VARIETIES OF INTERPERSONAL COMPATIBILITY OF BELIEFS

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1 Introduction

Since Lewis's (1969) and Aumann's (1976) pioneering contributions, the concepts of common knowledge and common belief have been discussed extensively in the literature, both syntactically and semantically[1]. At the individual level the difference between knowledge and belief is usually identified with the presence or absence of the Truth Axiom ($\Box_i A \to A$), which is interpreted as "if individual i believes that A, then A is true". In such a case the individual is often said to *know* that A (thus it is possible for an individual to believe a false proposition but she cannot know a false proposition). Going to the interpersonal level, the literature then distinguishes between common knowledge and common belief on the basis of whether or not the Truth Axiom is postulated at the individual level. However, while at the individual level the Truth Axiom captures merely a relationship between the individuals' beliefs and the external world, at the interpersonal level it has very strong implications. For example, the following is a consequence of the Truth Axiom: $\Box_i \Box_j A \to \Box_i A$, that is, if individual i believes that individual j believes that A, then individual i herself believes that A. Thus, in contrast to other axioms, the Truth Axiom does not merely reflect individual agents' "logic of belief". (The reason why the Truth Axiom is much stronger in an interpersonal context than appears at first glance is that it amounts to assuming that agreement of any individual's belief with the truth is common knowledge). Given its logical force, it is not surprising to find that it has strong implications for the logic of common knowledge. In particular, if each individual's beliefs satisfy the strongest logic of knowledge (namely S5 or KT5), the associated common knowledge operator satisfies this logic too. Such is not the case for belief: bereft of the Truth Axiom, even the strongest logic for individual belief (KD45) is insufficient to ensure the satisfaction of the "Negative Introspection" axiom for common belief: $\neg\Box_* A \to \Box_* \neg\Box_* A$ (where $\Box_*$ denotes the common belief operator). That is to say, it can happen that neither is A commonly believed nor is it common belief that A is not commonly believed. Indeed the Negative Introspection axiom for common belief implies restrictions on individual beliefs of an intersubjective nature. In this paper we consider a variety of intersubjective compatibility restrictions on the beliefs of the individuals and study their relationship. We also provide a characterization of Negative Introspection for common belief.

2 The basic system $\mathbf{K}_n^*$

We consider a multimodal system with $n+1$ operators $\Box_1, \Box_2, ..., \Box_n, \Box_*$ where, for $i = 1, ..., n$, the interpretation of $\Box_i A$ is "individual i believes that A", while $\Box_* A$ is interpreted as "it is common belief that A". The basic system $\mathbf{K}_n^*$ is given by a suitable axiomatization of Propositional Calculus together with the following axiom schemata and rules of inference:

[1] See, fo example, Fagin *et al* (1995) and references therein.

K $\Box(A \to B) \to (\Box A \to \Box B)$ $(\forall \Box \in \{\Box_1, ..., \Box_n, \Box_*\})$

CB1 $\Box_* A \to \Box_i A$ $(\forall i = 1, ..., n)$

CB2 $\Box_* A \to \Box_i \Box_* A$ $(\forall i = 1, ..., n)$

CB3 $\Box_*(A \to \Box_1 A \wedge ... \wedge \Box_n A) \to (\Box_1 A \wedge ... \wedge \Box_n A \to \Box_* A)$

MP (modus ponens) $\frac{A,\ A \to B}{B}$

RN (necessitation) $\frac{A}{\Box A}$ $(\forall \Box \in \{\Box_1, ..., \Box_n, \Box_*\})$

We now turn to the semantics. A *standard model* is a tuple

$$\mathcal{M} = \langle W, R_1, ..., R_n, R_*, V \rangle$$

where W is a non-empty set of *possible worlds*, $R_1, ..., R_n, R_*$ are binary *accessibility relations* on W and V is a *valuation*, that is, a function that associates with every atomic proposition p the set of possible worlds where p is true. The valuation is extended to the set of formulas in the usual way; we denote the fact that formula A is true at world w in model $\mathcal{M}$ by $\mathcal{M}, w \models A$. Thus, in particular, for $i = 1, ..., n$, $\mathcal{M}, w \models \Box_i A$ if and only if $\mathcal{M}, w' \models A$ for all w' such that wR_iw'. Similarly, $\mathcal{M}, w \models \Box_* A$ if and only if $\mathcal{M}, w' \models A$ for all w' such that wR_*w'. The following result is well-known (cf., for example, Bonanno, 1996).

Theorem 1 *The system* $\mathbf{K}_n^*$ *is sound and complete with respect to the class of standard models where* R_* *is the transitive closure of* $R_1 \cup ... \cup R_n$.[2]

A standard model where R_* is the transitive closure of $R_1 \cup ... \cup R_n$ will be called a *CB-model*.

We will investigate extensions of $\mathbf{K}_n^*$ obtained by adding one or more of the following axioms for individual beliefs:

D $\Box_i A \to \neg\Box_i \neg A$ $(i = 1, ..., n)$

T $\Box_i A \to A$ $(i = 1, ..., n)$

4 $\Box_i A \to \Box_i \Box_i A$ $(i = 1, ..., n)$

4c $\Box_i \Box_i A \to \Box_i A$ $(i = 1, ..., n)$

5 $\neg\Box_i A \to \Box_i \neg\Box_i A$ $(i = 1, ..., n)$

In the next section we list some theorems and derived rules of inference for $\mathbf{K}_n^*$ which will be used later.

3 Some properties of $\mathbf{K}_n^*$

For every modal operator $\Box \in \{\Box_1, ..., \Box_n, \Box_*\}$ we write $\Diamond$ for $\neg\Box\neg$ (thus, for example, $\Diamond_i A$ stands for $\neg\Box_i\neg A$). Furthermore, PL stands for "Propositional Logic".

[2] That is, aR_*b if and only if there are sequences $\langle w_1, ..., w_m \rangle$ and $\langle i_1, ..., i_m \rangle$ such that (1) $w_1 = a$, (2) $w_m = b$ and (3) for every $k = 1, ..., m-1$, $w_k R_{i_k} w_{k+1}$.

It is well-known (see, for example, Chellas, 1984, Lismont and Mongin, 1994) that $\mathbf{K}_n^*$ has the following theorems and rules of inference:

RK	$\frac{A \to B}{\Box A \to \Box B}$	for every $\Box \in \{\Box_1, ..., \Box_n, \Box_*\}$
RK$\Diamond$	$\frac{A \to B}{\Diamond A \to \Diamond B}$	for every $\Diamond \in \{\Diamond_1, ..., \Diamond_n, \Diamond_*\}$
M	$\Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B$	for every $\Box \in \{\Box_1, ..., \Box_n, \Box_*\}$
R1CB	$\frac{A \to (\Box_1 A \wedge ... \wedge \Box_n A)}{(\Box_1 A \wedge ... \wedge \Box_n A) \to \Box_* A}$	(apply RN to the hypothesis, then use **CB3** and MP)
R2CB	$\frac{A \to (\Box_1 A \wedge ... \wedge \Box_n A)}{A \to \Box_* A}$	(apply R1CB to the hypothesis, then use PL)

Proofs of the following lemma and corollary are given in the Appendix.

Lemma 2 *The following is provable in* $\mathbf{K}_n^*$*:*

$$\textbf{\textit{CB4}} \qquad (\Box_1 A \ \wedge \ ... \ \wedge \ \Box_n A) \wedge (\Box_1 \Box_* A \ \wedge \ ... \ \wedge \ \Box_n \Box_* A) \to \Box_* A$$

Corollary 3 *The following is provable in* $\mathbf{K}_n^* + \mathbf{4c}$*:*

$$\textbf{\textit{CB5}} \qquad (\Box_1 \Box_* A \ \wedge \ ... \ \wedge \ \Box_n \Box_* A) \to \Box_* A$$

Remark 1 *It is well-known (cf. Chellas, 1984) that axiom* **4c** *is provable in* $\mathbf{K}_n^* + \mathbf{5}$. *Hence* ***CB5*** *is provable in* $\mathbf{K}_n^* + \mathbf{5}$.

4 Interpersonal compatibility of beliefs

In general, the common belief operator $\Box_*$ does not inherit all the properties of the individuals' belief operators. Consider, for example, the counterpart for common belief of axiom **5** for individual beliefs:

$$\mathbf{5}^* \quad \neg\Box_* A \to \Box_* \neg \Box_* A.$$

Now, $\mathbf{5}^*$ is *not* provable in the system obtained by adding **D**, **4** and **5** to $\mathbf{K}_n^*$, which will be denoted by $\mathbf{K}_n^* + \mathbf{D45}$ (*a fortiori* it is not provable in a weaker system such as $\mathbf{K}_n^* + \mathbf{5}$). This is shown in the following example.

Example 1 *Consider the following CB-model:* $W = \{a, b\}$, $R_1 = \{(a, a), (b, b)\}$, $R_2 = \{(a, b), (b, b)\}$. *Thus* $R_* = \{(a, a), (a, b), (b, b)\}$. *Let* p *be an atomic proposition which is true at* b *and false at* a. *Then the formula* $(\neg\Box_* p \to \Box_* \neg \Box_* p)$, *which is an instance of* $\mathbf{5}^*$, *is false at* a. *Since the system* $\mathbf{K}_n^* + \mathbf{D45}$ *is sound with respect to the class of CB-models where* R_i *is serial, transitive and euclidean* ($i = 1, ..., n$) *and this model satisfies these properties, it follows that* $\mathbf{5}^*$ *is not provable in* $\mathbf{K}_n^* + \mathbf{D45}$.

It follows from Example 1 that axiom $\mathbf{5}^*$ must involve further restrictions on the beliefs of the individuals which presumably are intersubjective in nature. In this section we consider various requirements of intersubjective compatibility of beliefs and study their relationship.

The following axioms capture interpersonal restrictions of various strength on the beliefs of the individuals.

C	$\Box_i A \to \Diamond_j A$
TN	$\Box_i \Box_j A \to \Box_j A$
TP	$\Box_i \Diamond_j A \to \Diamond_j A$
IN	$\Box_i \Box_j A \to \Box_i A$
IP	$\Diamond_i \Box_j A \to \Diamond_i A$
SW	$\Box_i \Box_* A \to \Box_j \Box_* A$
$\mathbf{T}^{\mathrm{CB}}$	$\Box_i \Box_* A \to \Box_* A$

C rules out the possibility that one individual believes A and at the same some other individual believes $\neg A$. **TN** requires individuals to be correct in their beliefs about what others believe, while **TP** requires them to be correct in their beliefs about what others consider possible. By **IN** individuals must share the beliefs that they attribute to others, while **IP** is a weakening of this: if individual i considers it possible that individual j believes A then i himself must at least consider A possible. **SW** (for *Shared Worlds*) requires the individuals' beliefs about what is commonly believed to agree. Finally $\mathbf{T}^{\mathrm{CB}}$ requires individuals to be correct in their beliefs about common belief.

Remark 2 ***T****, the Truth Axiom for individuals beliefs, plays a much stronger role in multi-agent contexts than in single-agent ones. For a single individual the Truth Axiom captures merely a relationship between her beliefs and the external world; at the interpersonal level, on the other hand, it implies strong intersubjective compatibility of beliefs. Indeed, all of the above axioms (**C, TN**, **TP**, **IN**, **IP**, **SW** and $\mathbf{T}^{CB}$) are provable in* $\mathbf{K}_n^* + \mathbf{T}$*. The reason why the Truth Axiom has strong implications in an interpersonal context is that it amounts to assuming that agreement of any individual's beliefs with the truth is commonly believed.*

The following theorem establishes the relationship between the first five axioms, namely **C**, **TN**, **TP**, **IN** and **IP.**

Theorem 4

(i) $\mathbf{K}_n^* + \mathbf{IP} + \mathbf{TP} \vdash \mathbf{C}$
(ii) $\mathbf{K}_n^* + \mathbf{4} + \mathbf{C} \vdash \mathbf{IP}$
(iii) $\mathbf{K}_n^* + \mathbf{4} + \mathbf{C} \vdash \mathbf{TP}$
(iv) $\mathbf{K}_n^* + \mathbf{5} + \mathbf{C} \vdash \mathbf{TN}$

Proof. The proof of (i) is as follows.

1.	$\Diamond_i \Box_j \neg A \rightarrow \neg \Box_i A$	(**IP**)
2.	$\Box_i A \rightarrow \Box_i \Diamond_j A$	(1,PL)
3.	$\Box_i \Diamond_j A \rightarrow \Diamond_j A$	(**TP**)
4.	$\Box_i A \rightarrow \Diamond_j A$	(2,3,PL)

(ii) Next we prove that $\mathbf{K}_n^* + \mathbf{4} + \mathbf{C} \vdash \mathbf{IP}$:

1.	$\Box_i \neg A \rightarrow \neg \Box_j A$	(**C**)
2.	$\Box_i \Box_i \neg A \rightarrow \Box_i \neg \Box_j A$	(1,RK)
3.	$\Box_i \neg A \rightarrow \Box_i \Box_i \neg A$	(axiom **4**)
4.	$\Box_i \neg A \rightarrow \Box_i \neg \Box_j A$	(2,3,PL)
5.	$\Diamond_i \Box_j A \rightarrow \Diamond_i A$	(4,PL)

(iii) The proof of (iii) is as follows:

1.	$\Box_j \neg A \rightarrow \Box_j \Box_j \neg A$	(axiom **4**)
2.	$\Box_j \Box_j \neg A \rightarrow \neg \Box_i \neg \Box_j \neg A$	(**C**)
3.	$\Box_j \neg A \rightarrow \neg \Box_i \neg \Box_j \neg A$	(1,2,PL)
4.	$\Box_i \Diamond_j A \rightarrow \Diamond_j A$	(3,PL)

(iv) The proof that $\mathbf{K}_n^* + \mathbf{5} + \mathbf{C} \vdash \mathbf{TN}$ is as follows:

1.	$\neg \Box_j A \rightarrow \Box_j \neg \Box_j A$	(axiom **5**)
2.	$\Box_j \neg \Box_j A \rightarrow \neg \Box_i \Box_j A$	(**C**)
3.	$\neg \Box_j A \rightarrow \neg \Box_i \Box_j A$	(1,2,PL)
4.	$\Box_i \Box_j A \rightarrow \Box_j A$	(3,PL)

■

By Theorem 4, Compatibility (axiom **C**) has strong implications for the truth of intersubjective beliefs (**TP** and **TN**); moreover, it crucially involves some form of intersubjective agreement (**IP**). However, as the following example shows, intersubjective truth alone (**TP** and **TN**) fails to imply any intersubjective agreement (**IP** or **IN**) and therefore fails to imply **C**.

Example 2 *Consider the following model:* $W = \{a, b\}$, $R_1 = \{(a, b), (b, b)\}$, $R_2 = \{(a, a), (b, a)\}$. *Thus* R_* *(the transitive closure of* $R_1 \cup R_2$*) is the universal relation. Let* p *be an atomic proposition which is true at* a *and false at* b. *Then at* a *all of the following are true* $\Box_1 \Box_2 p$, $\Diamond_1 \Box_2 p$, $\neg \Box_1 p$, $\Box_1 \neg p$, $\Box_2 p$, *thus falsifying* ***IN****,* ***IP****, and* ***C****. On the other hand, both* ***TP*** *and* ***TN*** *are true at every world.*[3]

Similarly, the following example shows that intersubjective agreement (**IP** and **IN**) does not imply intersubjective truth (**TP** and **TN**) nor does it imply compatibility (**C**).

Example 3 *Let*

$$W = \{a, b, c\},\ R_1 = \{(a, a), (b, a), (c, c)\},\ R_2 = \{(a, a), (b, c), (c, c)\}.$$

[3] This is a consequence of the following fact, which is proved in the appendix. Axioms **TN** and **TP** are valid in the class of CB-models where, $\forall i, j \in N$, $\forall w, w', w'' \in W$, (1) R_i is serial and (2) if wR_iw' and $w'R_jw''$ then wR_jw''. The model of the above example satifies this property.

*Thus $R_* = \{(a,a),(b,a),(b,c),(c,c)\}$. Let p be an atomic proposition which is true at a and false at b and c. Here at world b all the following are true: $\Box_1\Box_2 p$, $\Box_1\Diamond_2 p$, $\neg\Box_2 p$, $\Box_2\neg p$ and $\Box_1 p$. Thus at b **TP**, **TN** and **C** are falsified. On the other hand both **IP** and **IN** are valid in this model.*[4]

We now turn to the relationship between **SW** and $\mathbf{T}^{\mathrm{CB}}$.

Theorem 5

$$\begin{array}{ll} \text{(i)} & \mathbf{K}_n^* + \mathbf{T}^{\mathrm{CB}} \vdash \mathbf{SW} \\ \text{(ii)} & \mathbf{K}_n^* + \mathbf{4c} + \mathbf{SW} \vdash \mathbf{T}^{\mathrm{CB}} \end{array}$$

Proof. For (i)

1.	$\Box_i\Box_* A \to \Box_* A$	($\mathbf{T}^{\mathrm{CB}}$)
2.	$\Box_* A \to \Box_j\Box_* A$	(**CB2**)
3.	$\Box_i\Box_* A \to \Box_j\Box_* A$	(1,2,PL)

For (ii)

1.	$\Box_i\Box_* A \to \Box_1\Box_* A$	(**SW**)
...	...	(**SW**)
n.	$\Box_i\Box_* A \to \Box_n\Box_* A$	(**SW**)
n+1.	$\Box_i\Box_* A \to (\Box_1\Box_* A \wedge ... \wedge \Box_n\Box_* A)$	(1,...,n,PL)
n+2.	$(\Box_1\Box_* A \wedge ... \wedge \Box_n\Box_* A) \to \Box_* A$	(**CB5**: cf. Corollary 3)
n+3.	$\Box_i\Box_* A \to \Box_* A$	(n+1, n+2, PL)

■

The next theorem relates $\mathbf{T}^{\mathbf{CB}}$ to $\mathbf{5}^*$.

Theorem 6

$$\begin{array}{ll} \text{(i)} & \mathbf{K}_n^* + \mathbf{D} + \mathbf{5}^* \vdash \mathbf{T}^{\mathbf{CB}} \\ \text{(ii)} & \mathbf{K}_n^* + \mathbf{5} + \mathbf{T}^{\mathbf{CB}} \vdash \mathbf{5}^* \end{array}$$

Proof. The proof of (i) is as follows:

1.	$\neg\Box_* A \to \Box_*\neg\Box_* A$	(axiom $\mathbf{5}^*$)
2.	$\Box_*\neg\Box_* A \to \Box_i\neg\Box_* A$	(**CB1**)
3.	$\Box_i\neg\Box_* A \to \neg\Box_i\Box_* A$	(**D**)
4.	$\neg\Box_* A \to \neg\Box_i\Box_* A$	(1,2,3,PL)
5.	$\Box_i\Box_* A \to \Box_* A$	(4,PL)

Next we prove (ii).

[4] This is a consequence of the following fact, which is proved in the Appendix. Axioms **IN** and **IP** are valid in the class of CB-models where, $\forall i,j \in N$, $\forall w \in W$, $\exists w' \in W$ such that (1) wR_iw' and (2) $\forall w'' \in W$, if wR_iw'' then $w'R_jw''$. The model of the above example satifies this property.

1. $\Box_i \Box_* A \to \Box_* A$ ($\mathbf{T^{CB}}$)
2. $\neg \Box_* A \to \neg \Box_i \Box_* A$ (1,PL)
3. $\neg \Box_i \Box_* A \to \Box_i \neg \Box_i \Box_* A$ (axiom **5**)
4. $\neg \Box_* A \to \Box_i \neg \Box_i \Box_* A$ (2,3,PL)
5. $\Box_* A \to \Box_i \Box_* A$ (**CB2**)
6. $\neg \Box_i \Box_* A \to \neg \Box_* A$ (5,PL)
7. $\Box_i \neg \Box_i \Box_* A \to \Box_i \neg \Box_* A$ (6,RK)
8. $\neg \Box_* A \to \Box_i \neg \Box_* A$ (4,7,PL)

Now, a repetition of steps 1-8 for every $i = 1, ..., n$ leads to (by 8 and PL)

9. $\neg \Box_* A \to (\Box_1 \neg \Box_* A \wedge ... \wedge \Box_n \neg \Box_* A)$
10. $\neg \Box_* A \to \Box_* \neg \Box_* A$ (9, R2CB)

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Remark 3 *Since (cf. Remark 1)* $\mathbf{K}_n^* + \mathbf{5} \vdash \mathbf{4c}$*, it follows from (ii) of Theorem 5 that* $\mathbf{K}_n^* + \mathbf{5} + \mathbf{SW} \vdash \mathbf{T}^{CB}$*. Thus, by Theorem 6,* $\mathbf{K}_n^* + \mathbf{5} + \mathbf{SW} \vdash \mathbf{5}^*$*. Furthermore, by Theorems 5 and 6,* $\mathbf{K}_n^* + \mathbf{D} + \mathbf{5}^* \vdash \mathbf{SW}$*. It follows from Theorem 6 that* $\mathbf{K}_n^* + \mathbf{D5} + \mathbf{SW}$ *and* $\mathbf{K}_n^* + \mathbf{D5} + \mathbf{5}^*$ *are the same system.*

We now turn to the relationship between axioms **TN**, **TP**, **IN**, **IP**, **C** and the Shared Worlds axiom (**SW**).

Proposition 7 *None of **TN**, **TP**, **IN**, **IP** and **C** is provable in* $\mathbf{K}_n^* + \mathbf{D45} + \mathbf{SW}$ *(thus, a fortiori, in a weaker system such as* $\mathbf{K}_n^* + \mathbf{SW}$*).*

Proof. First of all, it is straightforward to show that the system $\mathbf{K}_n^* + \mathbf{D45} + \mathbf{SW}$ is sound with respect to the class of CB-models where (1) $\forall i \in N$, R_i is serial, transitive and euclidean, and (2) $\forall i, j \in N$, $\forall x, y, z \in W$, if xR_jy and yR_*z then $\exists w \in W$ such that xR_iw and wR_*z. The following model belongs to this class: $W = \{a, b, c\}$, $R_1 = \{(a, b), (b, b), (c, c)\}$, $R_2 = \{(a, a), (b, c), (c, c)\}$, $R_3 = \{(a, a), (b, b), (c, a)\}$; thus R_* is the universal relation. Let p be an atomic proposition which is true at c and false at a and b. Then this model validates all the theorems of $\mathbf{K}_n^* + \mathbf{D45} + \mathbf{SW}$ (in particular **SW** itself)[5]. It is easily checked that

$\Box_1 \Box_2 p \to \Box_2 p$ (which is an instance of **TN**) is *false* at a
$\Box_1 \Diamond_2 p \to \Diamond_2 p$ (which is an instance of **TP**) is *false* at a
$\Box_1 \Box_2 p \to \Box_1 p$ (which is an instance of **IN**) is *false* at a
$\Diamond_1 \Box_2 p \to \Diamond_1 p$ (which is an instance of **IP**) is *false* at a
$\Box_2 p \to \neg \Box_3 \neg p$ (which is an instance of **C**) is *false* at b

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Theorem 8

[5]Since R_* is the universal relation, for every formula A and every world w, $w \models \Box_* A$ if and only if A is valid (i.e. true at every world). It follows from seriality of R_i that if individual i believed that A is common belief then A is indeed commonly believed (thus $\mathbf{T^{CB}}$ is valid). Hence everybody shares i's belief that A is common belief (thus **SW** is valid).

(i) $\mathbf{K}_n^* + \mathbf{TN} \vdash \mathbf{SW}$
(ii) $\mathbf{K}_n^* + \mathbf{D5} + \mathbf{TP} \vdash \mathbf{SW}$

Proof. For (i)

1.	$\Box_* A \to \Box_j \Box_* A$	(**CB2**)
2.	$\Box_i \Box_* A \to \Box_i \Box_j \Box_* A$	(1,RK)
3.	$\Box_i \Box_j \Box_* A \to \Box_j \Box_* A$	(**TN**)
4.	$\Box_i \Box_* A \to \Box_j \Box_* A$	(2,3,PL)

For (ii)

1.	$\Box_* A \to \Box_j \Box_* A$	(**CB2**)
2.	$\Diamond_j \Box_* A \to \Diamond_j \Box_j \Box_* A$	(1,RK$\Diamond$)
3.	$\neg \Box_j \Box_* A \to \Box_j \neg \Box_j \Box_* A$	(axiom **5**)
4.	$\Diamond_j \Box_j \Box_* A \to \Box_j \Box_* A$	(3,PL)
5.	$\Diamond_j \Box_* A \to \Box_j \Box_* A$	(2,4,PL)
6.	$\Box_j \Box_* A \to \Diamond_j \Box_* A$	(axiom **D**)
7.	$\Box_i \Box_* A \to \Box_i \Box_j \Box_* A$	(1,RK)
8.	$\Box_i \Box_j \Box_* A \to \Box_i \Diamond_j \Box_* A$	(6,RK)
9.	$\Box_i \Box_* A \to \Box_i \Diamond_j \Box_* A$	(7,8,PL)
10.	$\Box_i \Diamond_j \Box_* A \to \Diamond_j \Box_* A$	(**TP**)
11.	$\Box_i \Box_* A \to \Diamond_j \Box_* A$	(9,10,PL)
12.	$\Box_i \Box_* A \to \Box_j \Box_* A$	(5,11,PL)

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5 Concluding remarks

We have considered a variety of interpersonal compatibility restrictions on the beliefs of the individuals. The strongest restrictions are obtained by imposing the Truth Axiom on individual beliefs because of the implied common knowledge of the correctness of beliefs. Of the remaining conditions **C** is the strongest, since it implies all the others. The weakest of all is **SW**, which in the system $\mathbf{K}_n^* + \mathbf{D5}$ turns out to be equivalent to $\mathbf{5}^*$ Negative Introspection of Common Belief. The relationship among all the axioms considered in this paper (**T, C**, **TN**, **TP**, **IN**, **IP**, **SW**, $\mathbf{T^{CB}}$, $\mathbf{5}^*$) is summarized in Figure 1.

Weaker intersubjective restrictions on beliefs can be obtained from **C**, **TN**, **TP**, **IN**, **IP** and **SW** by replacing $\Box_j$ with $\Box_*$. In this case **TN** becomes $\mathbf{T}^{\mathrm{CB}}$, while the remaining axioms become

$\mathbf{C}^*$	$\Box_i A \to \Diamond_* A$
$\mathbf{TP}^*$	$\Box_i \Diamond_* A \to \Diamond_* A$
$\mathbf{IN}^*$	$\Box_i \Box_* A \to \Box_i A$
$\mathbf{IP}^*$	$\Diamond_i \Box_* A \to \Diamond_i A$
$\mathbf{SW}^*$	$\Box_i \Box_* A \to \Box_* \Box_* A$

It is straightforward to show that all of the above are provable in $\mathbf{K}_n^* + \mathbf{D45}$.[6]

[6]Indeed, $\mathbf{C}^*$and $\mathbf{TP}^*$ are provable in $\mathbf{K}_n^* + \mathbf{D}$, $\mathbf{IN}^*$ is provable in $\mathbf{K}_n^* + \mathbf{4c}$ and $\mathbf{IP}^*$ is

6 Appendix

Proof of Lemma 2.

1.	$(\Box_1 A \wedge \ldots \wedge \Box_n A) \wedge (\Box_1 \Box_* A \wedge \ldots \wedge \Box_n \Box_* A) \rightarrow$ $\Box_1(A \wedge \Box_* A) \wedge \ldots \wedge \Box_n(A \wedge \Box_* A)$	(**M**,PL)
2.	$\Box_* A \rightarrow \Box_1 A \wedge \ldots \wedge \Box_n A$	(**CB1**,PL)
3.	$\Box_* A \rightarrow \Box_1 \Box_* A \wedge \ldots \wedge \Box_n \Box_* A$	(**CB2**,PL)
4.	$\Box_* A \rightarrow \Box_1(A \wedge \Box_* A) \wedge \ldots \wedge \Box_n(A \wedge \Box_* A)$	(1,2,3,PL)
5.	$A \wedge \Box_* A \rightarrow \Box_* A$	(PL)
6.	$A \wedge \Box_* A \rightarrow \Box_1(A \wedge \Box_* A) \wedge \ldots \wedge \Box_n(A \wedge \Box_* A)$	(4,5,PL)
7.	$\Box_1(A \wedge \Box_* A) \wedge \ldots \wedge \Box_n(A \wedge \Box_* A) \rightarrow \Box_* (A \wedge \Box_* A)$	(6, R1CB)
8.	$\Box_* (A \wedge \Box_* A) \rightarrow \Box_* A \wedge \Box_* \Box_* A$	(**M**)
9.	$(\Box_1 A \wedge \ldots \wedge \Box_n A) \wedge (\Box_1 \Box_* A \wedge \ldots \wedge \Box_n \Box_* A) \rightarrow \Box_* A \wedge \Box_* \Box_* A$	(1,7,8,PL)
10.	$\Box_* A \wedge \Box_* \Box_* A \rightarrow \Box_* A$	(PL)
11.	$(\Box_1 A \wedge \ldots \wedge \Box_n A) \wedge (\Box_1 \Box_* A \wedge \ldots \wedge \Box_n \Box_* A) \rightarrow \Box_* A$	(9,10,PL)

■

Proof of Corollary 3.

1.	$\Box_* A \rightarrow \Box_i A$	(**CB1**)
2.	$\Box_i \Box_* A \rightarrow \Box_i \Box_i A$	(1,RK)
3.	$\Box_i \Box_i A \rightarrow \Box_i A$	(axiom **4c**)
4.	$\Box_i \Box_* A \rightarrow \Box_i A$	(2,3,PL)
5.	$(\Box_1 \Box_* A \wedge \ldots \wedge \Box_n \Box_* A) \rightarrow (\Box_1 A \wedge \ldots \wedge \Box_n A)$	(4,PL)
6.	$(\Box_1 \Box_* A \wedge \ldots \wedge \Box_n \Box_* A) \rightarrow (\Box_1 \Box_* A \wedge \ldots \wedge \Box_n \Box_* A)$	(PL)
7.	$(\Box_1 \Box_* A \wedge \ldots \wedge \Box_n \Box_* A) \rightarrow$ $(\Box_1 A \wedge \ldots \wedge \Box_n A) \wedge (\Box_1 \Box_* A \wedge \ldots \wedge \Box_n \Box_* A)$	(5,6,PL)
8.	$(\Box_1 A \wedge \ldots \wedge \Box_n A) \wedge (\Box_1 \Box_* A \wedge \ldots \wedge \Box_n \Box_* A) \rightarrow \Box_* A$	(**CB4**)
9.	$(\Box_1 \Box_* A \wedge \ldots \wedge \Box_n \Box_* A) \rightarrow \Box_* A$	(7,8,PL)

■

Lemma 9 *Axioms **TN** and **TP** are valid in the class of CB-models where, $\forall i, j \in N$, $\forall w, w', w'' \in W$, (1) R_i is serial and (2) if wR_iw' and $w'R_jw''$ then wR_jw''.*

Proof. First we prove validity for **TN**. Fix an arbitrary model that satisfies the above two properties. Fix arbitrary i, j, a and an arbitrary formula A. Suppose that $a \models \Box_i \Box_j A$. By seriality of R_i there exists a b such that aR_ib. Fix an arbitrary such b. Then $b \models \Box_j A$. By seriality of R_j, there exists a c such that bR_jc. Fix an arbitrary such c. Then $c \models A$. By property (2) aR_jc. Hence $a \models \Box_j A$.

(2) Next we prove validity of **TP**. Fix an arbitrary model that satisfies the above two properties. Fix arbitrary i, j, a and an arbitrary formula A. Suppose that $a \models \Box_i \Diamond_j A$. By seriality of R_i there exists a b such that aR_ib. Then $b \models \Diamond_j A$. Hence there exists a c such that bR_jc and $c \models A$. By property (2) aR_jc. Hence $a \models \Diamond_j A$. ■

provable in $\mathbf{K}_n^* + \mathbf{D4}$ or in $\mathbf{K}_n^* + \mathbf{D5}$.

Lemma 10 *Axioms **IN** and **IP** are valid in the class of CB-models where,* $\forall i, j \in N$, $\forall w \in W$, $\exists w' \in W$ *such that (1)* wR_iw' *and (2)* $\forall w'' \in W$, *if* wR_iw'' *then* $w'R_jw''$.

Proof. First we prove validity for **IN**. Fix an arbitrary model that satisfies the above properties. Fix arbitrary i, j, a and an arbitrary formula A. Suppose that $a \models \Box_i \Box_j A$. By the assumed properties, there exists a b such that aR_ib. Choose an arbitrary such b. Then $b \models \Box_j A$. Choose an arbitrary c such that aR_ig. By the assumed properties, bR_jc. Hence $c \models A$. Therefore $a \models \Box_i A$.

(2) Next we prove validity for **IP**. Fix an arbitrary model that satisfies the above properties. Fix arbitrary i, j, a and an arbitrary formula A. Suppose that $a \models \Diamond_i \Box_j A$. Then there exists a b such that aR_ib and $b \models \Box_j A$. By the assumed properties (choosing $c = b$), bR_jb. Hence $b \models A$. Therefore, $a \models \Diamond_i A$.
■

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