Analysis of Economics Data
Chapter 9: Models with Natural Logarithms

© A. Colin Cameron
Univ. of Calif. Davis

November 2022

CHAPTER 9: Models with Natural Logarithms

- Economists are often interested in measuring **proportionate changes**
  - e.g. price elasticity of demand
  - e.g. percentage change in earnings with one more year of education
  - **natural logarithms** are useful for this.
- Additional uses of the natural logarithm include
  - eliminating right skewness in data (chapter 2)
  - compounding and the rule of 72
  - linearizing exponential growth.

Outline

1. Natural Logarithm Function
2. Semi-elasticities and elasticities
3. Log-linear, Log-log and Linear-Log Models
4. Example: Earnings and Education
5. Further Uses of the Natural Logarithm
6. Exponential Function

Dataset: EARNINGS

9.1 Natural Logarithm Function

- A **logarithmic function** is the reverse operation to raising a number to a power
  - e.g. $10^2 = 100$ implies that $\log_{10} 100 = 2$
  - if 10 raised to the power 2 equals 100 then the logarithm to the base 10 of 100 is 2.
- More generally

$$a^b = x \quad \Rightarrow \quad \log_a x = b;$$

  - the logarithm to the base a of x equals b.
- Most obvious choice of the base a is base 10 (decimal system).
- Economics often uses **logarithm to base** e, the **natural logarithm**
  - where $e \simeq 2.71828....$ is a transcendental number like π

$$\ln x = \log_e(x), \quad x > 0.$$

Approximating Proportionate Changes

- $\Delta x = x_1 - x_0$ is the **change** in x when x changes from x_0 to x_1.
- The **proportionate change** in x is

$$\frac{\Delta x}{x_0} = \frac{x_1 - x_0}{x_0}.$$

- Example: Change from $x_0 = 40$ to $x_1 = 40.4$
  - $\Delta x = 40.4 - 40 = 0.4$
  - proportionate change in x is $\Delta x / x_0 = 0.4/40 = 0.01$
  - and percentage change is $100 \times 0.01 = 1\%$.

Approximating Proportionate Changes (continued)

- We have

$$\begin{array}{lrcll} & \frac{d \ln x}{dx} & = & \frac{1}{x} & \text{from calculus} \\ \Rightarrow & \frac{\Delta \ln x}{\Delta x} & \simeq & \frac{1}{x} & \text{for small } \frac{\Delta x}{x} \\ \Rightarrow & \Delta \ln x & \simeq & \frac{\Delta x}{x} & \text{rearranging} \end{array}$$

- For **small proportionate changes** we use the **approximation**

$$\Delta \ln \mathbf{x} \simeq \frac{\Delta \mathbf{x}}{\mathbf{x}} \quad \text{for small } \frac{\Delta x}{x} \; (\text{say } \frac{\Delta x}{x} < 0.1).$$

- Multiplying by 100 yields **percentage changes**, so equivalently

$$\mathbf{100} \times \Delta \ln \mathbf{x} \simeq \text{ Percentage change in } x.$$

- Example: Change from $x_0 = 40$ to $x_1 = 40.4$
  - approximation is $\ln(40.4) - \ln(40) = 3.69883 - 3.68888 \simeq 0.00995$
  - exact is $\Delta x / x_0 = (40.4 - 40)/40 = 0.01$.

9.2 Semi-elasticity and Elasticity

- The **semi-elasticity of** y **with respect to** x is the ratio of the **proportionate change** in y to the **change** in the level of x

$$Semi-elasticity_{yx} = \frac{\Delta y / y}{\Delta x}.$$

  - Multiplying by 100 gives the percentage change in y when x changes by one unit.
- Example: semi-elasticity of earnings with respect to years of schooling is 0.08
  - one more year of schooling is associated with a 0.08 proportionate change in earnings
  - one more year of schooling is associated with an 8% change in earnings.

Semi-elasticity and Elasticity (continued)

- The **elasticity of y with respect to x** is the **proportionate change** of y for a given **proportionate change** in x

$$Elasticity_{yx} = \frac{\Delta y / y}{\Delta x / x} = \frac{\Delta y}{\Delta x} \times \frac{x}{y}.$$

- Example price elasticity of demand for a good is -2
  - a one percent increase in price leads to a 2 percent decrease in demand.

Approximation of Semi-Elasticity and Elasticity

- Since $\dfrac{\Delta y}{y} \simeq \Delta \ln y$ and $\dfrac{\Delta x}{x} \simeq \Delta \ln x$ we obtain the following.
- Semi-elasticities and elasticities **can be approximated** as following

$$\begin{aligned} Semi-elasticity_{yx} &= \frac{\Delta y / y}{\Delta x} \simeq \frac{\Delta \ln y}{\Delta x} \\ Elasticity_{yx} &= \frac{\Delta y / y}{\Delta x / x} \simeq \frac{\Delta \ln y}{\Delta \ln x} \end{aligned}$$

- OLS regression of models that first transform variables to natural logarithms can directly estimate semi-elasticities and elasticities.
- Example: if $\ln y = a + b \ln x$ then the slope $b = \frac{\Delta \ln y}{\Delta \ln x} =$ the elasticity.
  - so we can obtain the semi-elasticity by regressing $\ln y$ on x.

9.3 Log-linear Model

- The **log-linear or log-level model** regresses $\ln y$ on x
  - with fitted value $\widehat{\ln y} = b_1 + b_2 x$
  - the slope coefficient $b_2 = \Delta\widehat{\ln y}/\Delta x$ is an estimate of the semi-elasticity of y with respect to x
  - we need $y > 0$ since only then is $\ln y$ defined.
- This is a very common model for right-skewed data such as individual earnings.

Log-log Model

- The **log-log model** regresses $\ln y$ on $\ln x$
  - with fitted value $\widehat{\ln y} = b_1 + b_2 \ln x$
  - the slope coefficient $b_2 = b_2 = \Delta\widehat{\ln y}/\Delta \ln x$ is an estimate of the elasticity of y with respect to x
  - we need $y > 0$ and $x > 0$ since only then are $\ln y$ and $\ln x$ defined.

Linear-log Model

- The **linear-log model** or **level-log** regresses $\ln y$ on $\ln x$
  - with fitted value $\widehat{y} = b_1 + b_2 \ln x$
  - $b_2/100$ is an estimate of the change in y in response to a one percent change in x.
  - we need $x > 0$ since only then is $\ln x$ defined.

Summary: Models with Logs

- We have

Model	Specification	Interpretation of b_2
Linear	$\widehat{y} = b_1 + b_2 x$	Slope: $\Delta y / \Delta x$
Log-Linear	$\widehat{\ln y} = b_1 + b_2 x$	Semi-elasticity: $(\Delta y / y)/\Delta x$
Log-log	$\widehat{\ln y} = b_1 + b_2 \ln x$	Elasticity: $(\Delta y / y)/(\Delta x / x)$
Linear-log	$\widehat{y} = b_1 + b_2 \ln x$	$\Delta y / (\Delta x / x)$

9.4 Example: Earnings and Education

- Dataset EARNINGS on 172 full-time male workers in 2010 aged 30 years.

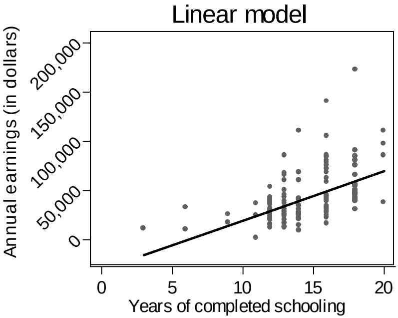
Variable	Definition	Mean	Standard Deviation	Min	Max
Earnings	Annual earnings in $	41413	25527	1050	172000
Lnearn	Natural logarithm of *Earnings*	10.46	0.62	6.96	12.05
Education	Years of completed schooling	14.43	2.73	3	20
Lneduc	Natural logarithm of *Education*	2.65	0.22	1.10	3.00
n	171				

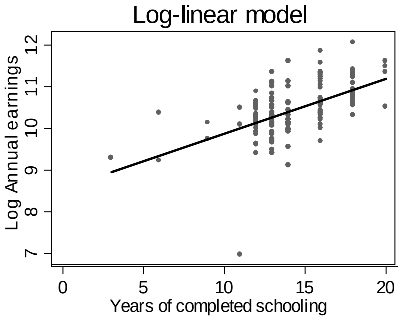
- OLS regression of *Earnings* (y) on *Education* (x) yields (t-statistics in parentheses)

$$\widehat{y} = \underset{(-3.49)}{-31056} + \underset{(8.30)}{5021}x, \quad R^2 = .290$$

Linear Model and Log-Linear Model

- Linear model: *Earnings* = -31056 + 5021 *Education*
- Log-linear model: ln(*Earnings*) = 8.561 + 0.131 *Education*





Comparison of Models with Earnings Data

- y is earnings and x is education (with t-statistics in parentheses).

Model	Estimates	R^2	Slope	Semi-elasticity	Elasticity
Linear	$\widehat{y} = \underset{(-3.49)}{-31056} + \underset{(8.30)}{5021}x$	0.289	5021		
Log-linear	$\widehat{\ln y} = \underset{(40.83)}{8.561} + \underset{(9.21)}{0.131}x$	0.334	-	0.131	
Log-log	$\widehat{\ln y} = \underset{(13.70)}{6.543} + \underset{(8.23)}{1.478}\ln x$	0.286	-	-	1.478
Linear-log	$\widehat{y} = \underset{(-5.05)}{-102767} + \underset{(7.11)}{54452}\ln x$	0.230	-	-	-

- Linear: one year more of education is associated with a \$5,021 increase in earnings
- Log-linear: one year more of education is associated with a 13.1% increase in earnings
- Log-log: 1% increase in education is associated with a 1.478% increase in earnings
- Linear-log: 1% increase in education is associated with a \$544 ($=54452/100$) increase in earnings.

9.5 Approximating Natural Logarithm

- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$
  - e.g. $\ln(1.1) = 1 - 0.1 + \frac{0.01}{2} - \frac{0.001}{3} + \cdots$
    $\simeq 1 - 0.1 + 0.005 - 0.00033 \simeq 0.0953.$
- So for small x we have the **approximation**

$$\ln(1+x) \simeq x, \quad \text{for, say, } x < 0.1.$$

- Approximation good for small x, but x increasingly overestimates $\ln(1+x)$
  - for $x < 0.10$ the approximation is within five percent of $\ln(1+x)$
  - for $x = 0.2$, for example, the approximation is ten percent larger than $\ln 1.2 = 0.1823$.

		"Small" x		**"'Larger" x**		
		x=0.05	x=0.10	x=0.15	x=0.20	x=0.50
True Value	ln(1+x)	0.0488	0.0953	0.1398	0.1823	0.4055
Approximation	x	0.05	0.10	0.15	0.20	0.50

Compounding and the Rule of 72

- **Rule of 72**: a series growing at percentage rate r takes approximately $72/r$ periods to double.
- Example: Invest at 4% per annum doubles in $72/4 = 18$ years.
- Reason:
  - After n periods at rate r investment is $(1+r)^n$ times larger.
  - Money doubles if n solves $(1+r)^n = 2$
  - Solution is $n = \ln 2/[\ln(1+r)]$.
  - Approximate: $\ln(1+r) \simeq r$ for small r.
  - Approximate: $\ln 2 = 0.6931 \simeq 0.72$.
  - So $n = \ln 2/[\ln(1+r)] \simeq \mathbf{0.72/r}$.
- Example: $r = 0.04$ (so 4%)
  - true value: $\ln 2/[\ln(1+0.04)] = 17.67$ so doubles in 17.67 years
  - rule of 72: $72/4 = 18$ so doubles in 18 years.
- More precisely can have rule of 70, or 69, or 69.3.

Linearizing Exponential Growth

- Many data series grow according to a power law, or **exponentially**, over time, rather than linearly.

$$x_t = x_0 \times (1 + r)^t$$

  - Here x_0 is value at time 0
  - x_t is value at time t
  - r is the constant growth rate (or decay rate if $r < 0$).

- Example: \$100 invested at 3% annual interest rate for 10 years.
  - annual growth rate is $r = 3/100 = 0.03$
  - investment worth $100 \times (1.03)^{10}$ or \$134.39 after ten years.

Linearizing Exponential Growth (continued)

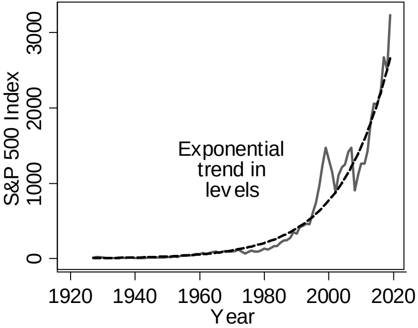
- Taking the natural logarithm of $x_t = x_0 \times (1+r)^t$ yields

$$\begin{aligned} \ln x_t &= \ln(x_0(1+r)^t) \\ &= \ln x_0 + \ln(1+r)^t \\ &= \ln x_0 + \ln(1+r) \times t \\ &\simeq \ln x_0 + r \times t \quad \text{for small } r. \end{aligned}$$

- **Exponential growth is linear growth in logs**!

Example: S&P 500

- Standard and Poor 500 Index 1927-2019
  - no inflation adjustment and no dividends
  - left panel: exponential growth in level
  - right panel: linear growth in logs
    - growth rate $\simeq \frac{7.8-1.8}{2019-1927} = \frac{6.0}{92} = 0.065$ or 6.5% per annum



9.6 Exponential Function

- What is e?
  - e is an irrational number that is approximately 2.7182818
  - e is a transcendental number, like $\pi \simeq 3.142$
  - unlike for π, there is no simple physical interpretation for e.
- The **exponential function** is denoted

$$\exp(x) = e^x.$$

- The natural logarithm is the reverse operation to exponentiation.
- Then

$$y = e^x \quad \Rightarrow \quad x = \ln y.$$

  - For example, $e^2 \simeq 7.38906$ so $\ln 7.38906 \simeq 2.0$.

Approximating Exponential

- $\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$
  - e.g. $\exp(1.1) = 1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6} + \cdots$
    $\simeq 1 + 0.1 + 0.005 + 0.00016 \simeq 1.1052.$
- So for small x

$$e^x \simeq 1 + x, \quad \text{for, say, } x < 0.1.$$

- Approximation good for small x, but increasingly underestimates e^x as x increases.

Table: Approximating exp(x) by 1+x.

		"Small" x		"Larger" x		
		x=0.05	x=0.10	x=0.15	x=0.20	x=0.50
True Value	exp(x)	1.0513	1.1052	1.1618	1.2214	1.6487
Approximation	1+x	1.05	1.10	1.15	1.20	1.50

Compound Interest Rates

- Consider compound for a year and find the annual percentage yield (APY).
- Suppose have 12% per annum then APY$=$ 12%.
- Suppose compound monthly at $12/12 = 1\%$ per month
  - $(1+0.01)^{12} = 1.12683$ so APY$=$ 12.683%.
- Suppose compound daily at $12/365\%$ per day
  - $(1+0.12/365)^{365} = 1.127547$ so APY$=$ 12.747%.
- If continuously compound for progressively smaller intervals at rate r

$$(1+r/n)^n \to e^r \text{ as } n \to \infty.$$

- Here $(1+0.12/n)^n \to \exp(0.12) = 1.12750$ or 12.750%.

Key Stata Commands

```
clear
use AED_EARNINGS.DTA
generate lnearn = ln(earnings)
generate lneduc = ln(education)
* Linear Model
regress earnings education, vce(robust)
* Log-linear Model
regress lnearn education
* Log-log Model
regress lnearn lneduc
* Linear-log Model
regress earnings lneduc
```

Some in-class Exercises

1. Consider numbers a and b with $\ln a = 3.20$ and $\ln b = 3.25$. Using only this information, what is the approximate percentage change in going from a to b?
2. Demand for a good falls from 100 to 90 when the price increases from 20 to 21. Compute the price elasticity of demand.
3. We estimate $\ln y = 3 + 0.5 \ln x$. Give the elasticity of y with respect to x.
4. We estimate $\ln y = 6 + 0.2x$. Give the response of y when x changes by one unit.
5. How long does it take for prices to double given 4% annual inflation?
6. Suppose $y = \alpha \times (1 + \beta)^x$. Explain how to estimate α and β using OLS.