RATIONAL CHOICE AND PATTERNS OF GROWTH IN A MONETARY ECONOMY*

By MIGUEL SIDRAUSKI
Massachusetts Institute of Technology

Concluding his excellent survey of recent monetary theory, Harry Johnson [4] suggested that future developments in this field should come from attempts "to break monetary theory loose from the mould of short-run equilibrium analyses, conducted in abstraction from the process of economic growth and accumulation, and to integrate it with the rapidly developing theoretical literature on economic growth."

This paper summarizes an attempt to deal with these issues. Like most theoretical work in rapidly growing fields, it is incomplete and the assumptions on which it is based are relatively crude abstractions. These abstractions, however, allow us to explore certain aspects of the interaction of the real and the monetary phenomena in a model of economic growth in which money, being government noninterest bearing debt, is introduced as an alternative asset to real capital.

Most of the recent work in this field[1] has centered on the analysis of the patterns of growth of a monetary economy by postulating alternative plausible saving functions and demand functions for money. What differentiates this product is the fact that, in line with Patinkin's [6] presentation of the neoclassical theory of money, and with the classical Fisherian theory of saving [2], it is based on an explicit analysis of individuals' saving behavior, viewed as a process of wealth accumulation aimed at maximizing some intertemporal utility function.

The first part of the paper describes the representative economic unit of an idealized economy and it analyzes the constraints imposed on its maximizing behavior. Section II is concerned with the optimizing conditions and presents the derivation of the demand functions for consumption, cash balances, and the stock of capital. The third section introduces an expectations formation hypothesis and presents a simple aggregative macroeconomic model in which the demand functions for assets and consumption are those which were derived from the analysis of the maximizing behavior of individual economic units. The final part of the paper considers the short-run and long-run effects of a change in the rate of monetary expansion as well as the stability of the equilibrium growth path in a monetary economy.

* This paper is a summary of my Ph.D. thesis presented at the University of Chicago. I am very grateful to the members of my committee, H. Uzawa, M. Friedman, and A. Harberger, as well as to my colleagues, M. Teubal and A. Treadway, for their helpful comments and suggestions.

[1] What I have in mind here is the work by Tobin [9], Gurley and Shaw [3], Johnson [5], and Sidrauski [7].

I. *The Model*

The basic economic unit in our model is the representative family. Its welfare at any point in time is measured by a time invariant utility function of the form

$$U_t = U(c_t, z_t) \tag{1}$$

where c_t stands for the flow of real consumption per unit of time, and z_t for the flow of services per unit of time derived from holdings of real cash balances, both variables being expressed in per capita terms. To simplify, we will assume that the flow of services derived from the holdings of real cash balances is proportional to the stock and, by an appropriate choice of units, we make the factor of proportionality equal to one.

$$z_t = m_t = M_t/p_tN_t \tag{2}$$

M_t represents the holdings of nominal cash balances by the economic unit, N_t the number of individuals in the economic unit and p_t the money price of the only commodity produced in our model. The instantaneous utility function can then be written

$$U_t = U(c_t, m_t) \tag{3}$$

It is assumed that the utility function is strictly concave with continuous first and second derivatives[2] and that both commodities are not inferior.[3] We further assume that the total welfare (W) associated with any particular time path (c_t, m_t) can be represented by the utility functional

$$W = \int_0^\infty [U(c_t, m_t)]e^{-\delta t}dt \tag{4}$$

$\delta > 0$ being the subjective rate of time preference of this family.

At each moment of time the behavior of the economic unit is subject to two constraints, one in terms of stocks and the other one in terms of flows. The stock constraint requires that the total endowment of real nonhuman wealth (a_t), be allocated between capital (k_t) and real cash balances in such a fashion that[4]

$$a_t = k_t + m_t \tag{5}$$

On the other hand, the flow constraint requires that at any time t disposable income has to be equal to consumption plus saving. Assuming that the production function is linear homogeneous, the capital stock produces an amount $y(k_t)$ of homogeneous output.[5] If we add to this

[2] This condition implies that $U_{cc} < 0$, $U_{mm} < 0$, and $J = U_{cc}U_{mm} - U_{mc}^2 > 0$.

[3] This requires: $J_1 = U_{mm} - U_{cm}U_m/U_c < 0$ and $J_2 = U_{cc}U_m/U_c - U_{cm} < 0$.

[4] In what follows all variables are expressed in per capita terms.

[5] We will assume that the production function is "well behaved": namely, $y(0) = 0$, $y(\infty) = \infty$, $y(0 < k < \infty) > 0$, $y'(0) = \infty$, $y'(\infty) = 0$, $y'(0 < k < \infty) > 0$, $y''(k) < 0$.

amount the real value of the net transfers that the economic unit receives from the government (v_t), we obtain the family's gross disposable income, which has to be equal to real consumption (c_t) plus gross real savings (s_t).

$$y(k_t) + v_t = c_t + s_t \tag{6}$$

Gross real saving is the sum of gross capital accumulation (i_t) plus the gross addition to the holdings of real cash balances (x_t).

$$s_t = i_t + x_t \tag{7}$$

Gross capital accumulation is equal to the net addition to the capital stock ($\dot{k}_t$) plus the replacement of the depreciated capital (uk_t) plus the amount of capital accumulation required to provide the newly born members of the economic unit with the same amount of capital as the amount with which the old members are endowed (nk_t); where u is the instantaneous rate of depreciation of capital and n is the instantaneous rate of growth of the number of individuals in the family.

$$i_t = \dot{k}_t + (u + n)k_t \tag{8}$$

Similarly, the gross accumulation of real cash balances is equal to

$$x_t = \dot{m}_t + (\pi_t + n)m_t \tag{9}$$

where π_t is the expected rate of change in prices.

Hence the flow constraint for this economic unit can be rewritten

$$y(k_t) + v_t - (\pi_t + n)m_t - (u + n)k_t - \dot{m}_t - \dot{k}_t - c_t = 0 \tag{10}$$

Differentiating equation (5) with respect to time and substituting into (10) we have

$$\dot{a}_t = y(k_t) + v_t - (\pi_t + n)m_t - (u + n)k_t - c_t \tag{11}$$

Equations (5) and (11) are the stock and flow constraints under which, given the initial condition (a_0) and the values of u, n, π_t and v_t, the rational economic unit will find the time path of consumption and accumulation that maximizes the utility functional (4).

II. *Maximization of the Utility Functional and Derivation of the Demand Functions*

In order to solve this maximization problem we form a new function (I) such that

$$\begin{aligned} I = \int_0^\infty \{ U(c_t, m_t) + \lambda_t[y(k_t) + v_t - (\pi_t + n)m_t \\ - (u + n)k_t - c_t - \dot{a}_t] + q_t[a_t - k_t - m_t]\} e^{-\delta t} dt \end{aligned} \tag{12}$$

where λ_t is the Lagrangian multiplier attached to the flow constraint (11) and q_t the Lagrangian multiplier attached to the stock constraint (5). The conditions for a maximum are given by the Euler equations (13)–(16) together with the transversality condition (17).

$$U_c(c_t, m_t) = \lambda_t \tag{13}$$

$$U_m(c_t, m_t) = \lambda_t(\pi_t + r_t + n) \tag{14}$$

$$y'(k_t) - (u + n) = r_t \tag{15}$$

$$\frac{\dot{\lambda}_t}{\lambda_t} = \delta - r_t \tag{16}$$

$$\lim_{t \to \infty} a_t \lambda_t e^{-\delta t} = 0 \tag{17}$$

where r_t is equal to the ratio of the two Lagrangian multipliers q_t/λ_t.

Equations (13)–(16) together with the constraints (5) and (11) form a system of six equations which, given the values of u, n, π_t, and v_t and given the initial stock of real wealth a_0, will describe the time path of the six endogenous variables c_t, m_t, k_t, a_t, λ_t, and r_t. The problem is therefore to find the path of these variables which, satisfying equations (5), (11) and (13)–(16) will also satisfy the transversality condition (17).[6]

From equations (13) and (14) we can solve for the quantities demanded of consumption and real cash as functions of the implicit price of consumption, λ, the implicit interest rate, r, and the expected rate of change in prices, π;

$$c = c^0(\lambda, r, \pi) \tag{18}$$

$$m = m^0(\lambda, r, \pi) \tag{19}$$

and from equation (15) we can solve for the quantity demanded of real capital as a function of the implicit interest rate.

$$k = k^0(r) \tag{20}$$

Considering now the stock constraint (7) we determine the implicit rate of interest as a function of the stock of wealth, the implicit price of consumption and the expected rate of inflation.

$$a = k^0(r) + m^0(\lambda, r, \pi) \tag{21}$$

namely

$$r = r(a, \lambda, \pi) \tag{22}$$

and substituting back into (18), (19) and (20) we can write the demand for consumption, capital and real cash as functions of the stock of real

[6] In what follows the time subscripts will be used only where necessary for a better understanding of the text.

nonhuman wealth, the implicit price of consumption and the expected rate of change in prices

$$c = c'(a, \lambda, \pi) \tag{23}$$

$$m = m'(a, \lambda, \pi) \tag{24}$$

$$k = k'(a, \lambda, \pi) \tag{25}$$

Finally, given the expected rate of inflation and government transfer payments, the pair of differential equations (11) and (16) determine the time path of the implicit price of consumption and the stock of wealth. The laws of motion of our system are shown in the phase diagram below. To verify that (λ^*, a^*) is a saddle point we solve the characteristic equation for the linear Taylor approximation to (11) and (16). For the characteristic roots to be real and opposite in sign, guaranteeing that (a^*, λ^*) is a local saddle point the following condition must hold[7]

$$(\pi + n)J_1 + J_2 < 0 \tag{26}$$

This condition will be satisfied for any expected rate of change in prices which is not smaller than the rate of growth of population. It should be noted that although the slope of the $\dot{a}=0$ schedule in the phase diagrams may be positive or negative, the solution (a^*, λ^*) is a local saddle point provided that (26) holds.

Therefore, given the initial holdings of assets a_0, there is only one time path of a and λ which will satisfy the Euler conditions (13)–(16), the constraints (5) and (11) and the transversality condition (17). This path

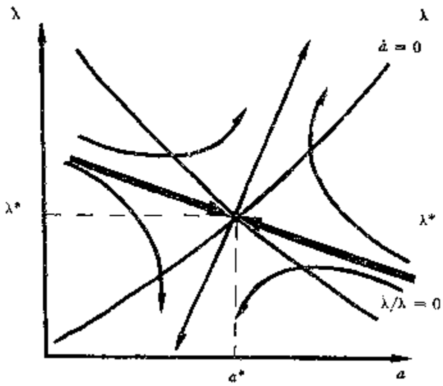

FIGURE 1a

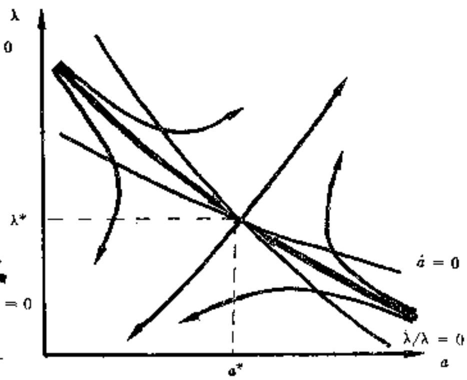

FIGURE 1b

is indicated by the heavy arrows in Figures 1a and 1b. All other paths (light arrows) fail to satisfy condition (17). For each total stock of wealth there is one implicit price of consumption λ which will determine

[7] This condition is also derived in [8].

the optimum allocation of the stock of wealth between capital and real money and the optimum allocation of the income flow between consumption and net saving. This implicit price is the value of λ on the heavy arrows that corresponds to the given level of wealth. Clearly, a change in the expected rate of inflation as well as a change in the real value of government transfers will shift the optimum path, hence

$$\lambda = \lambda(a, \pi, v) \tag{27}$$

By substituting (27) into (23), (24) and (25) we have shown that the quantities of capital, real money and consumption demanded are functions of total wealth, the expected rate of inflation and the net government transfers to the private sector.

$$c = c(a, \pi, v) \tag{28}$$

$$m = m(a, \pi, v) \tag{29}$$

$$k = k(a, \pi, v) \tag{30}$$

III. *The Expectations Hypothesis and the Macroeconomic Model*

It will be assumed that expectations are induced, i.e., that individuals take past rates of change in prices into account in forming their expectations about a "normal rate of change in prices," on the basis of which they determine the amounts of real cash balances, capital and consumption that they demand at each moment of time. In particular, our hypothesis will imply that when individuals realize that their expectations did not materialize they partially revise what they consider to be the "normal rate of change in prices." This is the so-called "adaptive expectations" model that was originally introduced by Cagan [1] and which can be expressed in terms of the following differential equation

$$\dot{\pi} = b(\dot{p}/p - \pi); \quad b > 0 \tag{31}$$

From the point of view of the economy as a whole, market equilibrium requires that at each moment of time the demand for money be equal to the total money supply. Thus, given the stock of money M, the equilibrium condition in the money market can be written as follows

$$\frac{M}{pN} = m(a, \pi, v) \tag{32}$$

where

$$a = \frac{M}{pN} + k \tag{33}$$

The government levies taxes and makes transfer payments to the private sector but it does not undertake any public expenditures. We assume that the excess of transfer payments over taxes is entirely financed by the creation of government noninterest bearing debt which we call money. Therefore, the excess of transfers over taxes is exactly equal to the amount of money issued per unit of time. We also assume that each economic unit in the economy receives the same amount of net transfers; namely, that taxes and transfers are of the per capita type. Hence we can write

$$v = \frac{\dot{M}}{Np} = \theta m \tag{34}$$

where $\theta = \dot{M}/M$. Equations (31)–(34) form a system of four equations in seven unknowns, M, N, k, p, a, v and π. In order to have a complete system that describes the behavior of the economy through time, three additional equations are required. To complete the system we assume that the government maintains a constant rate of monetary expansion and population grows at a constant rate

$$\dot{M}/M = \theta_0 \tag{35}$$

$$\dot{N}/N = n \tag{36}$$

Finally, since all output that is not consumed is necessarily used for capital accumulation, the rate of change in the capital stock is given by the following expression

$$\dot{k} = y(k) - (u + n)k - c(a, \pi, v) \tag{37}$$

We now have a system of seven equations in seven unknowns that describes the time path of our simplified economy. However, before we provide such a description it should be clear from the system of equations that a major difference arises when we go from the analysis of the individual economic unit to the analysis of the economy as a whole. While for the individual economic unit real wealth, the real value of the net government transfers and the expected rate of inflation are the variables that are given at each moment of time, for the economy as a whole these variables are the stock of capital, total population, the nominal quantity of money and the rate at which the government is increasing this quantity. Therefore, considering equations (32)–(34) we can write

$$\frac{M}{pN} = \hat{m}\,(k, \theta, \pi) \tag{38}$$

Thus, given M, N, k, θ, and π the price level is determined and therefore the real value of privately held cash is also determined.

Substituting equations (33), (34) and (38) into (37) the rate of change in the capital stock can then be written as

$$\dot{k} = y(k) - (u + n)k - \hat{c}(k, \theta, \pi) \tag{39}$$

where $\hat{c}$ is the consumption level desired for the price level p that equilibrates the money market at each moment of time.

The system of equations that describes the behavior of the economy through time is now given by (31), (35), (36), (38) and (39), the variables of the system being N, M, k, π and p. Differentiating equation (38) with respect to time and making use of (31), (35), (36) and (39) we express the rate of change in the expected rate of inflation as follows

$$\dot{\pi} = \frac{1}{\left[1 + b\dfrac{\partial \hat{m}}{\partial \pi}\dfrac{1}{\hat{m}}\right]} \cdot \left\{\theta - \pi - n - \left[y(k) - (u + n)k - \hat{c}(k, \theta, \pi)\right]\right\} \tag{40}$$

Given the rate of monetary expansion the system of differential equations (39), (40) describes the time path of the economy.

By setting $\dot{\pi}$ and $\dot{k}$ equal to zero in equations (39) and (40) it follows that along an equilibrium growth path consumption is equal to net output and the expected rate of inflation is equal to the difference between the rate of monetary expansion and the economy's rate of growth, which along such a path is equal to the rate of population growth. Hence,

$$c^* = y(k^*) - (u + n)k^* \tag{41}$$

$$\pi^* = \theta - n \tag{42}$$

IV. *Short-run and Long-run Effects of a Change in the Rate of Monetary Expansion and the Stability of Equilibrium Growth*

A question that naturally comes to mind is what are the conditions under which the equilibrium growth path characterized by equations (41) and (42) is stable? In what follows we will consider the local stability of our model and for this purpose we have to indicate first how changes in the capital stock, the expected rate of inflation and the rate of monetary expansion affect the demand for consumption and for real cash balances. It can be shown that, in the neighborhood of the equilibrium growth path, consumption and the real value of the stock of money are increasing functions of the stock of capital and of the rate of monetary expansion and decreasing functions of the expected rate of change in prices, namely

$$\frac{\partial \hat{c}^*}{\partial k^*} > 0 \qquad \frac{\partial \hat{c}^*}{\partial \theta} > 0 \qquad \frac{\partial \hat{c}^*}{\partial \pi^*} < 0 \tag{43}$$

$$\frac{\partial \hat{m}^*}{\partial k^*} > 0 \qquad \frac{\partial \hat{m}^*}{\partial \theta} > 0 \qquad \frac{\partial \hat{m}^*}{\partial \pi^*} < 0 \tag{44}$$

An increase in the capital stock raises consumption and the real value of the stock of cash of two accounts. First, because it raises disposable income and, second, because it is associated with an increase in the stock of real wealth. In addition, the increase in the stock of capital lowers its marginal product and therefore results in a decrease of the cost of holding cash, further raising the demand for the alternative asset. Net government transfers to the private sector are assumed to be financed entirely by money creation and therefore an increase in the rate of monetary expansion is equivalent to a rise in private disposable income due to higher government net transfers. The increase in real income raises the demand for both consumption and real cash balances. Finally, an increase in the expected rate of change in prices is equivalent to a rise in the rate of depreciation of one of the assets and it is therefore associated with a decrease in private disposable income which, in turn, lowers the demand for consumption and real money. In addition, the increase in the expected rate of inflation raises the opportunity cost of holding cash, thereby further reducing the demand for this asset.

With this information we now verify what are the conditions under which (k^*, π^*) is a stable solution to the pair of differential equations (39) and (40). For this purpose we solve the characteristic equations for the Taylor approximation to (39) and (40) at (k^*, π^*). From this exercise it follows that the necessary and sufficient conditions for the stability of the equilibrium growth path are

$$y'(k^*) - (u + n) - \frac{\partial \hat{c}^*}{\partial k^*} < 0 \tag{45}$$

$$\left(1 + \frac{\partial \hat{m}^*}{\partial \pi^*} \frac{1}{\hat{m}^*} b\right) > 0 \tag{46}$$

The condition in (45) reflects the fact that an increase in the capital stock has two opposite effects on the rate of capital accumulation. On the one hand, it increases real consumption, thereby lowering the rate of change of the capital stock, and, on the other hand, it raises net output and therefore stimulates capital accumulation. Stability in this model requires the rate of capital accumulation to be a decreasing function of the capital stock.

Since an increase in the expected rate of inflation reduces the demand for cash, it therefore results in a rise in prices. The increase in prices raises people's expected rate of inflation, further reducing the demand for money and causing a new rise in prices. Given that we assume that there is no lag in the adjustment of the actual to the desired stock of cash, the stability of the system depends on the existence as well as the magnitude of the expectations lag, as indicated by the expression in (46).

If the condition in (46) is satisfied, the solution (k^*, π^*) to the pair of differential equations (39) and (40), as indicated by the phase diagram in Figure 2, is a stable node. Provided the system is stable, a constant rate of monetary expansion will therefore guarantee a monotonic approach to the equilibrium growth path.

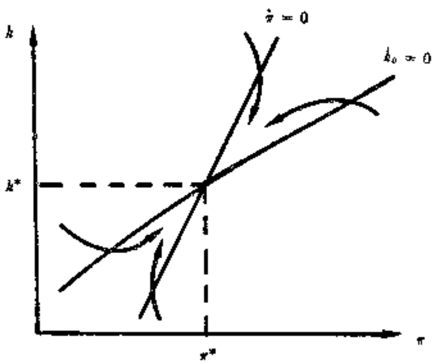

FIGURE 2

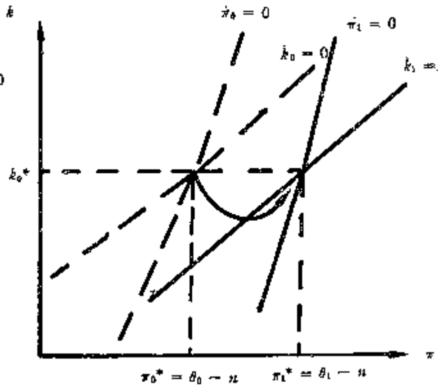

FIGURE 3

Consider now a situation in which the economy has reached its equilibrium growth path with $k^* = k_0^*$ and $\pi^* = \pi_0^*$ for $\theta = \theta_0 = \pi_0^* - n$. Suppose now that the government decides to change the rate of monetary expansion from θ_0 to θ_1, where $\theta_1 > \theta_0$. The first impact of this change is an increase in consumption which lowers capital accumulation, as well as an increase in the rate of change in prices which raises the rate of change of the expected rate of inflation. Both the $\dot k = 0$ and $\dot\pi = 0$ schedules shift to the right (Figure 3). Since we already know that the desired long-run stock of capital is determined only by its rate of depreciation, the rate of population growth and the subjective rate of time preference,[8] and, given that none of these variables are affected by the change in θ, the $\dot k = 0$ and the $\dot\pi = 0$ schedules will intersect at the same $k^* = k_0^*$ and at π_1^*, where $\pi_1^* = \theta_1 - n > \pi_0^*$. It therefore follows that while a change in the rate of monetary expansion lowers the short-

[8] This can be proved by setting equation (16) equal to zero and substituting into (15). The result of this operation is $y'(k^*) - u = (\delta + n)$.

run rate of capital accumulation, it does not affect the economy's long-run stock of capital.

V. *Conclusion*

Under the assumptions set out in Section I we have proved that in a growth model in which utility maximizing families are the basic economic unit of the system, the long-run capital stock of the economy is independent of the rate of monetary expansion. A rise in the rate of monetary expansion results in an equal absolute increase in the rate of change in prices; it reduces the stock of real cash but it does not affect steady state consumption. It therefore follows that the higher the rate of monetary expansion the lower will be the steady state level of utility. In the short run, an increase in the rate of monetary expansion is equivalent to a rise in government transfers to the private sector. It therefore results in an increase in consumption and a fall in the rate of capital accumulation.

Finally, we have also shown that in the absence of perfect foresight, utility maximization guarantees the fulfillment of only one of the steady state stability conditions; namely, that the rate of capital accumulation be a decreasing function of the capital stock. Since there is no lag in adjustment of the actual to the desired stock of cash, an additional, necessary but not sufficient, condition for the stability of the steady state is the existence of a lag in the formation of expectations.

References

1. P. Cagan, "The Monetary Dynamics of Hyperinflation," in *Studies in the Quantity Theory of Money*, ed., M. Friedman (Univ. of Chicago Press, 1956).
2. I. Fisher, *The Theory of Interest* (New York, 1930).
3. John G. Gurley and Edward S. Shaw, *Money in a Theory of Finance*, with a mathematical appendix by A. C. Enthoven (Washington, 1960).
4. Harry G. Johnson, "Monetary Theory and Policy," *A.E.R.*, June, 1962.
5. ———, "The neo-classical One Sector Growth Model, A Geometrical Exposition and Extension to a Monetary Economy," *Economica*, Aug., 1966.
6. D. Patinkin, *Money, Interest and Prices* (Row-Peterson, 1956).
7. M. Sidrauski, "Growth in a Monetary Economy" (Jan., 1966, mimeo.).
8. ———, "Inflation, Optimum Consumption and Real Cash Balances," a paper presented at the meetings of the Econometric Society, New York, Dec., 1965.
9. J. Tobin, "Money and Economic Growth," *Econometrica*, Oct., 1965.