Professor Salyer, Economics 137

Bond Pricing Formulas -

The basic bond pricing formula for payments made **annually** is:

$$P_b = \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + ... + \frac{C_N}{(1+i)^N} = \sum_{t=1}^{N} \frac{C_t}{(1+i)^t} \tag{1}$$

where: $C_t,\ t = (1,..,N)$ denotes the cash payment received in period t and N denotes the maturity of the bond. For a coupon bond, C_N will include both the coupon payment and the face value (denoted F) of the bond, while for an amortizing bond, $C_t = C$ for all t – payments are constant. For a discount bond, $C_t = 0$ for all $t < N$ and $C_N = F$. Note that the **yield to maturity,** i, is expressed as an annual yield.

Now consider a coupon bond with payments that are made semi-annually. That is, the annual coupon payment of $C = rF$ (where r is the **coupon rate**) is received twice a year. Then the bond pricing formula becomes:

$$P_b = \frac{C/2}{(1+i/2)} + \frac{C/2}{(1+i/2)^2} + ... + \frac{C/2}{(1+i/2)^{2N}} + \frac{F}{(1+i/2)^{2N}} = \sum_{t=1}^{2N} \frac{C/2}{(1+i/2)^t} + \frac{F}{(1+i/2)^{2N}}\ . \tag{2}$$

Note that the implied **effective annual yield** is $(1+i/2)^2$.

If payments are made at the rate of m times per year (so that all but the last payment is C/m), the formula becomes:

$$P_b = \sum_{t=1}^{mN} \frac{C/m}{(1+i/m)^t} + \frac{F}{(1+i/m)^{mN}}\ . \tag{3}$$

If $m = 2$, the formula (3) is the same as that in (2). Again the effective annual yield is $(1+i/m)^m$.

Continuous Compounding Suppose that the interest rate, i, is 100%. Then, using the above results, the future value, FV, of a dollar after one year which is compounded at the rate of m times per year is:

$$FV = \left(1 + \frac{1}{m}\right)^m\ . \tag{4}$$

If the rate becomes continuous, the future value is defined by the following limit:

$$FV = \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m = e = 2.71828.. \tag{5}$$

Hence continuous compounding of a 100% interest rate implies an effective annual yield of 171.828..%. If after 1 year a \$1 becomes \$$e$, then after 2 years the dollar becomes e^2. Hence, A dollars invested for t years becomes Ae^t.

If $i \neq 1$, then the future value formula under non-continuous compounding is, in general,

$$FV = \left(1+\frac{i}{m}\right)^m = \left[\left(1+\frac{i}{m}\right)^{\frac{m}{i}}\right]^i = \left[\left(1+\frac{1}{w}\right)^w\right]^i \quad (where\ w = \frac{m}{i}). \tag{6}$$

As before, continuous compounding is defined by the following limit:

$$FV = \lim_{w\to\infty}\left[\left(1+\frac{1}{w}\right)^w\right]^i = e^i. \tag{7}$$

Using the same reasoning as before, an investment of $\$A$ invested at rate i for t years will yield a future value of: $FV = Ae^{it}$.

Alternative representation of interest rates.

The payment of $1 after $1/m$ of a year (so that a continual payment would be received m times per year) at annual rate of i generates a future value of $\left(1+\frac{i}{m}\right)$. Taking natural logs (and recalling that $\ln(1+x) \simeq x$ if x is small), then the $\ln FV = \frac{i}{m}$. Alternatively, we could think of an annual interest rate of i being invested for $1/m$ of a year. Then the future value would be:

$$FV = (1+i)^{\frac{1}{m}}. \tag{8}$$

Again taking natural logs, $\ln FV = \frac{1}{m}\ln(1+i) \simeq \frac{i}{m}$. Hence the two representations are *approximately* equivalent:

$$FV = \left(1+\frac{i}{m}\right) = (1+i)^{\frac{1}{m}}. \tag{9}$$

To take an example, suppose you invest $\$10,000$ for $1/4$ of a year at a 10% (annual) interest rate. Then, according to eq. (4), this amount would become:

$$FV = \$10,000\left(1+\frac{0.10}{4}\right) = \$10,250.$$

Under the alternative characterization, the future value would be:

$$FV = \$10,000\,(1+0.10)^{0.25} = \$10,241$$