Economics 135

Bond Pricing and Interest Rates

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Bond Pricing Formulas - Interest Rates and Bond Prices

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- A bond defines (F, C, N) . Given the market price, P_b, (and the frequency of coupon payments), this determines i by the present discounted formula.

Basic Bond Pricing Formulas

If the coupon payments are received annually, the bond pricing formula is:

$$\begin{aligned} P_b &= \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + \ldots + \frac{C_N}{(1+i)^N} + \frac{F}{(1+i)^N} \\ &= \sum_{t=1}^{N} \frac{C_t}{(1+i)^t} + \frac{F}{(1+i)^N} \end{aligned}$$

Typically, the coupon payment is made semi-annually. With a constant annual coupon payment, C, the formula then becomes

$$\begin{aligned} P_b &= \frac{C/2}{(1+i/2)} + \frac{C/2}{(1+i/2)^2} + \ldots + \frac{C/2}{(1+i/2)^{2N}} + \frac{F}{(1+i/2)^{2N}} \\ \textit{or} \quad P_b &= \sum_{t=1}^{2N} \frac{C/2}{(1+i/2)^t} + \frac{F}{(1+i/2)^{2N}} \end{aligned}$$

A key relationship: Bond prices and interest rates are inversely related!!

The effective annual yield in this case is defined by:

$$(1+i/2)^2$$

That is, the yield includes interest on interest $= (i/2)^2$. If the number of compounding periods (defined as m) in a year grows, then the effective annual yield is determined by:

$$(1+i/m)^m$$

Suppose $i = 1$ (100% interest rate). What is the effective yield as $m \rightarrow \infty$? This is continuous compounding and yields the mysterious number e:

$$e = \lim_{m\rightarrow\infty} (1+1/m)^m = 2.71828...$$

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- $$P_b = \sum_{t=1}^{mN} \frac{C/m}{(1+i/m)^t} + \frac{F}{(1+i/m)^{mN}}$$

Common types of bonds

- Coupon bond (as above).
- Pure discount bond: $C_t = 0$ for all t. The entire amount is received at maturity. Example: 1 year Treasury Bill.

$$P_b = \frac{F}{(1+i)}$$

- Amortizing Bond: $F = 0$, $C_t = C$. The face value (or principal) is included in the coupon payment. Example: 4 year Car Loan with monthly payments.

$$P_b = \sum_{t=1}^{48} \frac{C}{(1+i/12)^t}$$

- Some more examples on the board.

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    1. This makes sense since bond prices are determined by the PDV of cash flows.
2. The greater the maturity, the greater the change in P_b for a given change in i.
    1. Cash received in the future is discounted at a greater rate. So a change in i is compounded more times.

The elasticity of bond prices with respect to interest rate changes

Calculating the Duration of a Bond

First - a little review of elasticity. Suppose we have a function $y = f(x)$ Recall that the elasticity of y with respect to x is defined as

$$\frac{\%\Delta y}{\%\Delta x} = \frac{\frac{dy}{y}}{\frac{dx}{x}} = \frac{dy}{dx}\frac{x}{y}$$

But note that this is the derivative of the logs

$$\frac{d\ln y}{d\ln x} = \frac{\frac{1}{y}dy}{\frac{1}{x}dx} = \frac{dy}{dx}\frac{x}{y}$$

So - the easy way to calculate elasticities is to take logs (natural) and then take the derivative.

The elasticity of bond prices with respect to interest rate changes

Calculating the Duration of a Bond

Recall the formula for the price of a pure discount bond:

$$P_b = \frac{F}{(1+i)^N}$$

Now take logs:

$$\ln P_b = \ln F - N \ln (1+i)$$

Now take the derivative with respect to i (note we don't want the percentage change in i but absolute change):

$$\frac{d \ln P_b}{di} = -N \left(\frac{1}{1+i} \right)$$

Hence the elasticity of bond prices is appoximately equal to the maturity of the bond.

Or, using discrete notation: $\%\Delta P_b = -N \left(\frac{\Delta i}{1+i} \right)$

But for a typical bond, this relationship is not so straightforward. But we can use the following insight:
Recall the formula for a coupon bond (with annual payments):

$$P_b = \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + ... + \frac{C_N}{(1+i)^N} + \frac{F}{(1+i)^N}$$

Note that each term can be thought of as the price of a pure discount bond for that period.Then a coupon bond can be interpreted as a portfolio of pure discount bonds. Furthermore, the elasticity of the coupon bond will be equivalent to a weighted average of the elasticities of the underlying coupon bonds. (Suppose $y = x + z$. Then $\Delta y = \Delta x + \Delta z$. Divide both sides by y and rewrite as: $\frac{\Delta y}{y} = \frac{x}{y}\frac{\Delta x}{x} + \frac{z}{y}\frac{\Delta z}{z}$)
This elasticity is defined as *Duration.*

The elasticity of bond prices with respect to interest rate changes

Calculating the Duration of a Bond

The formula for duration of a coupon bond is:

$$D = \frac{\frac{C}{(1+i)}}{P_b}(1) + \frac{\frac{C}{(1+i)^2}}{P_b}(2) + \ldots + \frac{\frac{C}{(1+i)^N}}{P_b}(N) + \frac{\frac{F}{(1+i)^N}}{P_b}(N)$$

Then, once we have D calculated, the elasticity of bond prices is given by the direct equivalent to a pure discount bond:

$$\%\Delta P_b = -D\left(\frac{\Delta i}{1+i}\right)$$

Duration Example

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2. $$D = \frac{\frac{98}{(1.14)}}{700}(1) + \frac{\frac{98}{(1.14)^2}}{700}(2) + \frac{\frac{798}{(1.14)^3}}{700}(3) = 2.65$$

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4. So the change in the price is $\Delta P_b = -0.0232\,(\$700) = -\$16.27$. (If you recalculate $P_b = \$684.02$ so not bad).