 2

Player 1		C	D
	A	$\begin{pmatrix} z_1 & z_2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	z_3
	B	$\begin{pmatrix} z_1 & z_3 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$	z_4

Player 2

Player 1		C		D	
	A	$\boxed{\frac{5}{12}}$	$\boxed{\frac{1}{2}}$	**1**	**0**
	B	$\boxed{\frac{17}{18}}$	**0**	$\boxed{\frac{5}{6}}$	$\boxed{\frac{1}{3}}$

If we don't want to deal with fractional numbers, we can multiply the payoffs of Player 1 by 36 and the payoffs of Player 2 by 6 to get the following game, which is the same game as the one given above:

Player 2

Player 1		C		D	
	A	**15**	**3**	**36**	**0**
	B	**34**	**0**	**30**	**2**

This game has no Nash equilibria in pure strategies, but – as we will see – it has a Nash equilibrium in mixed strategies.

Player 2

		C		D	
Player 1	A	15	3	36	0
	B	34	0	30	2

MIXED STRATEGIES

$$\left(\begin{pmatrix} A & B \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}\right)$$

$$\begin{pmatrix} \text{strategy profile} & (A,C) & (A,D) & (B,C) & (B,D) \\ \text{probability} & & & & \end{pmatrix}$$

next page →

30.95 < 31=B

Is $\left(\begin{pmatrix} A & B \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}\right)$ a Nash equilibrium?

Player 2

		C	D
Player 1	A	**15**, **3**	**36**, **0**
	B	**34**, **0**	**30**, **2**

$$\left(\begin{pmatrix} A & B \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}\right) \quad \begin{pmatrix} \text{strategy profile} & (A,C) & (A,D) & (B,C) & (B,D) \\ \text{probability} & \frac{1}{20} & \frac{3}{20} & \frac{4}{20} & \frac{12}{20} \end{pmatrix}$$

expected utility (or payoff) of Player 1 is

30.96

Player 2

		C	D
Player 1	A	**15** **3**	**36** **0**
	B	**34** **0**	**30** **2**

$$\left(\begin{pmatrix} A & B \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{6}{25} & \frac{19}{25} \end{pmatrix}\right)$$ **is** a Nash equilibrium $\begin{pmatrix} \text{strategy profile} & (A,C) & (A,D) & (B,C) & (B,D) \\ \text{probability} & & & & \end{pmatrix}$

expected utility (or payoff) of Player 1 is

Computing the mixed-strategy Nash equilibria

Theorem. At a Nash equilibrium in mixed strategies, a player must be indifferent between any two PURE strategies that she plays with positive probability.

Player 2

		C		*D*	
Player 1	*A*	**15**	**3**	**36**	**0**
	B	**34**	**0**	**30**	**2**

Theorem. At a Nash equilibrium in mixed strategies, a player must be indifferent between any two strategies that she plays with positive probability.

Provides a necessary, *but not sufficient*, condition for a mixed-strategy profile to be a Nash equilibrium

Player 2

Player 1		D		E	
	A	6	0	0	4
	B	0	4	6	0
	C	4	0	4	2

$$\left(\begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right)$$

Theorem: A strategy can be played with positive probability at a Nash equilibrium only if it survives the IDSDS procedure.

		C	D	E
	A	1 , 4	4 , 2	0 , 8
1	B	4 , 0	2 , 1	2 , 0
	C	2 , 3	6 , 4	1 , 6

		Player 2		
		D	*E*	*F*
	A	0, 1	3, 0	5, 0
Player 1	*B*	1, 2	1, 4	2, 3
	C	3, 0	0, 2	0, 1

Player 2

		D	*E*	*F*
	A	0, 1	3, 0	5, 0
Player 1	*B*	1, 2	1, 4	2, 3
	C	3, 0	0, 2	0, 1

c,g

Cardinal IDSDS (Iterated Deletion of Strictly Dominated **pure** Strategies)

		Player 2							
		D		*E*		*F*		*G*	
Player 1	*A*	6	8	4	2	2	4	2	3
	B	0	0	2	6	8	2	0	1
	C	2	8	2	8	4	12	0	11