Interpreting the Eigenvalues in a Symmetric Stochastic Matrix

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Consider the following n-state Markov process for the random variable, x_t:

$$x_t = \begin{cases} x_1 \\ x_2 \\ \vdots \\ x_n \end{cases} \tag{1}$$

The one-period transition probability matrix, with the entry in the ith row and jth column denoting the conditional probability of going from state i to state j, is:

$$\Pi = \begin{pmatrix} \pi & \frac{1-\pi}{n-1} & \frac{1-\pi}{n-1} & \cdots & \frac{1-\pi}{n-1} \\ \frac{1-\pi}{n-1} & \pi & \cdots & \cdots & \frac{1-\pi}{n-1} \\ \frac{1-\pi}{n-1} & \cdots & \ddots & ... & \frac{1-\pi}{n-1} \\ \frac{1-\pi}{n-1} & \cdots & \cdots & \pi & \frac{1-\pi}{n-1} \\ \frac{1-\pi}{n-1} & \cdots & \cdots & \cdots & \pi \end{pmatrix} \tag{2}$$

There will be n eigenvalues associated with the stochastic matrix but since the columns (and rows) sum to one, we know that one of the eigenvalues will be equal to 1. (The proof is easy: $\Pi \cdot \mathbf{1} = \mathbf{1}$). The unconditional probabilities, $\mathbf{p}' = (p_1, p_2, ..., p_n)$ are given by the solution to $\Pi^T \mathbf{p} = \mathbf{p}$. Since a matrix and its transpose have the same eigenvalues, we see that the eigenvector associated with the eigenvalue of unity is the vector of unconditional probabilities.

The remaining $n-1$ eigenvalues are not distinct. Let λ denote this eigenvalue and since the sum of the eigenvalues equal the trace of a matrix we have:

$$(n-1)\,\lambda = n\pi - 1$$

Or:

$$\lambda = \frac{n\pi - 1}{n-1} \tag{3}$$

Let the vector of realizations for x_t be symmetric around zero so that $\mathbf{p} \cdot \mathbf{x} = 0$. That is, the unconditional mean of $x_t, E(x_t) = 0$. I next show that the eigenvalue λ is the first-order autocorrelation of x_t. That is $\lambda = Corr(x_t, x_{t+1})$. Without proof, I first state that the vector $\mathbf{x}$ is an eigenvector associated with λ. (It can be shown that $\mathbf{x}$ can be expressed as a linear combination of the $n-1$ eigenvectors associated with the non-distinct eigenvalue λ.) Hence we have

$$\Pi \cdot \mathbf{x} = \lambda \mathbf{x}$$

The left-hand side is simply the vector of conditional expectations of x_{t+1}. Write this as:

$$\begin{pmatrix} E_1(x_{t+1}) \\ E_2(x_{t+1}) \\ \vdots \\ E_n(x_{t+1}) \end{pmatrix} = \lambda \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{n,t} \end{pmatrix} \tag{4}$$

Define the diagonal matrix D as:

$$D = \begin{pmatrix} x_{1,t} & 0 & \cdots & 0 \\ 0 & x_{2,t} & \cdots & 0 \\ 0 & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & x_{n,t} \end{pmatrix}$$

Then multiplying both sides of eq. (4) by D produces

$$\begin{pmatrix} E_1(x_{1,t}x_{t+1}) \\ E_2(x_{2,t}x_{t+1}) \\ \vdots \\ E_n(x_{n,t}x_{t+1}) \end{pmatrix} = \lambda \begin{pmatrix} x_{1,t}^2 \\ x_{2,t}^2 \\ \vdots \\ x_{n,t}^2 \end{pmatrix} \tag{5}$$

Mulitplying both sides by $\mathbf{p}'$ and using the fact that $E(x_t) = 0$, we have

$$Cov(x_t, x_{t+1}) = \lambda Var(x_t)$$

Which establishes that $\lambda = Corr(x_t, x_{t+1})$.