On the Structure of Strategy-Proof Social Choice*

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*This research started when one of us attended Salvador Barberá's lucid survey talk on strategy-proof social choice at the SCW conference 2000 in Alicante. Our intellectual debt to his work is apparent throughout. The paper is based on material from the unpublished manuscript Nehring and Puppe (2002).

Abstract *We define a general notion of single-peaked preferences based on abstract betweenness relations. Special cases are the classical examples of single-peaked preferences on a line, the separable preferences on the hypercube, the "multi-dimensionally single-peaked" preferences on the product of lines, but also the unrestricted preference domain. Generalizing and unifying the existing literature, we show that a social choice function is strategy-proof on a sufficiently rich domain of generalized single-peaked preferences if and only if it takes the form of "voting by committees" satisfying a simple condition called the "Intersection Property."*

Based on the Intersection Property, we show that the class of preference domains associated with "median spaces" gives rise to the strongest possibility results, thereby unifying and generalizing many existing possibility results in the literature. A space is a median space if, for any triple of elements, there is a fourth element that is between any pair of the triple; numerous examples are given, and the structure of median spaces and the associated preference domains is analyzed.

1 Introduction

By the Gibbard-Satterthwaite impossibility theorem non-degenerate social choice functions can be strategy-proof only on restricted domains. In response to this fundamental result, a large literature has taken up the challenge of determining domains on which possibility results emerge. In economic environments in which it is assumed that individuals care only about certain aspects of social alternatives, the well-known class of Groves mechanisms offers a rich array of strategy-proof social choice functions under the additional assumption of quasi-linear utility. By contrast, in contexts of "pure" social choice ("voting") individuals care about all aspects of the social state. Here, a path-breaking paper by Moulin (1980) demonstrated the existence of a large class of strategy-proof social choice functions in the Hotelling-Downs model in which social states can be ordered from left to right as in a line, and in which preferences are single-peaked with respect to that ordering. Moulin showed that all strategy-proof social choice functions can be understood as generalizations of the classical median voter rule. His result inspired a sizeable literature that obtained related characterizations for other particular domains or proved impossibility results (see, among others, Border and Jordan (1983) and Barberá, Sonnenschein and Zhou (1991)). Remarkably, it turned out that when a positive result could be obtained, the class of strategy-proof social choice functions had a structure similar to that uncovered by Moulin which we shall refer to as "voting by committees," following Barberá, Sonnenschein and Zhou (1991).

In this paper, we introduce a very large class of preference domains, referred to as "generalized single-peaked" domains, and show that strategy-proof social choice can be characterized in terms of voting by committees on these domains. This allows us then to determine exactly which domains admit strategy-proof social choice functions exhibiting fundamental additional properties such as non-dicatorship, anonymity, neutrality, and efficiency. While much of this work is left to companion papers (see Nehring and Puppe (2003a) and (2003b)), we shall identify here the class of domains on which the strongest possibility result obtains; these are characterized geometrically as "median spaces" and described in more detail below.

Generalized Single-peaked Domains

The basic idea underlying our approach is to describe the space of alternatives geometrically in terms of a three-place *betweenness* relation, and to consider associated domains of preferences that are *single-peaked* in the sense that individuals always prefer social states that are between a given state and their most preferred state, the "peak".

Following Nehring (1999), we shall conceptualize betweenness more specifically in terms of the differential possession of *relevant properties*: a social state y is between the social states x and z if y shares all relevant properties common to x and z. Single-peakedness means that a state y is preferred to a state z whenever y is between z and the peak x^*, i.e. whenever y shares all properties with the peak x^* that z shares with it (and possibly others as well). Throughout, it will be assumed that a property is relevant if and only if its negation is relevant. As further illustrated below, a great variety of preference domains that arise naturally in applications can be described as single-peaked domains with respect to such betweenness relations. In fact, to our knowledge almost all domains that have been shown to enable non-degenerate strategy-proof social choice in a voting context are generalized single-peaked domains in the sense. For instance, the standard betweenness relation in case of a line is derived from properties of the form "to the right (resp. left) of a given state." But single-peaked domains can also easily give rise to impossibility results. For instance, the unrestricted domain envisaged by the Gibbard-Satterthwaite theorem can be described as the set of all single-peaked preferences with respect to a vacuous betweenness relation that declares no social state between any two other states; the corresponding relevant properties are, for any social state x, "being equal to x," and "being different from x."

The Structure of Strategy-Proof Social Choice

Building on previous work culminating in Barberá, Massò and Neme (1997), our first result, Theorem 1, shows that strategy-proof social choice on generalized single-peaked domains can be described in a unified manner as "voting by committees." This structure has two aspects. First, the social choice depends on individuals' preferences through their most preferred alternative only, i.e. it satisfies "peaks only." Second, the social choice is determined by a separate "vote" on each property: an individual is construed as voting for a property over its negation if and only if her top-ranked alternative has the property. For example, in the special case in which voting by committees is anonymous and neutral it takes the form of "majority voting on properties"; that is, a chosen state has a particular property if and only if the majority of agents' peaks have that property.

Crucially, the voting by committees structure describes only an implication of strategy-proofness, not a characterization, since it does not by itself allow one to generate well-defined social choice functions. For without restrictions on the family of properties deemed relevant and/or the structure of committees, the properties chosen by the various committees may well be mutually incompatible. Consider, for example, majority voting on properties on a domain of three states, and take as relevant the six properties of being equal to or different from any particular of these states, corresponding to the unrestricted domain of preferences. If there are three agents with distinct peaks, a majority of agents votes for each property of the form "is different from state x." Since no social state is different from *all* social states (including itself), the social choice is therefore empty. A committee structure is called *consistent* if the properties chosen by each committee are always jointly realizable (irrespective of voters' preferences). We show that a committee structure is consistent if and only if it satisfies a simple condition, called the "Intersection Property." This leads to a unifying characterization of the class of all strategy-proof social choice functions on any generalized single-peaked domain, namely as voting by committees satisfying the Intersection Property (see Theorem 2 below). For any particular single-peaked domain, it allows one to describe the subclass of *anonymous* strategy-proof social choice functions in terms of a system of linear inequalities.

Median Spaces as Distinguished Domains

The restrictions imposed by the Intersection Property on the admissible committees reflect the structure of the underlying space. The Intersection Property thereby provides the crucial tool for determining on which single-peaked domains there exist *well-behaved* strategy-proof social choice functions, but it does not answer this question by itself.

This is the central concern of the two companion papers Nehring and Puppe (2003a) and (2003b). Here, we seek to determine those domains admitting a maximally rich class of strategy-proof social choice functions. Specifically, we ask which betweenness relations ensure consistency of *any* well-defined committee structure, or, in other words: when does the Intersection Property hold trivially? In this case, we shall say that voting by committees is *universally consistent*. We show that voting by committees is universally consistent if and only if the betweenness relation has the property that, for any three distinct states, there exists a state between any pair of them. Such a state is called a *median* of the triple, and the resulting space a *median space*. Universal consistency implies (and turns out to be equivalent to) the existence of anonymous, neutral and strategy-proof social choice functions amounting to "majority voting on properties." In the case of three agents, for example, majority voting on properties boils down to choosing the median of the agents' peaks. The median can be viewed as a natural compromise between the voters' preferences, since, by the single-peakedness, every voter ranks the median above the other two agents' peaks; thus, the median wins a majority vote against any voter's peak in pairwise comparison.

Median spaces represent the natural generalization and unification of the known cases in which well-behaved strategy-proof social choice functions have been shown to exist, the line and its multi-dimensional extensions on the one hand, and trees on the other; see Border and Jordan (1983) as well as Barberá, Gul and Stacchetti (1993) for the former, and Demange (1982) for the latter. In Nehring and Puppe (2003b) we show that efficiency presupposes an underlying median space structure, unless the social choice is dictatorial; thus, from this point of view as well, median spaces are central.

Since median spaces turn out to play such a distinguished role, one would like to characterize their associated preference domains directly, not merely indirectly via the associated betweenness geometry. To this behalf, we show that the class of single-peaked preferences on a median space can be described in terms of two fundamental types of preference restrictions, convexity and separability. In the special case of the line (or, more generally, in trees), the single-peaked preferences are simply the convex ones; in the case of the hypercube considered in Barberá, Sonnenschein and Zhou (1991), the single-peaked preferences are those that are separable. These are the two pure cases; in general, single-peaked preferences on a median space are characterized by a combination of convexity and separability restrictions.

Relation to the Literature

This paper was inspired by the remarkable paper Barberá, Massò and Neme (1997) which demonstrated that strategy-proof social choice functions can be characterized in terms of "voting by committees" much more generally than thought previously. These authors looked at the domain of all single-peaked preferences defined on a fixed product of lines, and considered subdomains of preferences by restricting the peaks to lie in arbitrary prespecified subsets interpreted as "feasible sets." By contrast, in this paper we assume an arbitrary fixed set of social states and consider a wide range of different preference domains over that set. This fixed set is understood to reflect all feasibility constraints that may be relevant. Our central assumption is that the "betweenness geometry" implicit in the domain can be described in terms of an abstract "property space." Sometimes these properties can be understood as Lancasterian characteristics, but at other times they are merely useful mathematical constructs.

Since states in a property space can typically be viewed as appropriately positioned points in a sufficiently high-dimensional hypercube, there is a close mathematical relationship between the setup of Barberá, Massò and Neme (1997) and ours. Indeed, for a subclass of the preference domains considered here, a crucial step in our first main result, the "peaks-only" property, could be derived from their corresponding result using an extension argument.[1] Here, we show that the "peaks-only" property applies more generally; for the precise relation of our work to theirs, see Appendix 1.

Barberá, Massò and Neme (1997) also provided a characterization of consistency in terms of a condition they called "intersection property" as well. Their condition is less transparent and workable than the one obtained here; for instance, in the anonymous case of "voting by quota," our condition directly translates into a system of linear inequalities, representing appropriate bounds on the quotas (see Section 3.3 below).

While median spaces are a well-known and well-studied object in abstract convexity theory (see e.g. van de Vel (1993)), they do not appear to have been considered anywhere in the strategy-proofness literature. Implicitly, however, the properties of median spaces play a central role in Barberá, Sonnenschein and Zhou (1991) and Barberá, Gul and Stacchetti (1993).

The remainder of the paper is organized as follows. Section 2 describes the preference domains to which our characterization results apply. In particular, it introduces the central concepts of single-peaked preference orderings with respect to general betweenness relations, and of betweenness relations derived from property spaces.

In Section 3, we use these concepts to provide a generalization and unification of the existing literature, including the main results of Moulin (1980), Barberá, Sonnenschein and Zhou (1991), Barberá, Gul and Stacchetti (1993) and Barberá, Massò and Neme (1997). Specifically, we show that any strategy-proof social choice function on a sufficiently rich domain of single-peaked preferences satisfying a weak condition of "voter sovereignty" must be voting by committees, i.e. in our framework: "voting by properties" (Theorem 1). We then derive a simple necessary and sufficient condition for the consistency of committee structures, the "Intersection Property." We thus obtain a unifying characterization of strategy-proof social choice on generalized single-peaked domains, namely as voting by committees satisfying the Intersection Property (Theorem 2).

Section 4 introduces the notion of a median space. We show that voting by committees is universally consistent if and only if the underlying domain of social states is a median space (Theorem 3). Median spaces thus give rise to the possibility of strategy-proof social choice in the strong sense that *any* well-defined voting by committees rule is consistent. Section 5 concludes, and all proofs are collected in Appendix 2.

2 Generalized Single-Peaked Domains

In this section, we describe the preference domains to which our later characterization of strategy-proof social choice functions applies. Throughout, we assume that the relevant preference restrictions are independent and identical across voters, so that the domains are n-fold Cartesian products of one common set of individually admissible preferences where n is the number of voters. The individual domains, in turn, can be

[1] This was done in an earlier version of this paper, see Nehring and Puppe (2002).

described as sufficiently rich sets of orderings that are "single-peaked" with respect to an appropriately defined betweenness relation. For expository convenience, we consider only the case of *linear* orderings here; the more general case of weak orderings and even partial orders is treated in the working paper version Nehring and Puppe (2002).

2.1 Single-Peakedness with Respect to General Beetweenness Relations

The classical example of a preference domain admitting non-dictatorial and strategy-proof social choice is the domain of all single-peaked preferences on a line. Suppose that the social alternatives are ordered from left to right as in Fig. 1a below. A preference ordering $\succ$ with top element x^* is single-peaked if $y \succ z$ whenever y is *between* z and the peak x^*. Here, the relevant notion of "betweenness" is of course the standard one corresponding to the left-to-right scale of the line. The aim of this paper is to study the structure of strategy-proof social choice on domains of preferences that are "single-peaked" with respect to more general betweenness relations. Formally, we will consider a ternary relation T on a finite universe X of social states or social alternatives with the interpretation that $(x, y, z) \in T$ if the social state y is **between** the social states x and z. By convention, let $(x, x, z) \in T$ and $(x, z, z) \in T$ for all x, z, i.e. any state is (weakly) between itself and any other state. The "betweenness" terminology will be justified in the sequel by the requirement of further axiomatic properties on the ternary relation.

Definition (Generalized Single-Peakedness) A preference ordering $\succ$ on X is *single-peaked with respect to* T if there exists $x^* \in X$ such that for all $y \neq z$,

$$(x^*, y, z) \in T \Rightarrow y \succ z. \tag{2.1}$$

Thus, in analogy to the standard definition, a preference is generalized single-peaked if any state y that is "T-between" the peak x^* and another state z is preferred to that state. The set of all linear orderings on X that are single-peaked with respect to T will be denoted by $\hat{\mathcal{S}}_{X,T}$.

As a first illustration, consider the three graphs in Figure 1 below with the nodes representing social states. To each graph one can associate the corresponding *graphic betweenness* according to which a social state y is between the two states x and z if y lies on some shortest path[2] connecting x and z. For instance, both y and y' are between x and z in Figures 1a and 1b, while w is not between x and z in Figures 1b and 1c. The graphic betweenness associated with the line in Fig. 1a is of course the standard betweenness and the corresponding notion of single-peakedness is the usual one. The graph in Fig. 1b can be viewed as the (3-dimensional) "hypercube" corresponding to the set $\{0,1\}^3$ of binary sequences of length 3. A preference is single-peaked with respect to the graphic betweenness on a hypercube if and only if it is *separable* in the sense of Barberá, Sonnenschein and Zhou (1991). Finally, the graph in Fig. 1c is the complete graph in which each state is connected to every other state by an edge. By consequence, the corresponding graphic betweenness is *vacuous* in the sense that no state is between any two other states. Clearly, *any* linear preference ordering is single-peaked with respect to this vacuous betweenness relation; therefore, the set of

[2] i.e. a path with a minimal number of edges

all generalized single-peaked preferences is the unrestricted preference domain in this case.

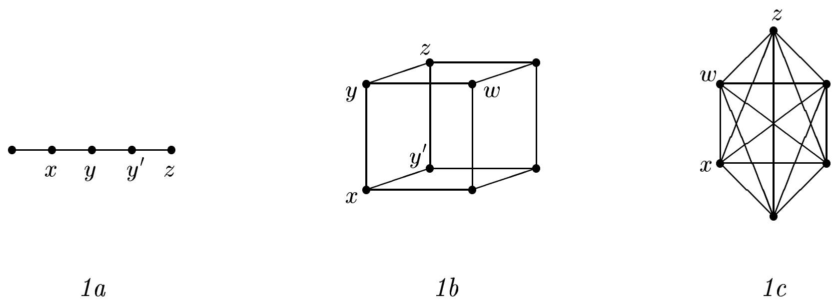


Figure 1: Three graphic betweenness relations

2.2 The Betweenness Structure of a Preference Domain

Obviously, any given preference ordering $\succ$ is single-peaked with respect to *some* appropriate betweenness relation (for instance, with respect to the betweenness relation according to which a state is between two other states if and only if it is intermediate in terms of the preference ordering $\succ$). The essence of the domain restrictions to be described here is of course that all voters' preferences be single-peaked with respect to *the same* betweenness relation. This raises the question of which preference domains can be described as generalized single-peaked domains, and if so, with respect to which betweenness relations.

Thus, we will now take preference domains as the primitive, and use betweenness relations in order to describe their relevant structure. Specifically, consider an arbitrary collection $\mathcal{D}$ of linear preference orderings on X, and define a ternary relation $T_{\mathcal{D}}$ as follows. For all x, y, z with $y \neq z$,

$$(x, y, z) \in T_{\mathcal{D}} :\Leftrightarrow [\ y \succ z \text{ for all } \succ \in \mathcal{D} \text{ with peak } x]. \qquad (2.2)$$

Also, by convention, $(x, z, z) \in T_{\mathcal{D}}$ for all x, z. By construction, one has $\mathcal{D} \subseteq \hat{\mathcal{S}}_{X,T_{\mathcal{D}}}$, i.e. any preference in $\mathcal{D}$ is single-peaked with respect to $T_{\mathcal{D}}$. In fact, $T_{\mathcal{D}}$ is easily seen to be the largest ternary relation with that property. On the other hand, note that $\mathcal{D}$ will in general not include *all* single-peaked orderings with respect to $T_{\mathcal{D}}$.

For instance, the betweenness relation associated with the unrestricted domain via (2.2) is the vacuous betweenness according to which no state is between any two other states. In fact, this is the betweenness associated with any domain such that, for every pair x, y, there is a preference with x as top element and y as second best element. As another example, consider the domain of all separable preferences on the hypercube (Barberá, Sonnenschein and Zhou (1991)). As is easily verified, the associated betweenness according to (2.2) is exactly the graphic betweenness of the hypercube described above.

Any ternary relation T on X can be viewed as a collection $\{T^x : x \in X\}$, where $yT^xz :\Leftrightarrow (x, y, z) \in T$. The binary relations T^x are sometimes referred to as the *base-point relations* associated with T (cf. van de Vel (1993, p.91)). For $T_{\mathcal{D}}$ as defined in (2.2), each base-point relation $T^x_{\mathcal{D}}$ is reflexive and transitive; moreover, if x is the peak of some preference ordering in $\mathcal{D}$, $T^x_{\mathcal{D}}$ is also antisymmetric, i.e. a partial order.

In order for the relevant qualitative structure of a preference domain $\mathcal{D}$ to be fully described by the associated betweenness relation $T_{\mathcal{D}}$, we need to make two types of assumptions described in the following two subsections: that the domain be rich relative to its betweenness relation, and that its betweenness relation have appropriate geometric structure.

2.3 Rich Domains

In the following, we will impose two "richness" conditions on domains of single-peaked preferences. Say that x and y are *neighbours* if $(x, w, y) \in T$ implies $[w = x \text{ or } w = y]$, i.e. x and y are neighbours if no other point is between them.

R1 For all neighbours x, y there is $\succ \in \mathcal{D}$ such that for all $w \in X \setminus \{x, y\}$, $x \succ y \succ w$.
R2 For all x, y, z with $(x, y, z) \notin T$, there is $\succ \in \mathcal{D}$ with peak x such that $z \succ y$.

Condition R1 requires that, for any pair of neighbours, there is a preference ordering that has one of them as peak and the other as the second best element. Condition R2 states that, for each triple x, y, z such that y is not between x and z, there is a preference with peak x that ranks z above y. Observe that, by definition, any domain $\mathcal{D}$ satisfies R2 with respect to the associated $T_{\mathcal{D}}$.

In the following, we will say that a domain $\mathcal{D} \subseteq \hat{\mathcal{S}}_{X,T}$ of single-peaked preferences is **rich** if it satisfies R1 and R2. Note, in particular, that a rich domain includes for each x at least one preference ordering with peak x. If all base-point relations T^x are partial orders, then the set $\hat{\mathcal{S}}_{X,T}$ of *all* single-peaked preference orderings with respect to T is rich. Moreover, for any T such that all base-point relations T^x are partial orders and for any $\mathcal{D} \subseteq \hat{\mathcal{S}}_{X,T}$ satisfying R2 with respect to T, one has $T = T_{\mathcal{D}}$.

2.4 Betweenness Relations Derived from Property Spaces

The requirement that the base-point relations be partial orders lends useful mathematical structure to the analysis, but not quite enough for the purpose of characterizing the class of all strategy-proof social choice functions on generalized single-peaked domains. Throughout, we will rely on the additional assumption that the betweenness relation can be derived from a "property space," as follows.

Suppose that the elements of X are distinguished by different *basic properties*. Formally, let these properties be described by a non-empty family $\mathcal{H} \subseteq 2^X$ of subsets of X where each $H \in \mathcal{H}$ corresponds to a property possessed by all alternatives in $H \subseteq X$ but by no alternative in the complement $H^c := X \setminus H$. The basic properties are thus identified *extensionally*: for instance, the basic property "the tax rate on labour income is 10% or less" is identified with the *set* of all social states in which the tax rate satisfies the required condition. We assume that the list $\mathcal{H}$ of basic properties satisfies the following three conditions.

H1 (Non-Triviality) $H \in \mathcal{H} \Rightarrow H \neq \emptyset$.
H2 (Closedness under Negation) $H \in \mathcal{H} \Rightarrow H^c \in \mathcal{H}$.
H3 (Separation) for all $x \neq y$ there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \notin H$.

Condition H1 says that any basic property is possessed by some element in X. Condition H2 asserts that for any basic property corresponding to H there is also the complementary property possessed by all alternatives not in H. We will refer to a pair (H, H^c) as an *issue*. Finally, condition H3 says that any two distinct elements are distinguished by at least one basic property. A pair $(X, \mathcal{H})$ satisfying H1-H3 will be called a **property space**.

Following Nehring (1999), a property space $(X, \mathcal{H})$ gives rise to a natural betweenness relation $T_{\mathcal{H}}$ as follows. For all x, y, z,

$$(x, y, z) \in T_{\mathcal{H}} :\Leftrightarrow [\text{ for all } H \in \mathcal{H} : \{x, z\} \subseteq H \Rightarrow y \in H]. \tag{2.3}$$

Thus, y is between x and z in the sense of $T_{\mathcal{H}}$ if y possesses all basic properties that are common to x and z (and possible some more). The ternary betweenness relation $T_{\mathcal{H}}$ induced by $(X, \mathcal{H})$ satisfies the following four conditions. For all x, y, z, x', z',

T1 (Reflexivity) $y \in \{x, z\} \Rightarrow (x, y, z) \in T$.
T2 (Symmetry) $(x, y, z) \in T \Leftrightarrow (z, y, x) \in T$.
T3 (Transitivity) $[(x, x', z) \in T$ and $(x, z', z) \in T$ and $(x', y, z') \in T] \Rightarrow (x, y, z) \in T$.
T4 (Antisymmetry) $[(x, y, z) \in T$ and $(x, z, y) \in T] \Rightarrow y = z$.

The reflexivity condition T1 and the symmetry condition T2 follow at once from the definition of $T_{\mathcal{H}}$. Note that it is the symmetry condition that justifies a geometric interpretation of T as "betweenness" relation. The transitivity condition T3 is also easily verified; it states that if both x' and z' are between x and z, and moreover y is between x' and z', then y must also be between x and z. Note that transitivity in this sense strengthens the requirement that all base-point relations be transitive, which corresponds to taking $x = x'$ in condition T3. Finally, the antisymmetry condition T4 is due to the separation property H3. Probably the biggest deal in terms of the entailed restrictions on the underlying preference domain is the symmetry condition that gives the betweenness relation its geometric flavor. Consider, for example, any domain consisting of exactly one linear ordering for each $x \in X$. While such a domain is always rich (relative to $T_{\mathcal{D}}$), one can show that its associated betweenness can never be symmetric.

Are the conditions T1-T4 sufficient to guarantee that a ternary relation T is induced by a property space via (2.3)? It turns out that one needs the following additional condition. Say that a set $A \subseteq X$ is **convex** if for all x, y, z,

$$[\{x, z\} \subseteq A \text{ and } (x, y, z) \in T] \Rightarrow y \in A. \tag{2.4}$$

Hence, in accordance with the usual notion of convexity in a Euclidean space, a set is convex if it contains with any two elements all elements that are between them. Furthermore, say that a subset $H \subseteq X$ is a **half-space** if both H and its complement H^c are non-empty and convex.

T5 (Separation) If $(x, y, z) \notin T$, then there exists a half-space H such that

$$H \supseteq \{x, z\} \text{ and } y \notin H.$$

Fact 2.1 *Let T be a ternary relation on X. There exists a collection $\mathcal{H}_T$ of basic properties satisfying H1-H3 such that $T = T_{(\mathcal{H}_T)}$, i.e. such that T is derived from $\mathcal{H}_T$ via (2.3), if and only if T satisfies T1-T5.*

Necessity of the conditions T1-T5 is easily verified (cf. Nehring (1999)); their sufficiency follows from defining the underlying property space $\mathcal{H}_T$ as the collection of all half-spaces induced by T. However, the underlying property space is not uniquely determined by T, and frequently it is not necessary to consider the collection of *all* half-spaces. Henceforth, we assume that the collection of basic properties $\mathcal{H}$ is sufficiently rich so that any half-space (with respect to $T_{\mathcal{H}}$) can be obtained as the intersection of a subfamily in $\mathcal{H}$.

The following result characterizes single-peakedness in terms of the basic properties from which the betweenness is derived.

Fact 2.2 *Let $(X, \mathcal{H})$ be a property space. A preference ordering $\succ$ is single-peaked with respect to $T_{\mathcal{H}}$ if and only if there exists a partition $\mathcal{H} = \mathcal{H}_g \cup \mathcal{H}_b$ with $\mathcal{H}_g \cap \mathcal{H}_b = \emptyset$ and $H \in \mathcal{H}_g \Leftrightarrow H^c \in \mathcal{H}_b$ such that*

(i) *$y \succ z$ whenever $y \neq z$ and $y \in H$ for all $H \in \mathcal{H}_g$ such that $z \in H$, and*

(ii) *there exists x^* such that $x^* \in H$ for all $H \in \mathcal{H}_g$.*

In view of condition (i), single-peakedness requires that it must be possible to partition all basic properties into a set of "good" properties (those in $\mathcal{H}_g$) and a set of "bad" properties (those in $\mathcal{H}_b$) in a *separable* way: a property is good or bad no matter with which other properties it is combined. In addition to separability, single-peakedness also requires, by condition (ii), that all good properties are jointly compatible, that is: possessed by some ideal point x^* which then represents the preference peak. The ordinally separable representation in Fact 2.2 suggests a natural cardinal strengthening, in which preferences have an additive utility representation of the form

$$u(x) = \sum_{H \in \mathcal{H}_g, H \ni x} \lambda_H,$$

where $\lambda_H > 0$ for all $H \in \mathcal{H}_g$. The domain $\mathcal{D}$ of all additive preferences in this sense is always rich relative to $T_{\mathcal{D}} = T_{\mathcal{H}}$.

Summarizing, there are two ways to think of the class of domains to which our characterization of strategy-proof social choice applies. First, one may start with a property space $(X, \mathcal{H})$ and consider any rich domain of single-peaked preferences with respect to the induced betweenness $T_{\mathcal{H}}$. In this case, conditions T1-T5 on the betweenness relation are automatically satisfied.

Alternatively, one can start with a domain $\mathcal{D}$ and consider the associated ternary relation $T_{\mathcal{D}}$ according to (2.2), assuming that $T_{\mathcal{D}}$ satisfies conditions T1-T5. The basic properties needed for the description of the common structure underlying all strategy-proof social choice functions (see Section 3 below) are then derived from $T_{\mathcal{D}}$ according to Fact 2.1. In this case, the richness condition R2 is automatically satisfied. Moreover, if any state is the peak of some preference ordering in $\mathcal{D}$, then the base-point relations are all partial orders. Thus, besides the richness requirement R1, the main "regularity" assumptions needed for our analysis turn out to be the symmetry condition T2 and the separation condition T5.

2.5 Examples

To illustrate the above concepts, consider the following examples of generalized single-peaked domains. The first three correspond to the graphic betweenness relations in Figure 1 above.

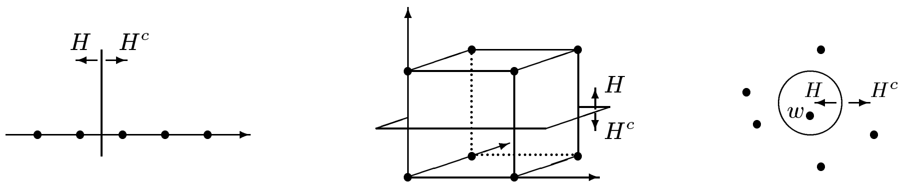
Example 1 (Single-Peakedness on Line) Let X be linearly ordered by $\geq$, and consider the betweenness relation T given by $(x, y, z) \in T :\Leftrightarrow [x \geq y \geq z \text{ or } z \geq y \geq x]$

(cf. Fig. 1a). This betweenness can be derived via (2.3) from the family $\mathcal{H}$ of all sets of the form $H_{\geq w} := \{y \geq w : \text{ for some } w \in X\}$ or $H_{\leq w} := \{y \leq w : \text{ for some } w \in X\}$. Each basic property is thus of the form "lying to the right of w" or "lying to the left of w" (see Figure 2a below). A subset is convex if and only if it is an interval, and the half-spaces are exactly the basic properties of the form $H_{\geq w}$ or $H_{\leq w}$. A preference $\succ$ is single-peaked with respect to T if and only if it is convex in the sense that all upper contour sets $\{y : y \succeq x\}$ are convex.

Example 2 (Separability on the Hypercube) Let $X = \{0,1\}^K$, which we refer to as the K-dimensional hypercube (cf. Fig. 1b). An element $x \in \{0,1\}^K$ is thus described as a sequence $x = (x^1, ..., x^K)$ with $x^k \in \{0,1\}$, and the natural betweenness is given by $(x,y,z) \in T :\Leftrightarrow [\text{ for all } k : x^k = z^k \Rightarrow y^k = x^k = z^k]$. As is easily verified, this betweenness coincides with the graphic betweenness in Fig. 1b above. Geometrically, y is between x and z if and only if y is contained in the "subcube" spanned by x and z (see Fig. 1b above and note, for instance, that the whole 3-hypercube is between w and y'). This betweenness can be derived from the basic properties of the form $H_1^k := \{x : x^k = 1\}$ and $H_0^k := \{x : x^k = 0\}$ for all k (see Figure 2b below which depicts the two basic properties corresponding to the vertical coordinate). In view of Fact 2.2, a preference $\succ$ is single-peaked with respect to T if and only if it is separable in the sense that, for all x, y and all k,

$$x \succ (x^{-k}, y^k) \Leftrightarrow (y^{-k}, x^k) \succ y.$$

Example 3 (The Unrestricted Domain) The vacuous betweenness on X, defined by $(x,y,z) \in T :\Leftrightarrow y \in \{x,z\}$, can be derived via (2.3) from the family $\mathcal{H}$ of all properties of the form $\{x\}$ ("being equal to x") and $X \setminus \{x\}$ ("being different from x") for all $x \in X$ (see Figure 2c below which depicts the property $H = \{w\}$). As noted above, any linear preference ordering is single-peaked with respect to the vacuous betweenness relation, i.e. the set of all single-peaked preferences is the unrestricted domain. Note that any subset of X is convex, hence in fact a half-space. Observe also that a domain is rich in the sense of R1 and R2 with respect to the vacuous betweenness relation if and only if, for any pair (x,y), there exists a preference ordering that ranks x on top and y as second-best alternative.



2a *2b* *2c*

Figure 2: Basic properties underlying Examples 1-3

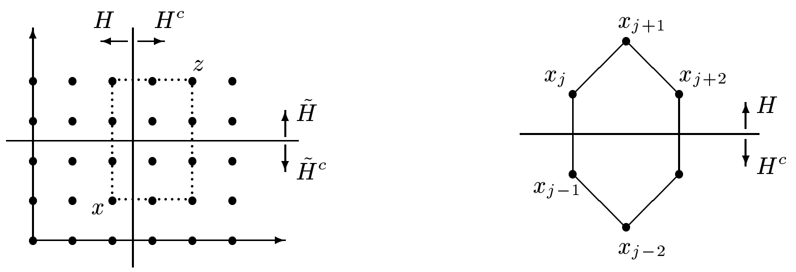
Example 4 (Products) The hypercube betweenness of Example 2 above is an instance of a product betweenness. More generally, let $X = X^1 \times ... \times X^K$, where the alternatives in each factor X^k are described by a list $\mathcal{H}^k$ of basic properties referring to coordinate

k. Let $\mathcal{H} := \{H^k \times \prod_{j \neq k} X^j$: for some k and $H^k \in \mathcal{H}^k\}$, and denote by $T_{\mathcal{H}^k}$ the betweenness relation on X^k induced by $\mathcal{H}^k$. The *product betweenness* T induced by $\mathcal{H}$ according to (2.3) is given by,

$$(x, y, z) \in T \Leftrightarrow [\text{ for all } k : (x^k, y^k, z^k) \in T_{\mathcal{H}^k}].$$

Figure 3a below depicts the product of two lines; the alternatives between x and z are precisely the alternatives contained in the dotted rectangle spanned by x and z.

Example 5 (Cycles) Let $X = \{x_1, ..., x_l\}$, and consider the l-cycle on X, i.e. the graph with the edges (x_i, x_{i+1}), where indices are understood modulo l so that $x_{l+1} = x_1$ (see Figure 3b for the case $l = 6$). The graphic betweenness on the l-cycle is derived from a property space as follows. If l is even, the basic properties are of the form $\{x_j, x_{j+1}, ..., x_{j-1+\frac{l}{2}}\}$. If l is odd, the family of basic properties consists of all sets of the form $\{x_j, x_{j+1}, ..., x_{j-1+\frac{l+1}{2}}\}$ or $\{x_j, x_{j+1}, ..., x_{j-1+\frac{l-1}{2}}\}$.



3a: Product of lines *3b: Cycles*

Figure 3: Further property spaces

Example 6 ("Doing the Opposite") Suppose that, for each state x, there exists a state $\bar{x}$ ("the opposite of x") such that $\bar{\bar{x}} = x$. Consider then the domain $\mathcal{D}$ of all linear orderings $\succ$ such that $x \succ y \Leftrightarrow \bar{y} \succ \bar{x}$, i.e. if a state is deemed better than another, then its opposite must be worse than that state's opposite. The canonical betweenness relation $T_{\mathcal{D}}$ associated with $\mathcal{D}$ according to (2.2) is given by

$$(x, y, z) \in T_{\mathcal{D}} \Leftrightarrow [z = \bar{x} \text{ or } y \in \{x, z\}],$$

i.e. any element is between opposite pairs, but no element is between two non-opposite states. As is easily verified $T_{\mathcal{D}}$ is the graphic betweenness corresponding to the graph in which each point is connected by an edge to all other points but to its opposite element. A subset $A \neq X$ is convex with respect to $T_{\mathcal{D}}$ if and only if $[x \in A \Rightarrow \bar{x} \notin A]$, and $H \neq X$ is a half-space if and only if $[x \in H \Leftrightarrow \bar{x} \notin H]$. In particular, $T_{\mathcal{D}}$ satisfies the separation condition T5; by Fact 2.1, it can thus be derived from a property space according to (2.3). In contrast to some of the examples above, however, the basic properties (i.e. the half-spaces) do not have a meaningful interpretation as "Lancasterian characteristics" here.

Observe that a preference is single-peaked with respect to $T_{\mathcal{D}}$ whenever the opposite of the peak is the least preferred alternative. By contrast, for a preference in $\mathcal{D}$, the ranking between *any* pair is uniquely determined by the ranking of the opposite pair. Thus, $\mathcal{D}$ is much smaller than the domain $\hat{\mathcal{S}}_{X,T_{\mathcal{D}}}$ of all single-peaked preferences. Nevertheless, the induced betweenness $T_{\mathcal{D}}$ is all what matters for the analysis of strategy-proofness.

3 Voting by Committees as Voting by Properties

3.1 Definition

Let $N = \{1, ..., n\}$ be a set of voters. Each voter i is characterized by a linear preference ordering $\succ_i$ in some domain $\mathcal{D}$; the best element of X with respect to $\succ_i$ is denoted by x_i^*. A *social choice function* is a mapping $F : \mathcal{D}^n \to X$ that assigns to each preference profile $(\succ_1, ..., \succ_n)$ in $\mathcal{D}^n$ a unique social alternative $F(\succ_1, ..., \succ_n) \in X$. In the following, we will assume that X is endowed with the structure of a property space; in the next subsection, we will then consider social choice functions defined on rich domains of generalized single-peaked preferences.

An important class of social choice functions are those that only depend on the peaks of voters' preferences; these are referred to as "voting schemes" (cf. Barberá, Gul and Stacchetti (1993)). A social choice function F is a *voting scheme* if there exists a function $f : X^n \to X$ such that for all $(\succ_1, ..., \succ_n)$, $F(\succ_1, ..., \succ_n) = f(x_1^*, ..., x_n^*)$, where x_i^* is voter i's peak. In this case, we say that F satisfies *peaks only.* With slight abuse of terminology, we will also refer to any $f : X^n \to X$ as a voting scheme, since any such function f naturally induces a social choice function satisfying peaks only.

Given a description of alternatives in terms of their properties, a natural way to generate a social choice is to determine the final outcome via its properties. This is described now in detail.

Definition (Committees) A *committee* is a non-empty family $\mathcal{W}$ of subsets of N satisfying $[W \in \mathcal{W}$ and $W' \supseteq W] \Rightarrow W' \in \mathcal{W}$. The coalitions in $\mathcal{W}$ are called *winning.*

For instance, *majority voting* corresponds to $\mathcal{W}_{\frac{1}{2}} = \{W \subseteq N : \#W > \frac{1}{2} \cdot n\}$. Majority voting is a special case of *voting by quota*: for any number $q \in (0, 1)$, voting by quota q corresponds to the committee $\mathcal{W}_q = \{W \subseteq N : \#W > q \cdot n\}$.

Definition (Committee Structures) A *committee structure* on a property space $(X, \mathcal{H})$ is a mapping $\mathcal{W} : H \mapsto \mathcal{W}_H$ that assigns a committee to each basic property $H \in \mathcal{H}$ satisfying the following two conditions.

CS1 $W \in \mathcal{W}_H \Leftrightarrow W^c \notin \mathcal{W}_{H^c}$.
CS2 $[H \subseteq H'$ and $W \in \mathcal{W}_H] \Rightarrow W \in \mathcal{W}_{H'}$.

As is easily verified, CS1 implies that, for any basic property H, the committees corresponding to H and H^c are interrelated as follows.

$$\mathcal{W}_H = \{W \subseteq N : W \cap W' \neq \emptyset \text{ for all } W' \in \mathcal{W}_{H^c}\}. \quad (3.1)$$

Consider now the following voting procedure, adapted to the present framework from Barberá, Sonnenschein and Zhou (1991).

Definition (Voting by Committees) Given a property space $(X, \mathcal{H})$ and a committee structure $\mathcal{W}$, *voting by committees* is the mapping $f_{\mathcal{W}} : X^n \to 2^X$ such that, for all $\xi \in X^n$,

$$x \in f_{\mathcal{W}}(\xi) :\Leftrightarrow \text{ for all } H \in \mathcal{H} \text{ with } x \in H : \{i : \xi_i \in H\} \in \mathcal{W}_H. \tag{3.2}$$

In our present framework, voting by committees amounts to "voting by properties" in that each committee decides whether or not the final outcome is to have one out of two complementary basic properties. Note that $f_{\mathcal{W}}(\xi) \subseteq X$ is not assumed to be non-empty; in particular, $f_{\mathcal{W}}$ does not yet define a voting scheme in the sense of the above definition.

Definition (Consistency) A committee structure $\mathcal{W}$ is called *consistent* if $f_{\mathcal{W}}(\xi) \neq \emptyset$ for all $\xi \in X^n$. If $\mathcal{W}$ is consistent, the corresponding voting procedure $f_{\mathcal{W}}$ will also be referred to as consistent.

Fact 3.1 *If $f_{\mathcal{W}}(\xi) \neq \emptyset$, then $f_{\mathcal{W}}(\xi)$ is single-valued. In particular, voting by committees defines a voting scheme whenever it is consistent.*

If $f_{\mathcal{W}}$ is consistent, one has for all H and ξ,

$$f_{\mathcal{W}}(\xi) \in H \Leftrightarrow \{i : \xi_i \in H\} \in \mathcal{W}_H \tag{3.3}$$

by (3.2) and CS1. Since $N \in \mathcal{W}_H$ for all H, this implies that $f_{\mathcal{W}}$ satisfies *unanimity*, i.e. for all $x \in X$, $f(x, x, ..., x) = x$. In particular, $f_{\mathcal{W}}$ is *onto* whenever it is consistent, i.e. each $x \in X$ is in the range of $f_{\mathcal{W}}$.

Voting by committees is characterized by the following monotonicity condition. Say that a voting scheme $f : X^n \to X$ is *monotone in properties* if, for all ξ, ξ', H,

$$[f(\xi) \in H \text{ and } \{i : \xi_i \in H\} \subseteq \{i : \xi'_i \in H\}] \Rightarrow f(\xi') \in H.$$

Monotonicity in properties states that if the final outcome has some property H and the voters' support for this property does not decrease, then the resulting final outcome must have this property as well.

Proposition 3.1 *A voting scheme $f : X^n \to X$ is monotone in properties and onto if and only if it is voting by committees with a consistent committee structure.*

A social choice function F is called *anonymous* if it is invariant with respect to permutations of individual preferences; similarly, a voting scheme f is called anonymous if $f(\xi_1, ..., \xi_n) = f(\xi_{\sigma(1)}, ..., \xi_{\sigma(n)})$ for any permutation $\sigma : N \to N$. The following fact shows that anonymous voting by committees takes the form of *voting by quota*; the first part is immediate from the definitions, the second part follows at once from (3.1).

Fact 3.2 *Voting by committees $f_{\mathcal{W}}$ is anonymous if and only if it is voting by quota, i.e. for all H there exists $q_H \in [0, 1]$ such that $\mathcal{W}_H = \{W : \#W > q_H \cdot n\}$ if $q_H < 1$ and $\mathcal{W}_H = \{N\}$ if $q_H = 1$.*[3] *If $f_{\mathcal{W}}$ is consistent, the quotas can be chosen such that, for all $H \in \mathcal{H}$, $q_{H^c} = 1 - q_H$.*

Observe that by Fact 3.2 and (3.1) voting by committees is anonymous and *neutral* in the sense that $\mathcal{W}_H = \mathcal{W}_{H'}$ for all $H, H' \in \mathcal{H}$ if and only if it is *issue-by-issue majority voting* with an odd number of voters, i.e. if and only if n is odd and, for all H, $\mathcal{W}_H$ corresponds to voting by quota $q_H = \frac{1}{2}$.

[3] Note that the quotas q_H are not uniquely determined.

3.2 The Equivalence of Strategy-Proofness and Voting by Committees

A social choice function $F : \mathcal{D}^n \to X$ is *strategy-proof* on $\mathcal{D}$ if for all i and $\succ_i, \succ_i' \in \mathcal{D}$,

$$F(\succ_1, ..., \succ_i, ..., \succ_n) \succeq_i F(\succ_1, ..., \succ_i', ..., \succ_n).$$

Furthermore, say that F satisfies *voter sovereignty* if F is onto, i.e. if any $x \in X$ is in the range of F. For any committee structure $\mathcal{W}$, denote by $F_{\mathcal{W}} : \mathcal{D}^n \to 2^X$ the mapping defined by $F_{\mathcal{W}}(\succ_1, ..., \succ_n) = f_{\mathcal{W}}(x_1^*, ..., x_n^*)$, where for each i, x_i^* is the peak of $\succ_i$ on X. The mapping $F_{\mathcal{W}}$ will also be referred as voting by committees.

Let now T be the betweenness relation derived from $\mathcal{H}$ according to (2.3), and denote by $\mathcal{S}_{X,T}$ any subset of single-peaked preferences with respect to T that is rich in the sense of conditions R1 and R2. When no confusion can arise, we will simply write $\mathcal{S}$ for $\mathcal{S}_{X,T}$ and refer to it as a *rich single-peaked domain.*

Proposition 3.2 *Let $\mathcal{S}$ be any rich single-peaked domain, and let $F : \mathcal{S}^n \to X$ be represented by the voting scheme $f : X^n \to X$. Then, F is strategy-proof on $\mathcal{S}$ if and only if f is monotone in properties.*

In combination with Proposition 3.1, this implies that a voting scheme is strategy-proof on a rich single-peaked domain if and only if it is voting by committees with a consistent committee structure. We now want to show that *any* strategy-proof social choice function $F : \mathcal{S}^n \to X$ satisfying voter sovereignty is voting by committees. For this, it remains to show that any such F is a voting scheme, i.e. that it satisfies peaks only.

Proposition 3.3 *Every strategy-proof social choice function $F : \mathcal{S}^n \to X$ on a rich single-peaked domain that satisfies voter sovereignty is a voting scheme, i.e. satisfies peaks only.*

Proposition 3.3 generalizes a corresponding result in Barberá, Massò and Neme (1997). While our proof in the appendix follows their proof in overall design, it augments it by a number of significant steps. Specifically, it applies to a larger class of property spaces and it makes explicit the needed richness conditions (see the appendix for the precise relation between their result and ours).

Combining Propositions 3.1 – 3.3 yields the following result.

Theorem 1 *A social choice function $F : \mathcal{S}^n \to X$ satisfies voter sovereignty and is strategy-proof on a rich single-peaked domain $\mathcal{S}$ if and only if it is voting by committees with a consistent committee structure.*

3.3 Consistent Committee Structures: The Intersection Property

By Theorem 1, a social choice function is strategy-proof on a rich domain of single-peaked preferences if and only if it is *consistent* voting by committes. It is, however, not self-evident whether a given committee structure is consistent. The needed characterization of consistency is provided in this subsection. As a simple example of an inconsistent committee structure, consider the vacuous betweenness on $X = \{x, y, z\}$, and assume that voting by committees takes the form of issue-by-issue majority voting among three voters. If all three peaks of the voters are distinct, each of the following basic properties gets a majority of two votes: $\{y, z\}$ ("being different from x"), $\{x, z\}$ ("being different from y"), and $\{x, y\}$ ("being different from z"). But clearly, $\{y, z\} \cap \{x, z\} \cap \{x, y\} = \emptyset$, i.e. the basic properties determined by the committees are jointly incompatible. Consistency of voting by committees requires that the committee structure be compatible with the structure of basic properties, as follows.

Definition (Critical Family) Say that a family $\mathcal{G} \subseteq \mathcal{H}$ of basic properties is a *critical family* if $\cap\mathcal{G} = \emptyset$ and for all $G \in \mathcal{G}$, $\cap(\mathcal{G} \setminus \{G\}) \neq \emptyset$.

The interpretation of a critical family is as an exclusion of a certain combination of basic properties. "Criticality" (i.e. minimality) means that this exclusion is not already entailed by a more general exclusion. More concretely, consider $\mathcal{G} = \{G_1, ..., G_l\}$; to say that $\mathcal{G}$ is a critical family is to say that for any combination of $l-1$ basic properties in $\mathcal{G}$ there are states possessing them jointly, but any state possessing $l-1$ of the basic properties cannot possess the remaining l-th property. Thus, critical families reflect the "entailment logic" of the underlying property space, a theme explored in more detail in Section 4.3 below. Trivial instances of critical families are all pairs $\{H, H^c\}$ of complementary properties. A non-trivial example of a critical family are the three basic properties $\{y, z\}$, $\{x, z\}$ and $\{x, y\}$ in the above example of the set $\{x, y, z\}$ endowed with the vacuous betweenness: any two of these basic properties have a non-empty intersection, while the intersection of all three is empty.

Intersection Property Say that voting by committtees $F_{\mathcal{W}}$ satisfies the *Intersection Property* if for any critical family $\mathcal{G} = \{G_1, ..., G_l\}$, and any selection $W_j \in \mathcal{W}_{G_j}$,

$$\bigcap_{j=1}^{l} W_j \neq \emptyset.$$

Using (3.1), it is easily verified that the Intersection Property applied to critical families with two elements yields precisely conditions CS1 and CS2 above.

Proposition 3.4 *Voting by committees is consistent if and only if it satisfies the Intersection Property.*

Combining this result with Theorem 1, we obtain the following characterization of all strategy-proof social choice functions on any rich single-peaked domain.

Theorem 2 *A social choice function $F : \mathcal{S}^n \to X$ satisfies voter sovereignty and is strategy-proof on a rich single-peaked domain $\mathcal{S}$ if and only if it is voting by committees satisfying the Intersection Property.*

Observe that by this result the class of all strategy-proof social choice functions on a domain $\mathcal{S}_{X,T}$ only depends on the "betweenness geometry" of the underlying property space in that any rich single-peaked domain defined on the same property space induces the same class of strategy-proof social choice functions.

Theorem 2 generalizes Corollary 3 in Barberá, Massò and Neme (1997) which applies to the domains $\hat{\mathcal{S}}_{X,T}$ where X is some subset of a product of lines. In that context, these authors also derive a condition called "intersection property" that can be viewed as relating families of "inconsistent properties" to admissible winning coalitions. However, the condition obtained here is much simpler and more powerful due to the restriction to *minimal* such families. For instance, in the anonymous case of voting by quota it simplifies to a system of linear inequalities.

Specifically, in the anonymous case, the Intersection Property can be formulated as follows. If, for any critical family $\mathcal{G}$,

$$\sum_{H \in \mathcal{G}} q_H \geq \#\mathcal{G} - 1, \tag{3.4}$$

then voting by quotas q_H for $H \in \mathcal{H}$ is consistent. Conversely, if anonymous voting by committees is consistent, then it can be represented by quotas satisfying (3.4). Observe that this immediately implies that issue-by-issue majority voting is consistent if and only if any critical family has two members. The remarkable consequences of this observation are explored in the following section.

To illustrate the intuition behind the Intersection Property, we verify the necessity of (3.4) in the special case of the vacuous betweenness on $X = \{x_1, ..., x_m\}$; from this it is straightforward to infer the non-existence of anonymous and strategy-proof social choice functions on an unrestricted domain if $m \geq 3$. Recall from Example 3 that the vacuous betweenness corresponds to the basic properties $H_j = \{x_j\}$ ("being equal to x_j") and $H_j^c = X \setminus \{x_j\}$ ("being different from x_j"), for $j = 1, ..., m$. The non-trivial critical families are $\{H_1^c, ..., H_m^c\}$ and, for all $j \neq k$, $\{H_j, H_k\}$. Consider the critical family $\{H_1^c, ..., H_m^c\}$, and suppose that (3.4) is violated, i.e. $\sum_j q_j^c < m - 1$, where q_j^c denotes the quota corresponding to H_j^c. If $q_j = 1 - q_j^c$ is the quota corresponding to H_j, one thus obtains $\sum_j q_j > 1$, say $\sum_j q_j = 1 + m \cdot \delta$ for some $\delta > 0$. Now assign to a fraction of $q_j - \delta$ voters the peak x_j. Since none of the basic properties $H_j = \{x_j\}$ reaches the quota, all complements are enforced; but since their intersection is empty, consistency is violated.

In Nehring and Puppe (2003a) we use the Intersection Property to characterize the domains that admit strategy-proof social choice functions satisfying fundamental additional properties such as non-dictatorship and anonymity. Thanks to the Intersection Property this task boils down to relating committee structures satisfying specified conditions to the underlying property space which is fully determined by the collection of its critical families. The characterizations can thus be formulated in terms of combinatorial conditions on the critical families of a property space.

4 Strong Possibility on Median Spaces

By Theorem 2 above, strategy-proof social choice on single-peaked domains takes the form of voting by committees satisfying the Intersection Property. For any *given* domain this yields a simple characterization of the class of all strategy-proof social choice functions. On the other hand, it does not answer the question for which property spaces there exist *well-behaved* strategy-proof social choice functions on the associated domain of single-peaked preferences. In this section, we derive a simple necessary and sufficient condition on a property space such that *all* well-defined committee structures are consistent. It turns out that the same condition also characterizes the class of spaces on which anonymous and neutral strategy-proof social choice functions, amounting to issue-by-issue majority voting, exist. The property spaces considered here thus enable well-behaved strategy-proof social choice in a strong sense. Weaker notions of "well-behavedness" are considered in Nehring and Puppe (2003a).

4.1 Universal Consistency of Median Spaces

In this subsection, we will study property spaces that admit a rich set of strategy-proof social choice functions in the sense of the following condition.

Definition (Universal Consistency) A property space $(X, \mathcal{H})$ is *universally consistent* if voting by committees $f_{\mathcal{W}}$ is consistent for any committee structure $\mathcal{W}$ (satisfying CS1 and CS2).

Obviously, universal consistency implies consistency of issue-by-issue majority voting. As an immediate consequence of (3.4), we have already seen that issue-by-issue majority voting is consistent if and only if every critical family has only two elements. What does that mean geometrically? To provide the intuition, consider three voters with peaks ξ_1, ξ_2, ξ_3 and denote by m the chosen state under issue-by-issue majority voting. Consider any basic property H possessed by both ξ_1 and ξ_2, i.e. assume that $\{\xi_1, \xi_2\} \subseteq H$. Then H gets a majority of at least two votes over H^c, hence we must have $m \in H$ (see Figure 4 below). By (2.3), this means that m is between ξ_1 and ξ_2. But the same argument applies to any basic property jointly possessed by ξ_1 and ξ_3, and to any basic property jointly possessed by ξ_2 and ξ_3. In other words, a necessary condition for issue-by-issue majority voting to be consistent is that any triple ξ_1, ξ_2, ξ_3 of social states admits a state $m = m(\xi_1, \xi_2, \xi_3)$ that is between any pair of them. Such a state will be called a "median" of the triple.

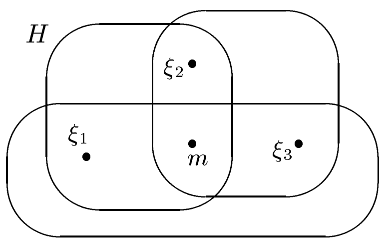


Figure 4: The median property

Definition (Median Space) A property space $(X, \mathcal{H})$ is called a *median space* if the induced betweenness relation T satisfies the following condition. For all $x, y, z \in X$ there exists an element $m = m(x, y, z) \in X$, called a *median* of x, y, z, such that m is between any pair of $\{x, y, z\}$, i.e. such that $\{(x, m, y), (x, m, z), (y, , m, z)\} \subseteq T$.

Median spaces are a classic topic in abstract convexity theory (see, e.g., Bandelt and Hedliková (1983) and the references in van de Vel (1993)).

Fact 4.1 *In a median space, any triple has a unique median.*

Median spaces can be characterized in terms of the underlying properties $\mathcal{H}$ as follows. Say that a family $\mathcal{A} \subseteq 2^X$ of subsets of X has the *pairwise intersection property* if for any collection $A_1, ..., A_l \in \mathcal{A}$ such that $A_k \cap A_h \neq \emptyset$ for all $k, h \in \{1, ..., l\}$, one has $\cap_{k=1}^{l} A_k \neq \emptyset$.

Proposition 4.1 *The following statements are equivalent.*
(i) $(X, \mathcal{H})$ *is a median space.*
(ii) $\mathcal{H}$ *has the pairwise intersection property.*
(iii) *For all critical families* $\mathcal{G}$, $\#\mathcal{G} = 2$.

Thus, a property space is a median space if and only if pairwise compatibility of a family of basic properties implies their joint compatibility. Note that in contrast to the Intersection Property for committees, the pairwise intersection property imposes a restriction only on the space $(X, \mathcal{H})$.

The existence of a median for any triple is not only necessary for the consistency of issue-by-issue majority voting but also sufficient, in fact it is even sufficient for universal consistency, as shown by the following result.

Theorem 3 *The following statements are equivalent.*
(i) $(X, \mathcal{H})$ *is a median space.*
(ii) $(X, \mathcal{H})$ *is universally consistent.*
(iii) *Issue-by-issue majority voting (among an odd number of voters) is consistent on* $(X, \mathcal{H})$.

The implication "(i) $\Rightarrow$ (ii)" is an easy consequence of two results that have already been established. By Proposition 4.1, all critical families in a median space have cardinality two. But for such critical families, the Intersection Property of Section 3.3 reduces to the requirements CS1 and CS2. Hence, by Proposition 3.4, any committee structure satisfying these two requirements is consistent.

Theorem 3 has the following obvious corollary which shows that median spaces admit a maximal class of strategy-proof social choice functions.

Corollary 4.1 *Let* $(X, \mathcal{H})$ *be a median space. A social choice function* $F : \mathcal{S}^n \to X$ *satisfies voter sovereignty and is strategy-proof on a rich single-peaked domain* $\mathcal{S}$ *if and only if* F *is voting by committees with an arbitrary well-defined committee structure.*

Universal consistency ensures the existence of a large class of strategy-proof social choice functions with a very simple structure. In particular, any "partial" committee structure satisfying the fundamental restrictions CS1 and CS2 can be extended to a complete, consistent committee structure. Formally, let $\mathcal{F}$ be any subset of $\mathcal{H}$ that is closed under complements. A *partial committee structure* is a mapping $\mathcal{W}$ that assigns to each $H \in \mathcal{F}$ a committee $\mathcal{W}_H$ satisfying CS1 and CS2.

Proposition 4.2 *Let* $(X, \mathcal{H})$ *be a median space, and let* $\mathcal{F} \subseteq \mathcal{H}$ *be closed under complements. Then, any partial committee structure on* $\mathcal{F}$ *can be extended to a consistent committee structure on* $\mathcal{H}$.

To appreciate the strength of this result, consider the case $\mathcal{F} = \{H, H^c\}$ for some $H \in \mathcal{H}$. Any property space verifying the assertion of Proposition 4.2 must admit non-dictatorial strategy-proof social choice functions, since the committees for H and H^c can be specified in such a way that no single voter ever forms a winning coalition. However, one can show that there are many non-dictatorial domains for which the assertion fails to hold. The result has also interesting consequences for non-degenerate subsets of basic properties. For example, one may ask when it is possible to require that all basic property in some subset $\mathcal{H}' \subseteq \mathcal{H}$ can only be chosen unanimously, i.e. when there exists a consistent committee structure such that $\mathcal{W}_H = \{N\}$ for all $H \in \mathcal{H}'$. One can easily infer from Proposition 4.2 that in a median space this is true if and only if the associated family of complementary properties is jointly compatible, i.e. if and only if $\cap\{H^c : H \in \mathcal{H}'\} \neq \emptyset$. While always necessary, this condition is in general not sufficient outside median spaces.

4.2 Examples of Median Spaces

The above results show how remarkably well-behaved median spaces are for the purposes of the analysis of strategy-proof social choice. It remains to understand this class better in itself. To this purpose, we will first present a range of examples, then show that all median spaces are characterized by a simple "entailment logic" (Subsection 4.3), and finally, moving from the median spaces themselves to the associated preference domains, show that all such domains can be described in terms of economically natural restrictions on preferences (Subsection 4.4).

First, consider the examples of Section 2 above. Since y is the median of x, y, z whenever y is between x and z, lines (Example 1 above) are median spaces with the middle point as the median of any triple. More generally, any graphic betwenness derived from a *tree* (i.e. connected and acyclic graph) gives rise to a median space. To see this, consider for any triple of points in a tree the (unique) shortest paths connecting any pair. By the acyclicity, these three shortest paths have exactly one point in common, namely the median of the triple.

Furthermore, all hypercubes (Example 2) are median spaces; a typical configuration is the triple x, z, w with the median y in Fig. 1b above. More generally, any distributive lattice is a median space (see van de Vel (1993)). In addition, products (Example 4) are median spaces if and only if every factor is a median space; indeed, the median on a product is simply given by taking the median in each coordinate. Summarizing, our analysis shows that the common source of the possibilities of strategy-proof social choice derived in Moulin (1980), Barberá, Sonnenschein and Zhou (1991) and Barberá, Gul and Stacchetti (1993) is that in each case the underlying space is a median space.

In contrast to these examples, the property spaces underlying Examples 3, 5 and 6 are not median spaces whenever $\#X \geq 3$ (with the exception of the 4-cycle which is isomorphic to the two-dimensional hypercube). For instance, the triple x, z, w in Fig. 1c above does not have a median. More generally, in Example 3 (the vacuous betweenness) *no* triple of pairwise distinct alternatives admits a median. The fact that cycles (Example 5) are not median spaces is exemplified by the triple x_{j-2}, x_j, x_{j+2} in Fig. 3b above. In Example 6 (doing the opposite) no triple of mutually non-opposite elements admits a median.

Further examples of median spaces are appropiate subdomains of median spaces.

Definition (Median Stability) A subset $Y \subseteq X$ of a median space $(X, \mathcal{H})$ is called *median stable* if $m(x, y, z) \in Y$ for all $\{x, y, z\} \subseteq Y$.

For instance, any subset of the form $\{x, y, z, m\}$ where m is the median of x, y, z is median stable. In general, one has the following characterization of median stability (cf. van de Vel (1993, p.130)).

Fact 4.2 *Let $(X, \mathcal{H})$ be a median space, and let $H, H' \in \mathcal{H}$. Then, the set $Y = X \setminus (H \cap H')$ is median stable. Moreover, all median stable subsets of X are obtained by sequentially deleting intersections of two basic properties.*

The following figure depicts a typical median stable subset of the product of two lines; its median stability follows at once from Fact 4.2.

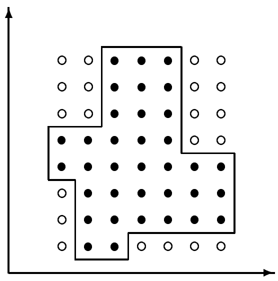


Figure 5: A median stable subset of the product of two lines

4.3 The Entailment Logic of Median Spaces

Our results suggest that the notion of a critical family of basic properties plays a key role for the understanding of voting by committees and thus of strategy-proof social choice on single-peaked domains. We have already noted that a critical family describes certain entailments among basic properties. Since a property space is uniquely identified through its critical families, this means that the critical families describe a property space in terms of its "entailment logic." To illustrate, consider the line, labelled by the natural numbers $1, ..., m$. The basic properties are $H_{\geq j}$ ("being greater than or equal to j") and $H_{\leq k}$ ("being smaller than or equal to k") for appropriate j and k in $\{1, ..., m\}$. All critical families have the form $\{H_{\geq j}, H_{\leq k}\}$ for some $k < j$. The interpretation is that "$\geq j$" logically entails "not $\leq k$" whenever $k < j$. Thus, the critical family corresponds to the statement "for all x, $x \geq j$ implies (not $x \leq k$)." In this case, the entailment is "simple" in that the antecedent of the implication consists of *one* basic property.

By contrast, consider the set $X = \{x_1, ..., x_m\}$ endowed with the vacuous betweenness. For each x_j, the set $H_j^c = X \setminus \{x_j\}$ corresponds to the basic property "being different from x_j." The critical family $\{H_1^c, ..., H_m^c\}$ thus describes the following entailment: "if an alternative is different from $m-1$ distinct elements of X, it cannot be different from the remaining m-th element." The antecedent of this implication is much more complex as it consists of $m-1$ conjunctions of basic properties.

The characterization of median spaces as those property spaces for which all critical families have cardinality two (cf. Proposition 4.1) thus says that median spaces are those property spaces with a simple entailment logic. This singles out median spaces as a fundamental class of property spaces.

As a more concrete illustration in a voting context, consider the following problem of *constitutional choice*. Suppose that a set of countries, say the EU member states, have to decide on the procedures for their collective choices, i.e. they have to decide on their joint constitution. Specifically, consider the problem of determining on which of the issues $K = \{1, ..., k\}$ future decisions are to be made on the basis of majority voting. Individual preferences are thus taken to be over subsets of K with the interpretation that $L \succ_i L'$ if country i prefers majority voting over exactly the issues in $L \subseteq K$ to majority voting over exactly the issues in $L' \subseteq K$. The assumption of single-peakedness does not seem implausible in that context; it requires that, for each single issue k, majority voting over issue k is preferred/not preferred independently of the corresponding preference over other issues. Observe, however, that this excludes a preference for the overall extent of majority voting (regardless on which issues), since in that case majority voting for one issue would be a substitute for majority voting over another issue.

In general, one cannot assume that the issues are independent from each other. In other words, one has to account for the "entailment logic" of the underlying problem. For instance, suppose that the issue k represents the joint defense policy of the countries, whereas k' represents their joint foreign policy. It is in general not possible to decide on defense policy by majority voting without also deciding at least on some foreign policy issues by majority voting. In particular, the set of all feasible constitutions will, in general, not be the entire power set 2^K. The entailment "majority voting over $k \Rightarrow$ majority voting over k'" thus corresponds to a critical family. As long as all such entailments are simple in the sense that their antecedent consists of only one basic property, the resulting space is a median space. By Theorem 3 above, any well-defined voting by committees procedure is applicable in that case.

4.4 Median Preference Domains

So far, we have described the preference domains on which possibility results emerge indirectly through their underlying geometry as median spaces. We now show that one can characterize these domains directly through appropriate convexity and separability conditions. In this sense, all preference domains associated with median spaces are economically meaningful in principle. Consider first the two basic instances of median spaces, the line and the hypercube. On a line, and more generally on any tree, the set of all single-peaked preferences coincides with the set of linear oderings that satisfy the "convexity" restriction

$$[x \succ y \text{ and } (x,y,z) \in T] \Rightarrow y \succ z \tag{4.1}$$

for all x, y, z with $y \neq z$.

By contrast, it is easily verified that no linear ordering on the hypercube can satisfy this condition for *all* triples x, y, z. On the other hand, the set of single-peaked preferences on the hypercube is generated by the following "separability" restrictions (cf. Sect. 2.5 above). Denote by $\mathcal{H}_{x\neg y} := \{H \in \mathcal{H} : x \in H \text{ and } y \notin H\}$ the basic properties possessed by x but not by y, and say that x and y are *immediate neighbours* if $\mathcal{H}_{x\neg y}$ consists of one single basic property.[4] If $\mathcal{H}_{x\neg y} = \mathcal{H}_{z\neg w} = \{H\}$ for some H, i.e. if (x, y) and (z, w) are two pairs of immediate neighbours separated by the same basic property, then

$$x \succ y \Leftrightarrow z \succ w. \tag{4.2}$$

By Fact 2.2 above, any single-peaked preference on any property space satisfies these separability restrictions. In general, however, they do not characterize the set of single-peaked preferences; for instance, in the case of the line they are vacuously satisfied. The following result shows that on a median space the set of all single-peaked preferences is generated by the separability conditions (4.2) together with an appropriate selection of convexity restrictions of the form (4.1).

[4] One can show that in median spaces all neighbours are immediate neighbours, which is not true in general.

Proposition 4.3 *Let $(X, \mathcal{H})$ be a median space. There exists a subset $\Psi \subseteq X^3$ such that the set of all single-peaked preferences coincides with the orderings that satisfy the separability restrictions (4.2) for any quadruple (x, y, z, w) such that $\mathcal{H}_{x\neg y} = \mathcal{H}_{z\neg w} = \{H\}$ for some H, and the convexity restrictions (4.1) for all triples $(x, y, z) \in \Psi$.*

As an illustration, consider the following figure which shows a median stable subset of the product of two lines. Any single-peaked preference ordering verifies the convexity condition (4.1) for the triple (x, y, z'), for instance. Indeed, $x \succ y$ implies that the peak of $\succ$ is either x or z, from which the conclusion $y \succ z'$ is immediate. By contrast, condition (4.1) need not hold for the triple (x, y, w), since $x \succ y$ and $w \succ y$ if z is the peak of $\succ$. On the other hand, the quadruple (x, y, z, w) obviously verifies the separability condition (4.2).

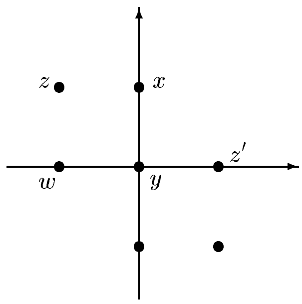


Figure 6: Convexity and separability in a median space

A conclusion similar to that of Proposition 4.3 fails often outside the class of median spaces. As an example, consider the 6-cycle; it is easily verified that any convexity restriction of the form (4.1) is violated by some single-peaked ordering,[5] while the relevant separability conditions are satisfied also by preferences that are not single-peaked.

5 Conclusion

In this paper, we have defined a general notion of single-peakedness based on abstract betweenness relations. We have shown that a social choice function is strategy-proof on a sufficiently rich single-peaked domain if and only if it takes the form of voting by committees satisfying the Intersection Property (Theorems 1 and 2). The concept of a median space, in which every triple of social states admits a fourth state that is between any pair of the triple, turned out to be fundamental for the existence of well-behaved strategy-proof social choice functions. Median spaces are distinguished from a number of different perspectives. Due to their universal consistency, median spaces give rise to a maximally rich class of strategy-proof social choice functions (Corollary 4.1). Moreover, they are exactly the spaces that admit anonymous and neutral strategy-proof social choice rules, amounting to issue-by-issue majority voting (Theorem 3), and they guarantee the extendability of any partial committee structure (Proposition 4.2). Finally, median spaces are characterized by their simple "entailment logic" (Proposition 4.1), and their associated domain of all single-peaked preferences can be exhaustively described by a family of convexity and separability restrictions (Proposition 4.3).

[5]Concretely, for $(x, y, z) \in T$, any single-peaked preference with peak opposite to y violates (4.1).

Appendix 1: "Voting under Constraints" as a special case

The following clarifies the relation of our work to Barberá, Massò and Neme's "Voting under Constraints" (1997). Let $(X, \mathcal{H})$ be a property space with $\hat{\mathcal{S}}_{X,T_{\mathcal{H}}}$ as the associated domain of all single-peaked preferences. Suppose that only a subset Y of social alternatives is in fact feasible, *and* that voters' ideal points are known to be feasible (as pointed out by Barberá, Massò and Neme (1997), the latter assumption is clearly restrictive). Formally, let $\mathcal{D} := \{\succ|_Y : \succ \in \hat{\mathcal{S}}_{X,T_{\mathcal{H}}}$ with peak of $\succ$ in $Y\}$. One can show that $\mathcal{D}$ consists exactly of the preferences on Y that are single-peaked with respect to the restriction of $T_{\mathcal{H}}$ to Y which is the betweenness associated with the relativization $(Y, \{H \cap Y : H \in \mathcal{H}\})$ of the underlying property space to Y. Any such domain $\mathcal{D}$ is therefore covered by our analysis. Barberá, Massò and Neme (1997) consider the special case in which the underlying property space $(X, \mathcal{H})$ is a product of lines.[6]

Appendix 2: Proofs

Proof of Fact 2.2 Let $\succ$ be single-peaked with repect to $T_{\mathcal{H}}$, and denote by x^* the peak of $\succ$; define $\mathcal{H}_g := \{H \in \mathcal{H} : x^* \in H\}$ and $\mathcal{H}_b := \{H \in \mathcal{H} : x^* \notin H\}$. Obviously, this partition of $\mathcal{H}$ satisfies all required properties.

Conversely, let the partition $\mathcal{H} = \mathcal{H}_g \cup \mathcal{H}_b$ satisfy (i) and (ii). It is straightforward to verify that $\succ$ is single-peaked with peak x^*.

Proof of Fact 3.1 Suppose that $x \in f_{\mathcal{W}}(\xi)$ and consider any $y \neq x$. By condition H3, there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \in H^c$. By definition of $f_{\mathcal{W}}$, $\{i : \xi_i \in H\} \in \mathcal{W}_H$. By CS1, $\{i : \xi_i \in H\}^c = \{i : \xi_i \in H^c\} \notin \mathcal{W}_{H^c}$, hence by definition, $y \notin f_{\mathcal{W}}(\xi)$.

Proof of Proposition 3.1 Since committees are by definition closed under taking supsersets, voting by committees is monotone in properties by (3.3). Furthermore, voting by committees is clearly onto since it satisfies unanimity.

Conversely, let $f : X^n \to X$ be onto and monotone in properties. For any $H \in \mathcal{H}$, define

$$\mathcal{W}_H := \{W \subseteq N : \exists \xi \text{ such that } \{i : \xi_i \in H\} = W \text{ and } f(\xi) \in H\}.$$

Note that by monotonicity of f, the definition of $\mathcal{W}_H$ does not depend on the choice of ξ. Since f is onto, $\mathcal{W}_H$ is non-empty. We verify that $\mathcal{W}_H$ is closed under taking supersets. Hence, suppose that $W \in \mathcal{W}_H$ and $W' \supseteq W$. Choose ξ such that $W = \{i : \xi_i \in H\}$ and $f(\xi) \in H$. Define ξ' as follows: $\xi_i = \xi_i$ whenever $i \in W$ or $i \in N \setminus W'$, and $\xi'_j \in H$ if $j \in W' \setminus W$. Then, $W' = \{i : \xi'_i \in H\}$ and, by monotonicity in properties, $f(\xi') \in H$. Hence, by definition, $W' \in \mathcal{W}_H$.

Next, we verify properties CS1 and CS2. It is easily seen that $W^c \notin \mathcal{W}_{H^c}$ implies $W \in \mathcal{W}_H$. To verify the converse implication, assume by way of contradiction that

[6] In the earlier working paper Nehring and Puppe (2002), we falsely asserted that *any* preference domain of the form $\hat{\mathcal{S}}_{X,T_{\mathcal{H}}}$, where $(X, \mathcal{H})$ is an arbitrary property space, is isomorphic to an appropriate relativized domain (in the above sense) embedded in a hypercube. However, this is true only when $T_{\mathcal{H}}$ satisfies an additional separation condition, known in the literature as the separation condition S3 (see van de Vel (1993)), which is slightly stronger than condition T5 above.

$W \in \mathcal{W}_H$ and $W^c \in \mathcal{W}_{H^c}$. Choose ξ with $\{i : \xi_i \in H\} = W$ and $f(\xi) \in H$, and ξ' with $\{i : \xi'_i \in H^c\} = W^c$ and $f(\xi') \in H^c$. Consider ξ'' defined by $\xi''_i = \xi_i$ for $i \in W$ and $\xi''_i = \xi'_i$ for $i \in W^c$. By monotonicity in properties, $f(\xi'') \in H$ and $f(\xi'') \in H^c$, a contradiction. This shows that $\mathcal{W}$ satisfies CS1. To verify CS2, let $H \subseteq H'$ and $W \in \mathcal{W}_H$. Choose ξ such that $\{i : \xi_i \in H\} = W$ and $f(\xi) \in H$. Consider ξ' with $\xi'_i = \xi_i$ for $i \in W$ and $\xi'_i \in H'^c$ for $i \notin W$. By monotonicity in properties, $f(\xi') \in H$, hence $f(\xi') \in H'$, and thus $W = \{i : \xi'_i \in H'\} \in \mathcal{W}_{H'}$.

The proof is completed by noting that $f = f_{\mathcal{W}}$. Indeed, by definition of $\mathcal{W}$, one clearly has $f(\xi) \in f_{\mathcal{W}}(\xi)$, but $f_{\mathcal{W}}$ is single-valued by Fact 3.1.

The following notation will be useful in the remaining proofs.

Definition (Segment) For all $x, z \in X$, denote by $[x, z] := \{y \in X : (x, y, z) \in T\}$ the *segment* spanned by x and z, i.e. all states between x and z.

Proof of Proposition 3.2 Suppose $f : X^n \to X$ is monotone in properties. Consider an individual j with true peak ξ_j who reports $\hat{\xi}_j$. Let $H \in \mathcal{H}$ be any basic property such that $\xi_j \in H$ and $f(\hat{\xi}_j, \xi_{-j}) \in H$. Clearly, $\{i : (\hat{\xi}_j, \xi_{-j})_i \in H\} \subseteq \{i : \xi_i \in H\}$, hence by monotonicity in properties $f(\xi) \in H$. This shows that $f(\xi) \in [\xi_j, f(\hat{\xi}_j, \xi_{-j})]$, i.e. $f(\xi)$ is between the true peak ξ_j and the outcome $f(\hat{\xi}_j, \xi_{-j})$. By single-peakedness, this implies that $f(\xi) \succ_j f(\hat{\xi}_j, \xi_{-j})$ whenever $f(\xi) \neq f(\hat{\xi}_j, \xi_{-j})$.

Conversely, suppose that f is not monotone in properties; then there exist ξ, ξ' and H such that $W := \{i : \xi_i \in H\} \subseteq W' := \{i : \xi'_i \in H\}$, $f(\xi) \in H$ but $f(\xi') \in H^c$. Without loss of generality, we may assume that $W' = W \cup \{j\}$ for some individual $j \notin W$. Since $f(\xi')$ is not between ξ'_j and $f(\xi)$, there exists by the richness condition R2, a preference $\succ_j \in \mathcal{S}$ with top ξ'_j such that $f(\xi) \succ_j f(\xi')$. Clearly, if $\succ_j$ is the true preference of j, this voter will benefit from reporting ξ_j. Hence, F is not strategy-proof.

Proof of Proposition 3.3 The following proof is based on the proof of Barberá, Massò and Neme (1997, Prop. 2) which it augments by a number of significant intermediate steps. Some of these additional arguments seem to be needed for their result, too (see, in particular, Facts A.1 and A.2 as well as Lemma A.2 below).

For $F : \mathcal{S}^n \to X$ and any voter i, define the "set of options given $\succ_i$" by

$$o_{-i}(\succ_i) := \{x \in X : \text{ there exists } \succ_{-i} \in \mathcal{S}^{n-1} \text{ such that } F(\succ_i, \succ_{-i}) = x\}.$$

Let now F be strategy-proof and onto. The proof of the "peaks-only" property proceeds by induction over the number of voters. Thus assume first $n = 2$. From the strategy-proofness of F it is immediate that

$$F(\succ_1, \succ_2) = \text{argmax}_{o_1(\succ_2)} \succ_1 = \text{argmax}_{o_2(\succ_1)} \succ_2, \tag{A.1}$$

i.e. $F(\succ_1, \succ_2)$ is the best element in $o_j(\succ_i)$ with respect to $\succ_j$.

Denoting by $\tau(\succ) \in X$ the peak of $\succ$, one has

$$[\tau(\succ_1) = \tau(\succ_2) = x] \Rightarrow F(\succ_1, \succ_2) = x. \tag{A.2}$$

For verification, suppose that x is the common peak of $\succ_1$ and $\succ_2$. Since F is onto, there exist $\succ'_1$ and $\succ'_2$ such that $F(\succ'_1, \succ'_2) = x$, i.e. $x \in o_1(\succ'_2)$. By (A.1), $F(\succ_1, \succ'_2) = x$, i.e. $x \in o_2(\succ_1)$, hence again by (A.1), $F(\succ_1, \succ_2) = x$.

The following fact plays a key role in the proof of Lemma A.1 below.

Fact A.1 *Suppose that $y \in o_2(\succ_1)$ and $y' \in [y, \tau(\succ_1)]$, then $y' \in o_2(\succ_1)$.*

To verify this, we can assume that y' is a neighbour of y; from this the general claim then follows by induction using the transitivity condition T3. Thus, assume by way of contradiction that $y' \in [y, \tau(\succ_1)]$ is a neighbour of y with $y' \notin o_2(\succ_1)$. By the richness condition R1, there exists $\succ$ such that $y' \succ y \succ w$ for all $w \notin \{y, y'\}$. By (A.1), $F(\succ_1, \succ) = y$, and by (A.2), $F(\succ, \succ) = y'$. By the single-peakedness of $\succ_1$, voter 1 can therefore manipulate at $(\succ_1, \succ)$ via $\succ$, a contradiction.

Lemma A.1 *If $\tau(\succ_1) = \tau(\succ_1')$, then $o_{-1}(\succ_1) = o_{-1}(\succ_1')$.*

Proof of Lemma A.1 We first prove the result for $n = 2$. Suppose, by way of contradiction, that $x = \tau(\succ_1) = \tau(\succ_1')$ and $y \in o_2(\succ_1)$ but $y \notin o_2(\succ_1')$. By (A.2), one must have $y \neq x$. First, we show that y cannot be a neighbour of x. Otherwise, one could choose, by R1, a preference $\succ$ with $y \succ x \succ w$ for all $w \notin \{y, x\}$; by (A.1) one would obtain $F(\succ_1, \succ) = y$ and $F(\succ_1', \succ) = x$, but then voter 1 could manipulate at $(\succ_1, \succ)$ via $\succ_1'$.

Thus, y is not a neighbour of x. Choose a neighbour $y' \in [x, y]$ of y. By Fact A.1, $y' \in o_2(\succ_1)$. Suppose that also $y' \in o_2(\succ_1')$. By R1, there exists a preference $\succ'$ with $y \succ' y' \succ' w$ for all $w \notin \{y, y'\}$. By (A.1), $F(\succ_1, \succ') = y$ and $F(\succ_1', \succ') = y'$. But by the single-peakedness of $\succ_1$, we have $x \succ_1 y' \succ_1 y$; therefore, voter 1 can manipulate at $(\succ_1, \succ')$ via $\succ_1'$. Thus, we must have $y' \in o_2(\succ_1)$ and $y' \notin o_2(\succ_1')$. Now replace y by y' and repeat the argument until a neighbour of x is reached to derive a contradiction.

To prove the assertion for general n, define a social choice function $E : \mathcal{S}^2 \to X$ by $E(\succ_1, \succ_2) := F(\succ_1, \succ_2, ..., \succ_2)$. It is easily verified that E inherits the strategy-proofness and voter sovereignty from F. Hence, by the above arguments,

$$[\tau(\succ_1) = \tau(\succ_1')] \Rightarrow o_2^E(\succ_1) = o_2^E(\succ_1').$$

The proof is thus completed by showing that, for all $\succ_1$, $o_2^E(\succ_1) = o_{-1}^F(\succ_1)$. Clearly, one has $o_2^E(\succ_1) \subseteq o_{-1}^F(\succ_1)$. To show the converse inclusion, take any $x \in o_{-1}^F(\succ_1)$ and choose $\succ_2, ..., \succ_n$ such that $x = F(\succ_1, \succ_2, ..., \succ_n)$. Consider any preference $\succ$ with $\tau(\succ) = x$. By the strategy-proofness of F,

$$x = F(\succ_1, ..., \succ_n) = F(\succ_1, ..., \succ_{n-1}, \succ) = ... = F(\succ_1, \succ, ..., \succ) = E(\succ_1, \succ),$$

hence $x \in o_2^E(\succ_1)$. This concludes the proof of Lemma A.1.

For the case $n = 2$, we can now complete the proof of the "peaks-only" property. Indeed, that property follows at once from the fact that

$$[o_2(\succ_1) = o_2(\succ_1')] \Rightarrow F(\succ_1, \succ_2) = F(\succ_1', \succ_2). \tag{A.3}$$

To verify (A.3), assume by way of contradiction, that $x = F(\succ_1, \succ_2) \neq F(\succ_1', \succ_2) = x'$. By assumption there exist $\succ$ and $\succ'$ such that $F(\succ_1, \succ') = x'$ and $F(\succ_1', \succ) = x$. But then voter 2 can either manipulate at $(\succ_1, \succ_2)$ via $\succ'$ (if $x' \succ_2 x$), or manipulate at $(\succ_1', \succ_2)$ via $\succ$ (if $x \succ_2 x'$).

The proof for $n = 2$ is thus complete. For the induction argument, we need the following definition and lemma.

Definition (Gated set) A subset $Y \subseteq X$ is called *gated* if, for all $x \in X$, there exists an element $\gamma(x) \in Y$ such that $\gamma(x) \in [x, y]$ for all $y \in Y$, i.e. such that $\gamma(x)$ is between x and any element of Y. The element $\gamma(x)$ is called the *gate of Y to x*.

Lemma A.2 *For all $\succ_1$, the set $o_{-1}(\succ_1)$ is gated.*

Proof of Lemma A.2 Given any element $x \in X$ choose $\succ$ with $\tau(\succ) = x$, and set $\gamma(x) := F(\succ_1, \succ, ..., \succ) = E(\succ_1, \succ)$. Assume, by way of contradiction, that $z \in o_{-1}(\succ_1)$ is such that $\gamma(x) \notin [x, z]$. By the richness condition R2, there exists $\succ'$ with $\tau(\succ') = x$ and $z \succ' \gamma(x)$. By the strategy-proofness of E and (A.1), $\gamma(x) \neq E(\succ_1, \succ')$; but this contradicts the "peaks-only" property of E.

Proof of Prop. 3.3 (cont.) For given $\succ_1$ define

$$G(\succ_2, ..., \succ_n) := F(\succ_1, ..., \succ_n),$$

and denote $Y := o_{-1}(\succ_1)$. Clearly, G is strategy-proof with range Y. Let G_Y denote the restriction of G to the profiles of preference orderings in $\mathcal{S}$ that have their peak in Y. Now observe that for any $\succ \in \mathcal{S}$ with peak x, the restriction $\succ |_Y$ is single-peaked (with respect to the induced betweenness on Y) with peak $\gamma(x)$. Moreover, the set of all restrictions is a rich domain on Y. Hence, by the induction hypothesis G_Y satisfies "peaks-only" and can therefore represented by a voting scheme $g : Y^{n-1} \to Y$.

Fact A.2 $G(\succ_2, ..., \succ_n) = g(\gamma(\xi_2), ..., \gamma(\xi_n))$, *where* $\xi_i = \tau(\succ_i)$.

This follows from $\gamma(\xi_i) = \text{argmax}_Y \succ_i$ and the observation that, by strategy-proofness, $G(\succ_2, ..., \succ_n) = G_Y(\succ_2 |_Y, ..., \succ_n |_Y)$.

We now complete the proof by deriving a contradiction from the assumption that there exist $\succ_1$ and $\succ_1'$ with $\tau(\succ_1) = \tau(\succ_1') =: x$ such that

$$y = F(\succ_1, \succ_2, ..., \succ_n) \neq F(\succ_1', \succ_2, ..., \succ_n) = y'.$$

By Lemma A.1, $o_{-1}(\succ_1) = o_{-1}(\succ_1') =: Y$. Define G, G_Y and g as above, and analogously, G', G_Y' and g'. By Fact A.2,

$$y = g(\gamma(\xi_2), ..., \gamma(\xi_n)) \neq g'(\gamma(\xi_2), ..., \gamma(\xi_n)) = y',$$

and by Propositions 3.1 and 3.2, g and g' are voting by committees on Y. Choose $H \in \mathcal{H}|_Y$ with $y \in H$, $y' \in H^c$ and, without loss of generality, $\gamma(x) \in H$. Let $W := \{i : \gamma(\xi_i) \in H\}$, $W' = \{i : \gamma(\xi_i) \in H^c\}$, and consider $\eta = (\eta_2, ..., \eta_n)$ where

$$\eta_i = \begin{cases} y' & \text{if} \quad i \in W' \\ \gamma(x) & \text{if} \quad i \notin W' \end{cases}.$$

Since g and g' are voting by committees, and since any basic property jointly possessed by y' and $\gamma(x)$ gets unanimous support, we have $\{g(\eta), g'(\eta)\} \subseteq [y', \gamma(x)]$. Moreover, $W = \{2, ..., n\} \setminus W'$ is winning for H in g, and W' is winning for H^c in g', hence $g(\eta) \in H$ and $g'(\eta) \in H^c$.

We show that $g'(\eta) \neq y'$. Otherwise, choose $\hat{\succ}_i$ with $\tau(\hat{\succ}_i) = \eta_i$ to obtain from $g(\eta) \in [y', \gamma(x)]$ and $g(\eta) \neq y'$,

$$g(\eta) = F(\succ_1, \hat{\succ}_2, ..., \hat{\succ}_n) \succ_1' F(\succ_1', \hat{\succ}_2, ..., \hat{\succ}_n) = y',$$

in contradiction to the strategy-proofness of F. Now repeat the argument replacing y' by $z' := g'(\eta) \in H^c$. The desired contradiction is then obtained by induction since the segment $[z', \gamma(x)]$ is strictly contained in $[y', \gamma(x)]$.

Proof of Proposition 3.4 Suppose $f_{\mathcal{W}}$ is consistent, and let $\mathcal{G} = \{G_1, ..., G_l\}$ be a critical family. For $j = 1, ..., l$, consider any selection $W_j \in \mathcal{W}_{G_j}$. We will show $\cap_{j=1}^{l} W_j \neq \emptyset$ by a contradiction argument. Thus, assume that $\cap_{j=1}^{l} W_j = \emptyset$. Then, for all $i \in N$, there exists j_i such that $i \notin W_{j_i}$. For each i, pick an element $\xi_i \in G_{j_i}^c \cap (\cap_{j \neq j_i} G_j) = \cap_{j \neq j_i} G_j$ (observe that the latter set is non-empty by definition of a critical family). By construction, if $i \in W_j$, then $j \neq j_i$, hence $\xi_i \in G_j$. This shows that, for all j, $W_j \subseteq \{i : \xi_i \in G_j\}$. Therefore, $\{i : \xi_i \in G_j\} \in \mathcal{W}_{G_j}$, hence by (3.3), $f_{\mathcal{W}}(\xi_1, ..., \xi_n) \in G_j$ for all $j = 1, ..., l$. However, this contradicts the fact that $\{G_1, ..., G_l\}$ is a critical family.

Conversely, suppose $f_{\mathcal{W}}$ is not consistent, i.e. for some ξ, $f_{\mathcal{W}}(\xi) = \emptyset$. By (3.2) and CS1, this implies that $\cap\{H \in \mathcal{H} : \{i : \xi_i \in H\} \in \mathcal{W}_H\} = \emptyset$. We show that $f_{\mathcal{W}}$ cannot satisfy the Intersection Property by contradiction. Thus assume $f_{\mathcal{W}}$ does satisfy the Intersection Property. Pick a critical family $\{G_1, ..., G_l\} \subseteq \{H \in \mathcal{H} : \{i : \xi_i \in H\} \in \mathcal{W}_H\}$. By the Intersection Property, $\cap_{j=1}^{l}\{i : \xi_i \in G_j\} \neq \emptyset$. Let $i_0 \in \{i : \xi_i \in G_j\}$ for all $j = 1, ..., l$. But then $\xi_{i_0} \in G_j$ for all j, contradicting the fact that $\{G_1, ..., G_l\}$ is a critical family.

Proof of Fact 4.1 Suppose, by way of contradiction, that x, y, z admit two distinct medians $m_1 \neq m_2$. By H3, these can be separated by a basic property H such that $m_1 \in H$ and $m_2 \in H^c$. Clearly, either H or H^c must contain at least two elements of $\{x, y, z\}$, say $\{x, y\} \subseteq H$. But then $m_2 \notin [x, y]$ in contradiction to the fact that it is a median.

Proof of Proposition 4.1 The equivalence of (ii) and (iii) is immediate since a critical family with more than two elements violates the pairwise intersection property; conversely, any minimal family of basic properties violating the pairwise intersection property must contain at least three elements and is by definition a critical family.

To prove the implication "(i) $\Rightarrow$ (ii)," take any collection $\{H_1, ..., H_l\} \subseteq \mathcal{H}$ such that $H_k \cap H_h \neq \emptyset$ for all $k, h \in \{1, ..., l\}$. We verify the pairwise intersection property by induction. For $l = 2$ it holds trivially; thus assume $l > 2$. Let $A := H_1 \cap ... \cap H_{l-2}$. Choose $x \in A \cap H_{l-1}$, $y \in A \cap H_l$ and $z \in H_{l-1} \cap H_l$, the first two intersections being non-empty by induction hypothesis, the latter by assumption. Consider the median $m = m(x, y, z)$; since A is convex, $[x, y] \subseteq A$, hence $m \in A$. Similarly, $m \in H_{l-1}$ and $m \in H_l$, hence $m \in \cap_{k=1}^{l} H_k$.

The converse implication "(ii) $\Rightarrow$ (i)" is verified as follows. For any pair x, y, denote by $\mathcal{H}_{\{x,y\}} := \{H \in \mathcal{H} : \{x, y\} \subseteq H\}$, and observe that $\cap\mathcal{H}_{\{x,y\}} = [x, y]$. Now simply note that, for any triple x, y, z, the family $\mathcal{H}_{\{x,y\}} \cup \mathcal{H}_{\{x,z\}} \cup \mathcal{H}_{\{y,z\}}$ has pairwise non-empty intersections. By assumption, its intersection is non-empty; since this intersection is contained in each of the segments $[x, y]$, $[x, z]$ and $[y, z]$ it contains a median of the triple.

Proof of Theorem 3 The implication "(i) $\Rightarrow$ (ii)" follows from Propositions 3.4 and 4.1, as shown in the main text; the implication "(ii) $\Rightarrow$ (iii)" is trivial. It remains to verify the necessity of the median property for the consistency of issue-by-issue majority voting. Thus, suppose that $(X, \mathcal{H})$ is not a median space. Specifically, let x, y, z be such that $[x, y] \cap [x, z] \cap [y, z] = \emptyset$. For odd $n \geq 3$, consider issue-by-issue majority voting, i.e. $\mathcal{W}_H = \{W : \#W > n/2\}$ for all H. Assume that voter's peaks are distributed as evenly as possible among the three points x, y and z. Thus, for instance, if n is divisible by 3, assume that exactly one third of the peaks are at x, y and z, respectively. Then, by definition, $f_{\mathcal{W}}(\xi) \in [x, y] \cap [x, z] \cap [y, z]$; but the latter set is empty, hence issue-by-issue majority voting is not consistent.

Proof of Proposition 4.2 By the universal consistency of median spaces, any committee structure satisfying CS1 and CS2 is consistent. By induction, it therefore suffices to show that, for any $H \in \mathcal{H}$, a partial committee structure on $\mathcal{F}$ can be extended to a partial committee structure on $\mathcal{F} \cup \{H, H^c\}$. We distinguish three cases. First, assume that $H \subseteq G$ for some $G \in \mathcal{F}$; then define $\mathcal{W}_H := \cap\{\mathcal{W}_G : G \in \mathcal{F} \text{ and } H \subseteq G\}$. If, on the other hand, for no $G \in \mathcal{F}$, $H \subseteq G$, but for some $G' \in \mathcal{F}$, $G' \subseteq H$, then set $\mathcal{W}_H := \cup\{\mathcal{W}_G : G \in \mathcal{F} \text{ and } G \subseteq H\}$. Finally, if for no $G \in \mathcal{F}$, $H \subseteq G$, and for no $G' \in \mathcal{F}$, $G' \subseteq H$, then assign an arbitrary committee $\mathcal{W}_H$ to H. In each case, define $\mathcal{W}_{H^c}$ according to (3.1) so that the pair (H, H^c) satisfies CS1. Condition CS2 is satisfied by construction, thus $\mathcal{W}$ is a partial committee structure on $\mathcal{F} \cup \{H, H^c\}$.

For the proof of Fact 4.2, we use the following lemma; in its statement, $medS$ denotes the smallest median stable set that contains S (the so-called "median stabilization" of S). Lemma A.3 is a straightforward reformulation of van de Vel (1993, Lemma 6.20, p.130); therefore its proof is omitted here.

Lemma A.3 *Let $(X, \mathcal{H})$ be a median space, and let $S \subseteq X$. Then $x \in medS$ if and only if for each pair $H, H' \in \mathcal{H}$ with $x \in H \cap H'$ one has $S \cap H \cap H' \neq \emptyset$.*

Proof of Fact 4.2 By Lemma A.3, it is clear that, for any median stable subset $Y \subseteq X$, the set $Y \setminus (H \cap H')$ is again median stable. To show that any median stable set has the required form, consider an arbitrary median stable subset $Y \subseteq X$, i.e. $medY = Y$. Let $X \setminus Y = \{x_1, ..., x_r\}$. Lemma A.3 implies that for any x_j there exist H_j, H'_j with $x_j \in H_j \cap H'_j$ such that $Y \cap H_j \cap H'_j = \emptyset$. Hence, $Y = (...(X \setminus (H_1 \cap H'_1)) \setminus ...) \setminus (H_r \cap H'_r)$.

Proof of Proposition 4.3 Define $\Psi \subseteq X^3$ as the set of all triples (x, y, z) such that $y \in [x, z]$, and for some $H, G \in \mathcal{H}$, $\mathcal{H}_{x \neg y} = \{H\}$, $\mathcal{H}_{y \neg z} = \{G\}$ and $\{H, G^c\}$ is critical. First, we show that any single-peaked preference ordering $\succ$ satisfies the convexity restriction (4.1) for any such triple. Using Fact 2.2, $x \succ y$ implies that $H \in \mathcal{H}_g$. Moreover, one has $G \in \mathcal{H}_g$ since $\{H, G^c\}$ is critical; this implies $y \succ z$, again by Fact 2.2.

To prove that, conversely, any linear preference ordering satisfying the stated restrictions is single-peaked, we use the following fact (cf. van de Vel (1993)). Let x, y, z be a triple of distinct elements in a median space such that $y \in [x, z]$. Then, there exists a "direct path" through y that connects x and z. Formally, there exists a sequence $y_0, y_1, ..., y_l, y_{l+1}$ with the following properties: $y_0 = x$, $y_{l+1} = z$, $y \in \{y_1, ..., y_l\}$, and for all $j = 0, ..., l$, y_j and y_{j+1} are immediate neighbours such that $y_{j+1} \in [y_j, z]$.

Now consider any linear preference ordering $\succ$ with peak x that satisfies the stated convexity and separability restrictions. We will show that $y \succ z$ for any $y \neq z$ with $y \in [x, z]$. As above, let $y_0, ..., y_{l+1}$ be a direct path through y connecting x and z. For all j, denote by Θ_j the set of immediate neighbours of y_j in $[y_j, z]$. We show by induction that, for all j,

$$y_j \succ w \text{ for all } w \in \Theta_j. \tag{A.4}$$

By transitivity of $\succ$, (A.4) implies $y_j \succ z$ for all j, since $z = y_{l+1} \in \Theta_l$. For $j = 0$, (A.4) holds trivially since $y_0 = x$ is the peak of $\succ$. Thus, assume that (A.4) holds for $j-1$, and consider y_j along with an immediate neighbour $w \in \Theta_j$. Let $\mathcal{H}_{y_{j-1} \neg y_j} = \{H\}$ and $\mathcal{H}_{y_j \neg w} = \{G\}$. There are two possible cases. First, if $\{H, G^c\}$ is critical, then the triple (y_{j-1}, y_j, w) belongs to Ψ since, by construction, $y_j \in [y_{j-1}, w]$. By induction hypothesis, $y_{j-1} \succ y_j$, hence by (4.1), $y_j \succ w$. Otherwise, if $\{H, G^c\}$ is not critical, there exists, using the median property, an immediate neighbour v of y_{j-1} in $H \cap G^c$. Since $v \in \Theta_{j-1}$, we have $y_{j-1} \succ v$ by the induction hypothesis. This implies $y_j \succ w$ by the separability restriction (4.2) corresponding to the quadruple (y_{j-1}, v, y_j, w).

References

[1] BANDELT, H.-J. and J. HEDLIKOVÁ (1983), Median Algebras, *Discrete Mathematics* **45**, 1-30.

[2] BARBERÁ, S., F. GUL and E. STACCHETTI (1993), Generalized Median Voter Schemes and Committees, *Journal of Economic Theory* **61**, 262-289.

[3] BARBERÁ, S., J. MASSÒ and A. NEME (1997), Voting under Constraints, *Journal of Economic Theory* **76**, 298-321.

[4] BARBERÁ, S., H. SONNENSCHEIN and L. ZHOU (1991), Voting by Committees, *Econometrica* **59**, 595-609.

[5] BORDER, K. and J. JORDAN (1983), Straightforward Elections, Unanimity and Phantom Voters, *Review of Economic Studies* **50**, 153-170.

[6] DEMANGE, G. (1982), Single Peaked Orders on a Tree, *Mathematical Social Sciences* **3**, 389-396.

[7] MOULIN, H. (1980), On Strategy-Proofness and Single-Peakedness, *Public Choice* **35**, 437-455.

[8] NEHRING, K. (1999), Diversity and the Geometry of Similarity, *mimeo.*

[9] NEHRING, K. and C. PUPPE (2002), Strategy-Proof Social Choice on Single-Peaked Domains: Possibility, Impossibility and the Space Between, *mimeo.*

[10] NEHRING, K. and C. PUPPE (2003a), Strategy-Proof Social Choice without Dictators, *mimeo.*

[11] NEHRING, K. and C. PUPPE (2003b), Efficient and Strategy-proof Voting Rules: A Characterization, *mimeo.*

[12] VAN DE VEL, M. (1993), *Theory of Convex Structures*, North Holland, Amsterdam.