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PREFERENCE FOR FLEXIBILITY IN A SAVAGE FRAMEWORK

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We study preferences over Savage acts that map states to opportunity sets and satisfy the Savage axioms. Preferences over opportunity sets may exhibit a *preference for flexibility* due to an implicit uncertainty about future preferences reflecting anticipated unforeseen contingencies. The main result of this paper characterizes maximization of the expected indirect utility in terms of an "Indirect Stochastic Dominance" axiom that expresses a preference for "more opportunities in expectation."

The key technical tool of the paper, a version of Möbius inversion, has been imported from the theory of nonadditive belief functions; it allows an alternative representation using Choquet integration, and yields a simple proof of Kreps' (1979) classic result.

KEYWORDS: Dynamic decision making, expected utility, unforeseen contingencies, opportunity set, option value, Möbius inverse.

1. INTRODUCTION

FLEXIA PLANS TO UNDERTAKE a plane trip; she has to decide whether to purchase an advance-reservation ticket now at a price p, or to wait until right before her intended date of departure and then to decide between staying home and purchasing a ticket at a higher price q. Flexia's present choice can be thought of as one among opportunity sets, here $\{\text{fly}\,@\,p, \text{stay}\,@\,p\}$ and $\{\text{fly}\,@\,q, \text{stay}\,@\,0\}$, from which her future choice is then made.

Flexia has fairly common present preferences over opportunity sets; if required to make a final choice now among basic alternatives (singleton opportunity sets), she would most prefer to purchase a ticket in advance ($\{\text{fly}\,@\,p\} \succ \{\text{stay}\,@\,0\}$). On the other hand, if possible, she would rather "wait-and-see" ($\{\text{fly}\,@\,p, \text{stay}\,@\,p\} \prec \{\text{fly}\,@\,q, \text{stay}\,@\,0\}$). Note that these preferences are not compatible with a ranking of opportunity sets according to their indirect utility (i.e. by ranking the sets as equivalent to their currently best element)[2]. They are naturally explained, however, as due to an uncertainty about her own future preferences between making the trip and staying at home.

Such *preference for flexibility* received its first axiomatic study in a classic paper by Kreps (1979) which characterized the class of preferences that rank opportunity sets in terms of their *expected indirect utility* (EIU). Let Ω denote a space of implicit states ω describing future preferences, v_ω the utility function in the

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[2] Assuming greater wealth to be preferred, of course.

implicit state ω with subjective probability λ_ω, and x a basic alternative; the states $\omega \in \Omega$ are *implicit* in that they are part of the representation, not of the set-up. Then the expected indirect utility of the opportunity set A can be written as $u(A) = \sum_{\omega \in \Omega} \lambda_\omega \max_{x \in A} v_\omega(x)$. Kreps showed that the implications of EIU maximization are surprisingly weak; in particular, a preference relation over opportunity sets has an EIU representation whenever it is strictly monotone in the sense that strictly larger opportunity sets are always strictly preferred.

The assumption that present choices determine future opportunity sets deterministically is very restrictive; for instance, if Flexia decides to wait, realistically she will need to reckon with the risk that seats may no longer be available later. The present paper characterizes EIU maximization in such more general situations in which the agent may be uncertain about the opportunity sets she is going to face. In formal terms, we will study preferences over Savage acts f ("*opportunity acts*") that map *explicit* states $\theta \in \Theta$ to opportunity sets $A \in \mathscr{A} = 2^X \setminus \emptyset$ of alternatives $x \in X$ and that satisfy all Savage axioms. Preference for flexibility corresponds to a violation of the following *indirect-utility property*:

$$\text{For all sets (constant acts) } A, B \in \mathscr{A}\text{: } A \succcurlyeq B \text{ implies } A \succcurlyeq A \cup B.$$

A particularly interesting interpretation of the distinction between explicit and implicit states has been given by Kreps (1992), where the explicit state-space models the space of *foreseen* contingencies determining the class of thought experiments relevant to the decision-maker's preference construction. Violation of a conditional version of the IU property[3] is viewed as reflecting *anticipated unforeseen* contingencies; for example, Flexia may explain her preference for flexibility by the expectation that "quite possibly *something* will interfere with my travel plans," without having a clear idea about specifically what is likely to interfere.

One may wonder whether a notion of maximizing expected (indirect) utility with respect to anticipated *unforeseen* contingencies makes any sense; after all, by definition, the decision-maker is supposed not to have/use an explicit representation of these, and can therefore neither assign subjective probabilities to them (in her mind), nor—from a more sophisticated revealed-preference perspective—can she be guided by the sure-thing principle applied to acts that are a function of the unforeseen contingencies.[4]

[3] A conditional version is relevant if preferences are allowed to be state-dependent, as in Kreps (1992); see Section 5 for a very brief discussion of this case. If preferences are state-independent (as assumed in this paper), the conditional and unconditional IU properties are equivalent.

[4] A conceptualization of anticipated unforeseen contingencies in terms of a set of implicit decision-relevant states with associated "implicit" subjective probabilities and expected utilities stands in interesting contrast with approaches in which "unforeseenness" is identified with *ignorance* of some kind, as in Ghirardato (1996) and Mukerjee (1997); for recent epistemic work on the related notion of awareness, see Modica-Rustichini (1994), and Dekel-Lipman-Rustichini (1996).

The question is therefore whether intrinsically appealing[5] axioms on preferences over opportunity acts (which are functions of explicit states only) can be formulated that yield an EIU representation with respect to implicit states. We propose an axiom called "Indirect Stochastic Dominance" which captures the notion that having "more opportunities in expectation" is better.[6] Combined with the Savage axioms, this yields the main result of the paper, a representation in which an act f is evaluated according to $\int(\sum_{\omega \in \Omega} \lambda_\omega \max_{x \in f(\theta)} v_\omega(x))\, d\mu$, where μ is the agents' subjective probability measure over the explicit state-space Θ.

In contrast to Kreps' (1979) result for a deterministic (and thus effectively ordinal) setting, the additive representation with respect to implicit states has substance here; indeed, it is not difficult to construct examples in which the decision-maker is uncertainty-averse in the sense of Gilboa-Schmeidler (1989) with respect to implicit states and fails to satisfy ISD. We note that in Kreps (1992), which considers preferences over opportunity acts also, the representation is merely monotone rather than additive in implicit state utilities. In addition to the cited contributions by Kreps, two other especially relevant references are Koopmans (1965) who suggested an EIU representation without axioms, as well as Jones-Ostroy (1984) who relate the value of flexibility to the amount of information to be received.

Besides extending the analysis of preference for flexibility to a stochastic setting, the second main goal of the paper is to introduce a new technical tool, (conjugate) Möbius inversion, which allows full and efficient use of the linear structure of the EIU representation and unifies the mathematical analysis. Beyond playing a key role in the demonstration of the main result, it allows a simpler and more transparent proof of Kreps' (1979) classical theorem. It also yields an explicit characterization of the class of EIU-maximizing utility functions over opportunity sets. In addition, the use of Möbius inversion establishes an interesting connection to the literature on nonadditive probability representations from which it has in fact been imported; in particular, we show in Section 3 that EIU maximization can alternatively be viewed as Choquet integration with respect to a "plausibility function" (conjugate belief function) in the sense of Dempster-Shafer theory.[7]

The remainder of this paper is organized as follows: Section 2 introduces a Savage framework with opportunity acts, presents the key axiom (Indirect Stochastic Dominance), and states the main representation result (Theorem 2). Conjugate Möbius inversion is introduced in Section 3. Theorem 2 is proved, and an alternative representation in terms of Choquet integration is given. Section 4 revisits Kreps (1979) and gives a direct and intuitive proof of his result

[5]In contrast to axioms that characterize EIU-maximization in a mathematical sense, but cannot themselves be understood as primitive conditions.

[6]As such, it is genuinely an axiom of preference for *flexibility* as it excludes Ulyssean motivations for reducing one's future opportunities for the sake of precommitment; the classical reference on the latter is Strotz (1955).

[7]See Dempster (1967) and Shafer (1976).

based on conjugate Möbius inversion. The paper concludes with some remarks on the possibility and interest of allowing for state-dependent preferences, and on the (non-)uniqueness of the representation; the reader is referred to Nehring (1996) for details. All remaining proofs are collected in the Appendix.

2. PREFERENCE FOR FLEXIBILITY IN A SAVAGE FRAMEWORK

Three basic types of *explicit* uncertainty can be distinguished in a context of two-stage decision-making: the agent may be uncertain as to which opportunity set results from a particular present choice (e.g., in Flexia's case, the availability of a ticket if she does not buy one now), the agent may receive information about the comparative value of alternative final choices (e.g., if Flexia is worried about the health of her child, her final decision may depend on his body temperature), and thirdly the final choice itself may be one under uncertainty (e.g., at the time of her final decision, Flexia may still not know whether the child will fall seriously ill). In this paper, we will focus on uncertainty about the future opportunity set. Uncertainty of the second kind, which is associated with state-dependent preferences, is straightforwardly integrated into the analysis (cf. the concluding remarks in Section 5). Uncertainty that is not resolved before the final choice is not explicitly modeled here; doing so promises to be a worthwhile and nontrivial task for future research.

After recalling the Savage framework and axioms (applied to acts that have opportunity sets as consequences), we will introduce a novel axiom capturing "preference for flexibility" and conclude by stating and commenting on the main result of the paper, a representation theorem for Expected Indirect Utility maximizing preferences. Its proof and further analysis will be deferred to the following section, following a presentation of the key technical tool of the analysis, conjugate Möbius inversion.

Some notation and definitions first.

X: the finite set of alternatives.

$\mathscr{A}$: the set of nonempty subsets of X ("opportunity sets").

Θ: the space of explicit states θ.

$\mathscr{F}$: the class of opportunity acts $f: \Theta \to \mathscr{A}$.

$\mathscr{F}^{const}$: the subclass of constant acts, typically denoted by the constant prize.

$[f, E; g, E^c]$: the act h such that, for $\theta \in \Theta$,

$$h_\theta = \begin{cases} f_\theta & \text{if } \theta \in E \\ g_\theta & \text{if } \theta \in E^c \end{cases} \quad (\text{"}f\text{ on }E\text{ and }g\text{ on }E^c\text{"}).$$

$\succcurlyeq$: a preference relation on $\mathscr{F}$.

$f \succcurlyeq_E g$: whenever $[f, E; h, E^c] \succcurlyeq [g, E; h, E^c]$ for some $h \in \mathscr{F}$ ("f is weakly preferred to g *given* the event E").[8]

[8] While $\succcurlyeq_E$ is well-defined formally, only in the presence of Axiom 2, the sure-thing principle, does it have a meaningful interpretation.

E is null if $f \succcurlyeq_E g$ for all $f, g \in \mathscr{F}$.

The following are the first six of Savage's seven axioms.[9]

AXIOM 1 (Weak Order): *$\succcurlyeq$ is transitive and complete.*

AXIOM 2 (Sure-Thing Principle): *For all $f, g, h, h' \in \mathscr{F}, E \subseteq \Theta$: $[f, E; h, E^c] \succcurlyeq [g, E; h, E^c]$ if and only if $[f, E; h', E^c] \succcurlyeq [g, E; h', E^c]$.*

AXIOM 3 (State Independence): *For non-null $E \subseteq \Theta$ and all $f, g \in \mathscr{F}^{const}$: $[f, E; h, E^c] \succcurlyeq [g, E; h, E^c]$ if and only if $f \succcurlyeq g$.*

AXIOM 4 (Comparative Probability): *For all $E, F \subseteq \Theta$ and $f, g, f', g' \in \mathscr{F}^{const}$ such that $f \succ g$ and $f' \succ g'$: $[f, E; g, E^c] \succcurlyeq [f, F; g, F^c]$ if and only if $[f', E; g', E^c] \succcurlyeq [f', F; g', F^c]$.*

AXIOM 5 (Nontriviality): *There exist $f, g \in \mathscr{F}^{const}$: $f \succ g$.*

AXIOM 6 (Archimedean): *For all $f, g \in \mathscr{F}$ such that $f \succ g$ and all $h \in \mathscr{F}^{const}$, there exists a finite partition $\mathscr{H}$ of Θ such that, for all $H \in \mathscr{H}$:*
(i) $[h, H; f, H^c] \succ g$,
(ii) $f \succ [h, H; g, H^c]$.

With μ denoting a finitely additive measure on 2^Θ, define, for any finitely-ranged function $x: \Theta \to \mathbb{R}$,

$$\int x(\theta)\, d\mu = \sum_{\xi \in x(\Theta)} \xi \mu(\{\theta \in \Theta | x(\theta) = \xi\}).$$

THEOREM 1 (Savage): *The preference relation $\succcurlyeq$ on $\mathscr{F}$ satisfies Axioms 1 through 6 if and only if there exists a finitely additive, convex-ranged[10] probability measure $\mu: 2^\Theta \to \mathbb{R}$ and a nonconstant utility function u such that*

$$f \succcurlyeq g \quad \text{if and only if} \quad \int u(f(\theta))\, d\mu \geq \int u(g(\theta))\, d\mu, \qquad \text{for all } f, g \in \mathscr{F}.$$

The key ingredient to the axiom capturing preference for flexibility is an "indirect stochastic dominance relation" which relies on the comparative probability relation revealed by the decision-maker's preferences. Thus, let $\geq$ be the more-likely-than relation on 2^Θ defined by

$$E \geq F \text{ if, for any constant acts } f, g \text{ such that } f \succ g: [f, E; g, E^c] \succcurlyeq [f, F; g, F^c].$$

[9]The last, P7, is not needed, since all opportunity acts are finitely-ranged due to the maintained assumption of a finite domain of alternatives X.

[10]μ is said to be *convex-ranged* if, for all $E \subseteq \Theta$ and all ρ: $0 \leq \rho \leq 1$, there exists $F \subseteq E$ such that $\mu(F) = \rho\mu(E)$.

Note that by Axiom 4, "any" can be replaced by "all" in the definition of $\geq$, and that $E \geq F$ if and only if $\mu(E) \geq \mu(F)$.

DEFINITION 1: The opportunity act f indirectly stochastically dominates g with respect to the weak order R on X ("$f \trianglerighteq_R g$") *if and only if*, for all $x \in X$:

$$\{\theta \in \Theta | f(\theta) \cap \{y \in X | yRx\} \neq \emptyset\} \geq \{|\theta \in \Theta | g(\theta) \cap \{|y \in X | yRx\} \geq \emptyset\}.$$

f indirectly stochastically dominates g ("$f \trianglerighteq g$") if it indirectly stochastically dominates g with respect to every weak order R on X.

The opportunity act f indirectly stochastically dominates g, if, for any *hypothetical* weak order over alternatives R, it is subjectively at least as likely to reach any given level set of R through the opportunity sets resulting from f than through those resulting from g.[11] The Indirect Stochastic Dominance relation restricted to constant acts coincides with set-inclusion; in a stochastic setting, it is however much richer in content.

EXAMPLE 1: Let $X = \{x, y, z\}$, E be any event that is $\geq$-equally likely to its complement (hence with subjective probability $\mu(E) = \frac{1}{2}$), and define opportunity acts $f = [\{x, y\}, E; \{x, z\}, E^c]$ and $g = [\{x\}, E; \{x, y, z\}, E^c]$. Then $f \trianglerighteq g$, but not $g \trianglerighteq f$.

This is easily verified. If x is a best alternative with respect to R, it is available with subjective probability one under f and g, and thus $f \trianglerighteq_R g$ as well as $g \trianglerighteq_R f$. If, on the other hand, x is not a best alternative with respect to R, the[12] R-best alternative is available with probability one half under each. Under f, the at-least-second-best alternative is always available, and thus $f \trianglerighteq_R g$ again. However, if x is worst with respect to R, with probability one half not even the second-best option is available under g, and thus not $g \trianglerighteq_R f$ for such R. It follows that $f \trianglerighteq g$, but not $g \trianglerighteq f$.

AXIOM 7 (ISD): *For all $f, g \in \mathscr{F}$: $f \succsim g$ whenever f indirectly stochastically dominates g.*

ISD can also be expressed purely in preference terms: if any bet on attaining under f any level set of any weak order, i.e., any bet on an event of the form $\{\theta \in \Theta | f(\theta) \cap \{y \in X | yRx\} \neq \emptyset\}$, is preferred to the corresponding bet based on g, then f itself is weakly preferred to g. Note the Savage axioms themselves imply the restriction of ISD to single-valued acts which is ordinary stochastic dominance.

[11] In other words, f indirectly stochastically dominates g if, given any (ordinal) indirect utility function u, the probability distribution of indirect utilities $\mu \circ (u \circ f)^{-1}$ induced by f first-order stochastically dominates (in the ordinary sense) the analogously defined probability distribution $\mu \circ (u \circ g)^{-1}$ induced by g.

[12] Breaking ties arbitrarily throughout.

To capture formally uncertainty about future tastes in the intended representation, let Ω denote a (finite) set of preference-determining contingencies ω with associated utility-function v_ω, and let $\lambda \in \Delta^\Omega$ denote a probability distribution over the *implicit state space* Ω. The preference relation $\succcurlyeq$ is "EIU-rationalizable" if the ex-ante utility of an opportunity set $u(S)$ can be accounted for as an expectation of maximally achievable future utility, allowing for uncertainty of future preferences, i.e., if $u(S)$ can be written as $\sum_{\omega \in \Omega} \lambda_\omega \max_{x \in X} v_\omega(x)$.

DEFINITION 2: The preference relation $\succcurlyeq$ is *EIU-rationalizable* if there exists a finitely additive, convex-ranged probability measure μ, a finite set Ω, a probability-measure λ on Ω and utility-functions $\{v_\omega\}_{\omega \in \Omega}$ such that, for all $f, g \in \mathscr{F}$:

$$f \succcurlyeq g \Leftrightarrow \int \Big(\sum_{\omega \in \Omega} \lambda_\omega \max_{x \in f(\theta)} v_\omega(x) \Big)\, d\mu \geq \int \Big(\sum_{\omega \in \Omega} \lambda_\omega \max_{x \in g(\theta)} v_\omega(x) \Big)\, d\mu.$$

REMARK 1: In order to preserve generality, we have allowed in this definition the implicit state-space Ω to be arbitrary (finite), herein following Kreps (1979). It is debatable whether this is really meaningful; one may want to restrict attention to a canonical space of states that is logically constructed from the data, i.e., ultimately from the universe of alternatives X. A natural candidate for such a canonical state-space is the set of all weak orders on X. However, fixing Ω in this way is not enough to ensure uniqueness of the representation.

REMARK 2: The representation entails that the distributions of future preferences and opportunity sets be subjectively independent; this strong assumption corresponds to the state-independence Axioms 3 and 4 and is substantially relaxed in Nehring (1996).

THEOREM 2: *A nontrivial preference relation* $\succcurlyeq$ *over opportunity acts satisfies Axioms* 1–4, 6 *as well as ISD if and only if it is EIU-rationalizable.*

REMARK 1: The richness of the state-space implied by Axiom 6 and characteristic of the Savage-framework is critical to the validity of the result. The result would cease to hold with additively separable preferences and a finite state space as in Kreps (1992); it is easily verified, for instance, that Theorem 2 would become false if Θ consisted of only one state, for then ISD coincides with monotonicity with respect to set inclusion which is not enough according to Theorem 3 below.

REMARK 2: Viewed as a statement about the induced preference-relation on probability-distributions over opportunity sets $\succcurlyeq^*$,[13] Theorem 2 belongs to a family of decision-theoretic results that obtain an additively separable represen-

[13] $\succcurlyeq^*$ is formally defined in the proof of Theorem 2 in Section 3.

tation by appropriately augmenting the von Neumann-Morgenstern axioms. These include in particular Harsanyi's (1955) utilitarian representation theorem, as well as Anscombe-Aumann's (1963) characterization of SEU maximization. The role of ISD (induced on $\succcurlyeq^*$) is played by a Pareto condition in the former and by an (implicit, see Kreps (1988, p. 107)) "only marginals matter" condition in the latter. The analogy to Harsanyi's theorem is particularly close, in that ISD is a monotonicity condition analogous to the Pareto condition there. Jaffray's (1989) mixture-space approach to belief-functions, by contrast, enhances the von Neumann-Morgenstern axioms in a rather different direction.

REMARK 3: Recent work by Gilboa-Schmeidler (1995) and Marinacci (1996) on Möbius inversion[14] in infinite settings allow generalization of Theorem 2 to infinite sets of alternatives X roughly as follows. Let an opportunity act be a mapping from Θ to the nonempty sets of some algebra of opportunity sets; require ISD of all simple acts, and add Savage's Axiom P7. Then Theorem 2 generalizes, with EIU-rationalizability defined in terms of a probability measure on an appropriately defined measure-space of weak orders.

3. THE SIMPLE ALGEBRA OF EXPECTED INDIRECT UTILITY

In this section, the key technical tool of the analysis, (conjugate) Möbius inversion, is presented. Its relevance is due to the observation (Proposition 1) that the structure of EIU functions is closely related to that of "plausibility functions" (conjugate belief functions) in the literature on nonprobabilistic belief representations, in which Möbius inversion (originally due to Shapley (1953) and, in greater generality, to Rota (1964)) occupies a central place.[15] Conjugate Möbius inversion is employed in this section to characterize the class of utility functions on opportunity sets with an EIU representation, to prove Theorem 2, and to obtain a representation of EIU-rationalizable preferences in terms of a Choquet integral.

Let $\mathscr{A}^* = 2^X \setminus (\emptyset \cup \{X\})$. $\#S$ is the cardinality of a set S, with $\#X = n$, and $\subsetneqq$ denotes the *strict* subset relation. $\mathbf{1}$: $\mathscr{A} \to \mathbb{R}$ is the constant function equal to 1, $\mathbf{1}_{\{S\}}$: $\mathscr{A} \to \mathbb{R}$ is the indicator-function of the singleton $\{S\}$. Functions from $\mathscr{A}$ to $\mathbb{R}$ will often be viewed as vectors in $\mathbb{R}^{\mathscr{A}}$. $\Delta^{\mathscr{A}}$ denotes the probability simplex in $\mathbb{R}^{\mathscr{A}}$.

A function u: $\mathscr{A} \to \mathbb{R}$ is an *indirect utility* (*IU*) *function* if it has the form $u(A) = \max_{x \in A} u(\{x\})$ for all $A \in \mathscr{A}$. A function u: $\mathscr{A} \to \mathbb{R}$ is an *expected indirect utility* (*EIU*) *function* if it is a convex combination of IU-functions: $u(A) = \sum_{\omega \in \Omega} \lambda_\omega v_\omega(A) = \sum_{\omega \in \Omega} \lambda_\omega \max_{x \in A} v_\omega(\{x\})$ for all $A \in \mathscr{A}$, for some finite collection of IU-functions $\{v_\omega\}_{\omega \in \Omega}$ and some set of coefficients $\{\lambda_\omega\}_{\omega \in \Omega}$ such that $\lambda_\omega \geq 0$ for all $\omega \in \Omega$ and $\sum_{\omega \in \Omega} \lambda_\omega = 1$. Thus, preferences over opportunity acts

[14] Cf. Section 3.

[15] The classical references on belief-functions are Dempster (1967) and Shafer (1976); for a recent thorough study of Möbius inversion, the key technical tool, see Chateauneuf-Jaffray (1989).

are EIU-rationalizable if and only if they have a Savage representation in terms of an EIU function u. An IU function is *dichotomous* if it takes the values 0 and 1 only, i.e., if $u(\mathscr{A}) \subseteq \{0,1\}$. It is easily verified that a function $u: \mathscr{A} \to \mathbb{R}$ is a dichotomous IU function if and only if it is *simple*, i.e., if $u = \chi_S$ for some $S \in 2^X$, with $\chi_S: \mathscr{A} \to \mathbb{R}$ defined by

$$\chi_S(A) = \begin{cases} 1 & \text{if } A \cap S \neq \emptyset, \\ 0 & \text{if } A \cap S = \emptyset, \end{cases} \quad \text{for } A \in \mathscr{A}.$$

The following observation characterizes IU and EIU functions as equivalent to certain linear combinations of simple functions.

PROPOSITION 1: (i) *u is an IU-function if and only if $u = \sum_{S \in \mathscr{A}} \lambda_S \chi_S$, for $\lambda \in \mathbb{R}^{\mathscr{A}}$ such that $\lambda_S \geq 0$ for all $S \neq X$, and such that $\lambda_S > 0$ and $\lambda_T > 0$ imply $S \subseteq T$ or $S \supseteq T$.*

(ii) *u is an EIU-function if and only if $u = \sum_{S \in \mathscr{A}} \lambda_S \chi_S$, for $\lambda \in \mathbb{R}^{\mathscr{A}}$ such that $\lambda_S \geq 0$ for all $S \neq X$.*

EXAMPLE 2: Let $X = \{1,2,3\}$, and let u be the IU-function defined by $u(S) = \max_{x \in S} x^2$. Then $u = \chi_{\{1,2,3\}} + 3\chi_{\{2,3\}} + 5\chi_{\{3\}}$.

Mathematically, the key is the observation that the set of simple functions is a linear basis of the space $\mathbb{R}^{\mathscr{A}}$. How simple functions combine (in particular to yield EIU functions) is described by the "*conjugate Möbius operator*" $\Psi: \mathbb{R}^{\mathscr{A}} \to \mathbb{R}^{\mathscr{A}}$ which maps vectors of "weights" $(\lambda_S)_{S \in \mathscr{A}}$ on the simple functions χ_S to set-functions u. Ψ is defined by $\lambda \mapsto u = \sum_{S \in \mathscr{A}} \lambda_S \chi_S$, and thus $u(A) = \Psi(\lambda)(A) = \sum_{S \in \mathscr{A}:\, S \cap A \neq \emptyset} \lambda_S$, for $A \in \mathscr{A}$. Setting $u(\emptyset) = 0$ as a matter of convention, Ψ is conjugate to the standard "*Möbius operator*" Φ: $\mathbb{R}^{\mathscr{A}} \times \{0\} \to \mathbb{R}^{\mathscr{A}} \times \{0\}$: $\lambda \mapsto l$ with $l(A) = \sum_{S \in 2^X:\, S \subseteq A} \lambda_S$.[16] Note that u and l are related by the conjugation relations $l(A) = u(X) - u(A^c)$ and $u(A) = l(X) - l(A^c)$, for $A \in 2^X$; this relationship allows us to translate results on the standard Möbius operator into results on its conjugate in a straightforward manner.

Basic is the following lemma.

LEMMA 1: *The conjugate Möbius operator $\Psi: \mathbb{R}^{\mathscr{A}} \to \mathbb{R}^{\mathscr{A}}$ is a bijective linear map. Its inverse Ψ^{-1} is given by*

$$\Psi^{-1}(u)(A) = \sum_{S \in 2^X:\, S \subseteq A} (-1)^{\#(A \setminus S)+1} u(S^c) \quad \textit{for} \quad A \in \mathscr{A}.$$

EXAMPLE 3: Let $X = \{x, y, z\}$, and let u be an EIU-function. Lemma 1 yields

$$\Psi^{-1}(u)(\{y,z\}) = -u(\{x,y,z\}) + u(\{x,z\}) + u(\{x,y\}) - u(\{x\}).$$

Combined with Proposition 1, the lemma allows a straightforward characterization of EIU-functions in terms of $2^n - 2$ linear inequalities.

[16] $l(A)$ may be interpreted as the utility-loss from lacking the set of alternatives A.

PROPOSITION 2: *u is an EIU function if and only if* $\Psi^{-1}(u)(A) \geq 0$ *for all* $A \in \mathscr{A}^*$.

In Example 3, applying Proposition 2 to the set $A = \{y, z\}$ yields

$$u(\{x,z\}) + u(\{x,y\}) \geq u(\{x,y,z\}) + u(\{x\}).$$

Note that this inequality is the utility-analogue to the preference-implication of ISD in Example 1. Indeed, it is clear from Theorem 2 that the characterizing condition of Proposition 2 must be the utility-analogue to ISD. Drawing on the literature on belief functions, this condition can be made more intelligible by generalizing it to the following, effectively equivalent pair of conditions.

DEFINITION 3: (i) u: $\mathscr{A} \to \mathbb{R}$ is *monotone* if $A \subseteq B$ implies $u(A) \leq u(B)$ $\forall A, B \in \mathscr{A}$.

(ii) u: $\mathscr{A} \to \mathbb{R}$ is *totally submodular* if, for any finite collection $\{A_k\}_{k \in K}$ in $\mathscr{A}$ such that $\bigcap_{k \in K} A_k \neq \emptyset$,

$$u\left(\bigcap_{k \in K} A_k\right) \leq \sum_{J:\, J \subseteq K,\, J \neq \emptyset} (-1)^{\#J+1} u\left(\bigcup_{k \in J} A_k\right).$$

The conjunction of monotonicity and total submodularity differs from "infinite monotonicity" in the sense of Choquet (1953) in two ways: infinite monotonicity would result if in the definition of total submodularity the inequality were reversed and if the nonempty-intersection clause were dropped. The former difference is due to the conjugate definition of the relevant operator, the latter to the absence of the empty set from the domain of u. The following is an adaptation of standard results.

PROPOSITION 3: *u is an EIU function if and only if it is monotone and totally submodular.*

Total submodularity is understood most easily by considering the case of $\#K = 2$, where it specializes to the following standard "submodularity" condition:

$$u(A \cap B) + u(A \cup B) \leq u(A) + u(B) \qquad \forall A, B \in \mathscr{A} \text{ such that } A \cap B \neq \emptyset,$$

or equivalently:

$$u(A' \cup B') - u(A') \geq u(A' \cup B' \cup C') - u(A' \cup C') \qquad \forall A', B', C' \in \mathscr{A}.$$

In this version, submodularity says that the incremental value of adding some set to a given set of alternatives (the set B' to A') never increases as other alternatives (the set C') are added. Submodularity implies that opportunity subsets are *substitutes* in terms of flexibility value.

(Total) submodularity can be translated into a characterization of the risk attitudes towards opportunity acts implied by EIU maximization. Submodularity in particular translates into the following condition of *opportunity risk-aversion* which generalizes Example 1:

For all $A'', B'', C'' \in \mathscr{A}$ such that $A'' \supsetneqq B'' \cup C''$ and $B'' \cap C'' = \emptyset$, and any $f, g \in \mathscr{F}$ such that $\mu \circ f^{-1} = \frac{1}{2}\mathbf{1}_{\{A'' \setminus B''\}} + \frac{1}{2}\mathbf{1}_{\{A'' \setminus C''\}}$ and $\mu \circ g^{-1} = \frac{1}{2}\mathbf{1}_{\{A''\}} + \frac{1}{2}\mathbf{1}_{\{A'' \setminus (B'' \cup C'')\}}$: $f \succcurlyeq g$.

Thus, losing one of the opportunity subsets B'' or C'' for sure (each with equal odds) is weakly preferred to facing a fifty-percent chance of losing both B'' and C''. The higher-order instances of total submodularity may also be given a (rather more tenuous) opportunity risk-aversion interpretation.

We are now in a position to demonstrate the main result, Theorem 2; the two lemmas to which it appeals are proved in the Appendix. A quick overview first. We begin by deriving from the preference relation $\succcurlyeq$ an induced von Neumann-Morgenstern ordering of probability distributions on opportunity sets $\succcurlyeq^*$ on $\Delta^{\mathscr{A}}$. Using conjugate Möbius inversion, $\succcurlyeq^*$ is shown to have an additive representation of the form

$$p \succcurlyeq^* q \Leftrightarrow \sum_{A \in \mathscr{A}^*} \lambda_A p(\{S | S \cap A \neq \emptyset\}) \geq \sum_{A \in \mathscr{A}^*} \lambda_A q(\{S | S \cap A \neq \emptyset\}), \quad \text{for all } p, q \in \Delta^{\mathscr{A}},$$

in which the coefficient vector λ denotes the conjugate Möbius inverse of the utility-function u from the Savage representation of $\succcurlyeq$. According to Proposition 2, u is an EIU-function if and only if λ_A is nonnegative for all $A \neq X$. The latter is exactly what ISD delivers. Here are the details.

PROOF OF THEOREM 2: By Savage's representation theorem, there exists a probability measure μ and a utility function $u: \mathscr{A} \to \mathbb{R}$ such that

$$f \succcurlyeq g \quad \text{if and only if} \quad \int u(f(\theta))\, d\mu \geq \int u(g(\theta))\, d\mu, \quad \text{for all } f, g \in \mathscr{F}.$$

We need to show that u is an EIU-function if and only if $\succcurlyeq$ satisfies ISD.

From the convex-rangedness of μ and the definition of $\mathscr{F}$, it follows that $\mu \circ f^{-1}$ has "full range," i.e., that $\{\mu \circ f^{-1} | f \in \mathscr{F}\} = \Delta^{\mathscr{A}}$. Define an induced preference relation on "opportunity lotteries" $\succcurlyeq^*$ on $\Delta^{\mathscr{A}}$ by setting $\mu \circ f^{-1} \succcurlyeq^* \mu \circ g^{-1}$ if and only if $f \succcurlyeq g$, for all $f, g \in \mathscr{F}$. $\succcurlyeq^*$ has a von Neumann-Morgenstern representation

$$p \succcurlyeq^* q \Leftrightarrow \sum_{S \in \mathscr{A}} p_S u(S) \geq \sum_{S \in \mathscr{A}} q_S u(S), \quad \text{for all } p, q \in \Delta^{\mathscr{A}}.$$

Note first that $p(\{S | S \cap A \neq \emptyset\}) = \sum_{S \in \mathscr{A}:\ S \cap A \neq \emptyset} p_S = \Psi(p)(A)$. The conjugate Möbius operator Ψ thus maps opportunity prospects p to their *characteristic profiles* $\Psi(p)$, establishing a linear isomorphism between $\Delta^{\mathscr{A}}$ and the space of characteristic profiles $\Gamma^{\mathscr{A}} := \Psi(\Delta^{\mathscr{A}})$. The desired proof is given by studying the induced isomorphic preferences $\hat{\succcurlyeq}$ over characteristic profiles $\pi \in \Gamma^{\mathscr{A}}$ defined by

$$\pi \hat{\succcurlyeq} \pi' \Leftrightarrow \Psi^{-1}(\pi) \succcurlyeq^* \Psi^{-1}(\pi').$$

Set $\lambda_A = \Psi^{-1}(u)(A)$ for all $A \in \mathscr{A}$. The expected-utility representation of $\succcurlyeq^*$ translates into an additive representation for the induced preference relation $\hat{\succcurlyeq}$. In it, the coefficients λ_A describe the marginal ex-ante utility of an incremental probability of reaching the set $A \in \mathscr{A}^*$, i.e., of an increase in $p(\{S | S \cap A \neq \varnothing\})$.

LEMMA 2: *$\pi \hat{\succcurlyeq} \pi'$ if and only if $\Sigma_{A \in \mathscr{A}^*} \lambda_A \pi(A) \geq \Sigma_{A \in \mathscr{A}^*} \lambda_A \pi'(A)$, for all $\pi, \pi' \in \Gamma^{\mathscr{A}}$.*

The preference relation $\hat{\succcurlyeq}$ is *monotone* if $\pi \geq \pi' \Rightarrow \pi \hat{\succcurlyeq} \pi'$. The proof is completed by showing that ISD of $\succcurlyeq$ translates into monotonicity of $\hat{\succcurlyeq}$, which is equivalent to nonnegativity of the coefficients λ_A, which in turn is equivalent to u being an EIU-function in view of Proposition 1.

LEMMA 3: (i) *The preference relation $\hat{\succcurlyeq}$ is monotone if and only if $\succcurlyeq$ satisfies ISD.*

(ii) *The preference relation $\hat{\succcurlyeq}$ is monotone if and only if $\lambda_A \geq 0$ for all $A \in \mathscr{A}^*$.* *Q.E.D.*

REMARK 1: Setting $\pi(\varnothing) = 0$ by convention, standard results imply that $\Psi(\Delta^{\mathscr{A}}) \times \{0\} = \{\pi \in \mathbb{R}^{2^X} | \pi$ is monotone, totally submodular on 2^X and $\pi(X) = 1\}$.[17] Thus characteristic profiles can be viewed as "plausibility functions" in the sense of the theory of nonadditive belief-functions (Shafer (1976)); however, on the expected-utility account given so far, a characteristic profile π as defined does not express a nonadditive belief, but rather probabilistic beliefs about events of the form $\{S | S \cap T \neq \varnothing\} \subseteq 2^{2^X}$.

REMARK 2: On the other hand, characteristic profiles can also profitably be viewed as *capacities* with the particular structure of a plausibility function expressing a nonadditive belief about the state-space X; this yields a representation of EIU-maximization in terms of Choquet integration.[18] The following remarks are intended for readers familiar with the latter.

The idea is to redescribe an agent's future *opportunity* of choosing from some set S epistemically as ignorance about the future *choice* from S, an ignorance

[17] $\pi: 2^X \to \mathbf{R}$ is *totally submodular on* 2^X if, for any finite collection $\{A_k\}_{k \in K}$ in 2^X:

$$\pi\Big(\bigcap_{k \in K} A_k\Big) \leq \sum_{J:\, J \subseteq K,\, J \neq \varnothing} (-1)^{\#J+1} \pi\Big(\bigcup_{k \in J} A_k\Big).$$

Note that *on* 2^X the intersection of $\{A_k\}_{k \in K}$ is allowed to be empty.

[18] This interpretation has been suggested by an anonymous referee.

captured nonadditively by the plausibility-function $\chi_S\colon 2^X \to \mathbb{R}$, with $\chi_S(A)$ representing the upper probability that the future choice from S lies in the set A; note that $\chi_S = \Psi(\mathbf{1}_{\{S\}})$. More generally, a probability distribution over opportunity sets p can be associated with the capacity $\Psi(p)$ over future choices. Let Δ^S denote the set of probability measures on X with support in S, and set $\Pi(p) = \Sigma_{S \in \mathscr{A}} p_S \Delta^S \subseteq \Delta^X$,[19] the convex set of choice distributions consistent with the opportunity lottery p. Then the capacity $\Psi(p)$ has an upper-probability representation given by $\Psi(p)(E) = \max\{q(E) | q \in \Pi(p)\}$, for $E \in 2^X$.

For a given future indirect utility function v_ω, the expected indirect utility taken with respect to the *explicit* uncertainty Θ generated by the act f and given by $\int \max_{x \in f(\theta)} v_\omega(\{x\})\, d\mu$ can be written as a Choquet integral $\int_x v_\omega(\{x\})\, d\nu$, with $\nu = \Psi(\mu \circ f^{-1})$, or equivalently as upper expectation $\max\{\Sigma_{x \in X} q(\{x\}) v_\omega(\{x\}) | q \in \Pi(\mu \circ f^{-1})\}$.[20] The atypical use of an upper rather than lower expectation, respectively of a submodular rather than a supermodular (or "convex") capacity, reflects an agent's expectation that future choices will be made in line with his current interests. One can incorporate uncertainty over future preferences by jointly integrating over ω and x with respect to a capacity defined on $\Omega \times X$.[21]

REMARK 3: ISD restricted to singleton-valued acts amounts to ordinary stochastic dominance in preference terms. A stochastic-dominance like axiom,[22] "Cumulative Dominance," has been used by Sarin-Wakker (1992) to axiomatize Choquet expected utility preferences in a Savage setup. Cumulative dominance and ISD do quite different work, though. Cumulative dominance "calibrates" explicitly ambiguous acts in terms of equivalent unambiguous ones and is responsible for the rank-dependent character of the integration, while neutral to the nature of the capacity. By contrast, ISD pertains to the implicit uncertainty of future choice and preference, and in effect singles out a particular type of capacity, namely, a plausibility-function as explained in Remark 2.

4. KREPS (1979) REVISITED

The analysis of Sections 2 and 3 allows us to throw new light on Kreps' (1979) classic analysis of preference for flexibility in a setting without explicit uncertainty; in particular, we will use conjugate Möbius inversion to provide a new and simplified proof of his main result, a characterization of EIU-rationalizable preference orders defined over the class of opportunity *sets*.

Let $\succcurlyeq_0$ be a weak order on $\mathscr{A}$; in the Savage setting of Section 2, $\succcurlyeq_0$ can be identified with the restriction of $\succcurlyeq$ to the class of constant acts.

[19] Using Minkowskian set-addition.

[20] Equivalences of this type are well-known; see, for example, Chateauneuf-Jaffray (1989, p. 275).

[21] Then the overall expected indirect utility is given by $\int v_\omega(\{x\})\, d\bar{\nu}$, with $\bar{\nu}$ defined by

$$\bar{\nu}(E) = \sum_{\omega \in \Omega} \lambda_\omega \cdot \Psi(\mu \circ f^{-1})(\{x \in X | (\omega, x) \in E\}) \quad \text{for} \quad E \subseteq \Omega \times X.$$

[22] See, however, the discussion in Nehring (1994).

Definition 4: (i) $\succcurlyeq_0$ is *monotone* if $A \supseteq B$ implies $A \succcurlyeq_0 B$, for all $A, B \in \mathscr{A}$.
(ii) $\succcurlyeq_0$ is *strictly monotone* if $A \supsetneqq B$ implies $A \succ_0 B$, for all $A, B \in \mathscr{A}$.
(iii) $\succcurlyeq_0$ is *ordinally EIU-rationalizable* if there exists an EIU function u: $\mathscr{A} \to \mathbb{R}$ such that $A \succcurlyeq_0 B$ if and only if $u(A) \geq u(B)$ for all $A, B \in \mathscr{A}$.

In view of Theorem 2, it seems natural to conjecture that ordinal EIU-rationalizability is equivalent to the restriction of indirect stochastic dominance to constant acts which is monotonicity with respect to set inclusion. This conjecture proves to be half-true. On the one hand, it is borne out for the class of ordinal preferences that contain no nontrivial indifferences: any strictly monotone ordering $\succcurlyeq_0$ turns out to be ordinally EIU-rationalizable. Nontrivial indifferences are quite plausible, however; they occur whenever some alternative is preferred to another at every implicit state ω.[23] In this case, ordinal EIU-rationalizability requires additional restrictions captured by the following condition of ordinal submodularity.

Definition 5: The preference relation $\succcurlyeq_0$ is *ordinally submodular* if $A \succcurlyeq_0 A \cup B$ implies $A \cup C \succcurlyeq_0 A \cup B \cup C$ for all $A, B, C \in \mathscr{A}$.

Within a Savage-framework, ordinal submodularity is implied axiomatically by ISD, state-independence and the existence of events equally likely to their complement. The details follow. Take any $A, B \in \mathscr{F}^{const}$ such that $A \succcurlyeq A \cup B$. Let E be any event $\geq$-equally likely to its complement. State independence (Axiom 3) implies that $[A \cup C, E; A, E^c] \succcurlyeq [A \cup C, E; A \cup B, E^c]$; ISD yields $[A \cup C, E; A \cup B, E^c] \succcurlyeq [A \cup B \cup C, E; A, E^c]$, whence by transitivity $[A \cup C, E; A, E^c] \succcurlyeq [A \cup B \cup C, E; A, E^c]$; by state independence again, $A \cup C \succcurlyeq A \cup B \cup C$, as desired. Thus, within a Savage framework, ordinal submodularity is not such a simple condition! Interestingly, but in line with its ordinal character, in its derivation no reference has been made to the sure-thing principle.

Kreps (1979) has shown that monotonicity and ordinal submodularity together are sufficient to ensure EIU-rationalizability.

Theorem 3 (Kreps): *A weak order $\succcurlyeq_0$ is ordinally EIU-rationalizable if and only if it is monotone as well as ordinally submodular.*

To facilitate the discussion, we restate the result as one about ordinal utility functions.

$$\text{(OSM)} \quad u(A) \geq u(A \cup B) \Rightarrow u(A \cup C) \geq u(A \cup B \cup C) \quad \forall A, B, C \in \mathscr{A}.$$

Theorem 4 (Kreps, restated): *For any function u: $\mathscr{A} \to \mathbb{R}$, there exists a strictly increasing transformation τ: $\mathbb{R} \to \mathbb{R}$ such that $\tau \circ u$ is an EIU function if and only if u is monotone and satisfies OSM.*

[23] For example, $\{x, y\} \sim \{x\}$ if and only if $v_\omega(x) \geq v_\omega(y)$ for all ω with positive probability λ_ω.

A comparison of Proposition 3 and Theorem 4 suggests a proof heuristic as follows. The former implies that the *cardinal* content of EIU-maximization is given by monotonicity plus *total* submodularity, whereas the latter asserts that the *ordinal* content of EIU-maximization is given by monotonicity plus *ordinal* submodularity. In other words, to prove Theorem 4, one needs to show that for any monotone and ordinally submodular function $u: \mathscr{A} \to \mathbb{R}$, there exists a strictly increasing transformation $\tau: \mathbb{R} \to \mathbb{R}$ such that $\tau \circ U$ is totally submodular. Note that in view of the essentially cardinal nature of total submodularity, one would expect its ordinal implications to be weak; the restated theorem looks therefore not that surprising after all.

Moreover, total submodularity is a condition with concave flavor: the incremental utility of an opportunity subset decreases weakly as the set to which it is joined becomes larger. This suggests that by defining transformations τ that concavify u "sufficiently strongly," one should be able to transform any ordinally submodular utility function into a totally submodular one. The proof given in the Appendix follows this line of argument by showing that for sufficiently concave transformations of u, the conjugate Möbius inverse of $\tau \circ u$ becomes nonnegative. Two cases need to be considered. For "instances of strict monotonicity," i.e., for any S such that $u(S^c \cup \{x\}) > u(S^c)$ for all $x \in S$, the value of the conjugate Möbius inverse $\Psi^{-1}(\tau \circ u)(S)$ becomes strictly positive for sufficiently concaving τ. On the other hand, for "instances of indifference," i.e., all other S, the value of the conjugate Möbius inverse $\Psi^{-1}(\tau \circ u)(S)$ is zero for any monotone τ and ordinally submodular u.

5. CONCLUDING REMARKS

The analysis of this paper can be generalized to situations in which some of the uncertainty about future preferences (for example due to the foreseen receipt of new information) is incorporated into the *explicit* state space Θ. Since such uncertainty is associated with state-dependent preferences over opportunity acts, the generalization needs to invoke Karni-Schmeidler's (1993) state-dependence extension of Savage's (1954) classical representation theorem; the details have been worked out in Nehring (1996). Note that explicit uncertainty about future preferences leads to violations of the indirect-utility property; hence, in a state-dependent setting, preference for flexibility due to anticipated unforeseen contingencies is appropriately identified with the following *conditional* indirect-utility property, as proposed in Kreps (1992):

For all sets (constant acts) $A, B \in \mathscr{A}$, and all acts $f \in F$: $[A, E; f, E^c] \succcurlyeq [B, E; f, E^c]$ implies $[A, E; f, E^c] \succcurlyeq [A \cup B, E; f, E^c]$.

EIU representations are not unique; the precise extent of the uniqueness achieved is described in Nehring (1996). It is also shown that uniqueness can be obtained if the set of a-priori possible future preference relations is sufficiently restricted: in particular, it suffices that no two different possible future preference relations have any upper-contour set in common. Such situations occur quite naturally with infinite domains X, as for example with the class of EU preferences on a lottery space X of the form $X = \Delta^Y$. In a Savage framework, preferences of this type arise from uncertainty that is not resolved in the second stage.

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APPENDIX: REMAINING PROOFS

PROOF OF PROPOSITION 1: (i) $\Leftarrow$. Consider $u = \Sigma_{S \in \mathscr{A}} \lambda_S v_S$, for $\lambda \in \mathbb{R}^{\mathscr{A}}$ such that $\lambda_S \geq 0$ for $S \neq X$, and such that $\lambda_S > 0$ and $\lambda_T > 0$ imply $S \subseteq T$ or $S \supseteq T$. Define $\Lambda = \{S \in \mathscr{A} | \lambda_S > 0 \text{ or } S = X\}$. Then $u(\{x\}) = \Sigma_{S \in \Lambda:\ S \ni x} \lambda_S$, for all $x \in X$, and $u(A) = \Sigma_{S \in \Lambda:\ S \cap A \neq \emptyset} \lambda_S = u(\{y\})$ for any $y \in \cap \{S \cap A | S \in \Lambda, S \cap A \neq \emptyset\}$; such y exist by the assumed ordering property of Λ. Since clearly $u(A) \geq u(\{z\})$ for all $z \in A$, $u(A) = \max_{x \in A} u(\{x\})$; u is thus an IU-function.

(i) $\Rightarrow$. If u is an IU-function, let $\{x_k\}_{k=1,\ldots,n}$ be an enumeration of X such that $u(\{x_k\}) \geq u(\{x_{k+1}\})$ for $k = 1, \ldots, n-1$. Then

$$w = \sum_{k=1}^{n-1} (u(\{x_k\}) - u(\{x_{k+1}\}))\chi_{\{x_j | j \leq k\}} + u(\{x_n\})\chi_X$$

denotes a function of the desired form. By the first part of the proof, w is an IU-function. To show its equality to u, it thus suffices to show equality for singleton-sets, as follows: $w(\{x_l\}) = \Sigma_{k=l}^{n-1}(u(\{x_k\}) - u(\{x_{k+1}\})) + u(\{x_n\}) = u(\{x_l\})$.

(ii) $\Leftrightarrow$. Immediate from the above implications. *Q.E.D.*

PROOFS OF LEMMA 1 AND PROPOSITION 3: The proofs can be obtained from straightforward adaptations of standard results[24] and are therefore omitted. Details can be found in Nehring (1996).

PROOF OF LEMMA 2: Setting $p = \Psi^{-1}(\pi)$ and noting that $\pi(A) = p(\{S | S \cap A \neq \emptyset\}) = \Sigma_{S \in \mathscr{A}} p_S \chi_A(S)$ for all $A \in \mathscr{A}$, the claim follows from the following series of equalities:

$$\sum_{A \in \mathscr{A}^*} \lambda_A \pi(A) = \sum_{A \in \mathscr{A}^*} \lambda_A \left(\sum_{S \in \mathscr{A}} p_S \chi_A(S) \right) = \sum_{S \in \mathscr{A}} p_S \left(\sum_{A \in \mathscr{A}^*} \lambda_A \chi_A(S) \right)$$

$$= \sum_{S \in \mathscr{A}} p_S u(S). \qquad \textit{Q.E.D.}$$

PROOF OF LEMMA 3: (i) *Only if*: Consider π and π' such that $\pi \geq \pi'$. By the full-rangedness of $\mu \circ f^{-1}$, there exist acts f and g such that $\Psi(\mu \circ f^{-1}) = \pi$ and $\Psi(\mu \circ g^{-1}) = \pi'$. Noting that $f \trianglerighteq_R g$ is equivalent by definition to $\Psi(\mu \circ f^{-1})(A) \geq \Psi(\mu \circ g^{-1})(A)$ for all $\hat{A}$ of the form $\{x | xRy\}$ (for some $y \in X$), this implies $f \trianglerighteq g$. By ISD, $f \succcurlyeq g$, which translates into $\pi \mathrel{\hat{\succcurlyeq}} \pi'$ via $\mu \circ f^{-1} \succcurlyeq^* \mu \circ g^{-1}$.

[24] See, for example, Chateauneuf-Jaffray (1989).

The converse is straightforward.

(ii) *Only if*: Define $\overline{\pi} \in \Gamma^{\mathscr{A}}$ by $\overline{\pi}(T) = \#\{S \in \mathscr{A} | S \cap T \neq \varnothing\}/\#\{S \in \mathscr{A}\}$, for $T \in \mathscr{A}$. Clearly,

$$\Psi^{-1}(\overline{\pi}) = \frac{1}{2^n - 1} \cdot \mathbf{1} \gg 0. \tag{1}$$

To ascertain that $\overline{\pi}$ is in the interior of $\Gamma^{\mathscr{A}}$, an auxiliary equation (2) is needed. By Lemma 1, for any $A, T \in \mathscr{A}$:

$$\Psi^{-1}(\mathbf{1}_{\{A\}})(T) = \begin{cases} (-1)^{\#(T \setminus A^c)+1} & \text{if } A^c \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

This yields

$$\sum_{T \in \mathscr{A}} \Psi^{-1}(\mathbf{1}_{\{A\}})(T) = \sum_{T \in \mathscr{A}:\, A^c \subseteq T} (-1)^{\#(T \setminus A^c)+1} = 0; \tag{2}$$

the latter equality is standard (cf. Chateauneuf-Jaffray (1989, p. 281)).

By the continuity of Ψ^{-1}, (1) and (2) imply, for any $A \in \mathscr{A}^*$ and small enough $\varepsilon > 0$, $\Psi^{-1}(\overline{\pi} + \varepsilon \mathbf{1}_{\{A\}}) \in \Delta^{\mathscr{A}}$, i.e., $\overline{\pi} + \varepsilon \mathbf{1}_{\{A\}} \in \Gamma^{\mathscr{A}}$. By the monotonicity of $\hat{\succcurlyeq}$ and Lemma 2, $\lambda_A \geq 0$.

The converse is immediate, noting that $\Psi(p)(X) = 1$ for all $p \in \Delta^{\mathscr{A}}$. *Q.E.D.*

Proof of Theorem 4: Necessity is straightforward.

For sufficiency, assume w.l.o.g. that $u(X) = 0$, and hence that $u(S) \leq 0$ for all $S \in \mathscr{A}$ by monotonicity. Let $u_m: \mathscr{A} \to \mathbb{R}_-$ be defined by $u_m(S) = -(-u(S))^m$. Let λ^m denote its conjugate Möbius inverse $\lambda^m = \Psi^{-1}(u_m)$; note that $-u_m(S) = \Sigma_{A \subseteq S^c} \lambda_A^m$.

We want to prove that, for some sufficiently large m, u_m is an EIU-function. By Proposition 1(ii), it thus needs to be shown that for some sufficiently large m: $\lambda_S^m \geq 0$ for all $S \neq X$. Since X is finite, it suffices to demonstrate that for any given $S \neq X$, $\lambda_S^m \geq 0$ for all sufficiently large m. Take $S \neq X$.

Case 1: For some $x \in S$: $u(S^c \cup \{x\}) = u(S^c)$.

We will show that $\lambda_S^m = 0$. Note that this result is straightforward if u is already known to be an EIU function; however, we only know that u is ordinally submodular.

In Case 1, $u_m(S^c \cup \{x\}) = u_m(S^c)$; since, moreover, u_m satisfies ordinal submodularity because u does, it follows that $u_m(T^c \cup \{x\}) = u_m(T^c)$ for all T such that $x \in T \subseteq S$. We will show by induction on the size of T that $\lambda_T^m = 0$ for all T such that $x \in T \subseteq S$, hence in particular that $\lambda_S^m = 0$.

This claim evidently holds for $T = \{x\}$. Suppose, then, that it holds for all A such that $x \in A \subsetneqq T \subseteq S$, which implies $\Sigma_{A:\, x \in A \subsetneqq T} \lambda_A^m = 0$. From this one obtains

$$\lambda_T^m = \sum_{A:\, x \in A \subseteq T} \lambda_A^m = u_m(T^c \cup \{x\}) - u_m(T^c) = 0. \qquad Q.E.D.$$

Case 2: For all $x \in S$: $u(S^c \cup \{x\}) > u(S^c)$.

We will show that $\lim_{m \to \infty} \lambda_S^m / -u_m(S^c) = 1$, hence that $\lambda_S^m > 0$ for sufficiently large m.

From the result for Case 1, by the monotonicity of u_m and the fact that $u_m(X) = 0$, we know that $\lambda_T^m = 0$ whenever $u_m(T^c) = 0$.

From $\lambda_S^m = -u_m(S^c) - \Sigma_{T \subsetneqq S} \lambda_T^m = -u_m(S^c) - \Sigma_{T \subsetneqq S:\, u_m(T^c) \neq 0} \lambda_T^m$, one obtains

$$\frac{\lambda_S^m}{-u_m(S^c)} = 1 - \sum_{T \subsetneqq S:\, u_m(T^c) \neq 0} \frac{\lambda_T^m}{-u_m(T^c)} \frac{-u_m(T^c)}{-u_m(S^c)}. \tag{3}$$

From the definition of λ^m as the conjugate Möbius inverse of u_m, it follows that

$$\frac{\lambda_T^m}{-u_m(T^c)} \leq \frac{\sum_{A \subseteq T} |u_m(A^C)|}{-u_m(T^c)} \leq 2^{\#T}$$

in view of the monotonicity and nonpositivity of u_m.

The claim follows thus from (3), since by the definition of u_m, the condition "for all $x \in S$: $u(S^c \cup \{x\}) > u(S^c)$" implies $(u_m(T^c)/u_m(S^c)) \to 0$ as $m \to \infty$, for all T such that $T \subsetneq S$. *Q.E.D.*

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[Footnotes]

[6] **Myopia and Inconsistency in Dynamic Utility Maximization**
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