The Impossibility of a Paretian Rational: A Bayesian Perspective

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Abstract

If a group takes a collective decision on the basis of separately aggregated group judgments on the probabilities of independent events, there may not exist any anonymous aggregation rule that respects individuals' unanimous outcome preferences at all profiles of beliefs.

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1. INTRODUCTION

Consider the following fairly general model of reason-based group choice. A group of agents $i \in I$ has agreed to base its choice among social alternatives $y \in Y$ on a set of group judgments $x_I^k \in X^k$ on "issues" $k \in K$ according to a group decision function $\Phi : \prod_{k \in K} X^k \to Y$. The group judgments on the k-th issue, in turn, are derived from the individual judgments on that issue according to some aggregation rule $f_k : \left(X^k\right)^I \to X^k$. The issues in question might concern, for example, beliefs regarding specified events or propositions, the valuation of some alternative according to various criteria, the weight of those criteria, etc. .

It is natural to wonder in which cases reason-based group choices agree with unanimous individual outcome preferences; if such agreement is guaranteed, we will say that the aggregation rule f is "Pareto consistent with Φ". Formally, the aggregation rule $f = (f_k)_{k \in K}$ is *Pareto consistent with* Φ iff, for all profiles $(x_i)_{i \in I}$ and all $y \in Y$, $\Phi\left(f\left((x_i)_{i \in I}\right)\right) = y$ whenever $\Phi(x_i) = y$ for all $i \in I$.

In Nehring (2005), we have analyzed this question exhaustively for the case of binary judgments on issues and binary decisions on outcomes, i.e. with $X^k = Y = \{0, 1\}$ for all k, and a monotone decision function Φ. It is shown there in particular that only dictatorial aggregation rules are Pareto consistent whenever Φ is sufficiently complex. In those cases a "Paretian rational" is impossible.

In the present note, we consider a Bayesian version of the problem, allowing the group judgment to take on continuous values, with $Y = [0, 1]$ and $X^k = D \subseteq [0, 1]$ for all k. We demonstrate that the basic thrust of the results in the discrete setting of Nehring (2005) is robust by showing that there exist natural decision rules Φ for which anonymous Pareto consistent aggregation rules do not exist.

A significant dimension of generality is the consideration of restricted domains $D \subsetneq [0, 1]$. Such domains are of interest in situations in which the group decision mechanism elicits only coarse information about the agents' beliefs, as assumed in the standard discrete judgment aggregation set-up. For example, in a legal setting, the agents (jury members) may only be asked whether they are "sufficiently confident" about whether E_k has occurred or not. This can be given a Bayesian interpretation by interpreting a positive answer as a probabilistic estimate α that is higher than the one associated with a negative answer, β, resulting in a domain $D = \{\alpha, \beta\}$. The Bayesian interpretation represents a significant shift from the acceptance-rejection formulation that is standard in the judgment aggregation literature[1] in that it naturally gives rise to graded group beliefs intermediate between β and α and increasing continuously with the number of agents affirming sufficient confidence in the occurrence of a particular event.

2. A BAYESIAN MODEL OF REASON-BASED GROUP CHOICE

To formulate a Bayesian model of reason-based group choice, suppose that a group decision is to be taken on the basis of individual agents' probability judgments $\mathbf{p}_i^k$ on K subjectively stochastically independent contingencies E_k. Thus each agent's beliefs $\mathbf{p}_i$ are described by a product measure $\otimes_k \mathbf{p}_i^k$ on the state space $\{0,1\}^K$, with $E_k = \{1\} \times \{0,1\}^{K \backslash k}$, where $\mathbf{p}_i^k$ is uniquely specified by the number $p_i^k = \mathbf{p}_i^k(E_k)$, the subjective probability of agent i that the k-th contingency E_k materializes. The assumption of stochastic independence is clearly rather special, but the basic points of the following discussion would easily generalize to conditional independence structures. As demonstrated by the explosive growth of "Bayes' nets" and "graphical models" in Bayesian theory and applications over the last 15 years, these are of extremely wide applicability and fundamental importance; see, for example, Pearl (1988) and Cowell et al. (1999).

The individual beliefs are aggregated by an aggregation rule f mapping profiles of individual beliefs into a group beliefs $\mathbf{p}_I$, where $\mathbf{p}_I$ is a probability measure on $\{0,1\}^K$. The aggregation rule is **anonymous** if it is invariant under any permutation of beliefs across agents.

The group needs to make a Yes-No-decision on the basis of the aggregated group probabilities. In a Bayesian setting, it is natural to assume that the group uses an expected utility criterion described by an agreed-upon group utility function $u : 2^K \to \mathbf{R}$, where $u(\omega)$ is the (possibly negative) utility gain in state ω of having chosen "Yes" rather than "No". A given utility function u induces the decision function Φ_u, with

$$\Phi_u(\mathbf{p}) = 1 \text{ if and only if } \sum_{\omega \in 2^K} u(\omega)\, \mathbf{p}(\omega) > 0.$$

[1] See, for example, List-Pettit (2002).

Of particular interest are utility functions of the form $u = 1_S - \tau$, where S is an event in 2^K; in this case, the decision function $\Phi_{(1_S-\tau)}$ simplifies to

$$\Phi_{(1_S-\tau)}(\mathbf{p}) = 1 \text{ if and only if } \mathbf{p}(S) > \tau.$$

That is, the decision is "Yes" if and only if the group assessment of the probability of the event S exceeds some threshold value τ. An aggregation rule f is **Pareto consistent with respect to** u if, for all profiles $P = (p_i)_{i \in I}$ and $x \in \{0,1\}$, $\Phi_u(f(P)) = x$ whenever $\Phi_u(\mathbf{p}_i) = x$ for all $i \in I$.

Reason-basedness of the group choice is expressed by separability of the aggregation rule. The aggregation rule f is **separable** if $\mathbf{p}_I$ is a product measure $\mathbf{p}_I = \otimes_k \mathbf{p}_I^k$, and if the group belief over the partition $\{E_k, E_k^c\}$ is determined by the corresponding individual beliefs, $\mathbf{p}_I^k = f_k\left(\left(\mathbf{p}_i^k\right)_{i \in I}\right)$ for some appropriate component rule f_k; since the component state-spaces are binary, we will write more simply $p_I^k = f_k\left(\left(p_i^k\right)_{i \in I}\right)$, viewing f_k as a mapping from D^I to $[0,1]$, where D is contained in $[0,1]$ and contains a least two elements.

Separability seems eminently sensible in view of the agreed-upon epistemic independence of the contingencies E_k : since all agents agree that there is nothing to learn about the likelihood of E_k by being informed about the occurrence or non-occurrence of E_ℓ, it is hard to see how one could justify the possibility of such learning if all agents are assumed to be rational while disagreeing in their assessments.[2]

As shown by the following example, Pareto consistency becomes an issue already in the simplest of group choice problems.

Example 1. Suppose two expected-value maximizing agents share the profits from a potential investment equally. The success of this investment depends on the joint realization of two independent events E_1 and E_2. The investment is successful if and only if both events materialize; in this case, the investment recoups the initial

[2] If, on the other hand, one interprets the disagreement as reflecting the irrationality/bias of at least some agent, it may seem prima-facie plausible to postulate an ability of the group to learn from the occurrence or non-occurrence of E_ℓ about the relative bias of different agents; this could motivate a change in the conditional group belief about E_k. However, one needs to ask why an analogous reasoning failed to occur at the individual level.

outlays tenfold; in the alternative, it is completely wasted. Thus we have $u = 1_S - \tau$, where $S = E_1 \cap E_2 = \{(1,1)\}$ and $\tau = \frac{1}{10}$.

Consider now the following profile of probability judgments illustrated in table 1 below. Agent 1 believes that the first contingency will materialize with 90% probability, but the second only with 10% probability; the investment will therefore succeed with 9% probability, implying a negative expected return. Agent 2 likewise believes that the investment will succeed with 9% probability, but for different reasons. While she thinks that the second contingency will materialize with 90% probability, she gives only a 10 % chance to the first. Pareto consistency thus counsels against investing in the project.

By contrast, aggregating the probability judgments for the two contingencies directly by the arithmetic mean, for example, entails a group probability of 50% for each in view of the symmetry of the individual 90% and 10%=100%-90% estimates. This entails a 25% probability for the investment to succeed, hence a clear decision to invest.

	$p^1(E_1)$	$p^2(E_2)$	$p(E_1 \cap E_2)$	Decision
Agent 1	0.9	0.1	0.09	Don't Invest
Agent 2	0.1	0.9	0.09	Don't Invest
Group {1,2}	0.5	0.5	0.25	Invest

Table 1

The example shows that even in the simplest group decision problems, well-motivated aggregation rules f may fail to be Pareto consistent. On the other hand, in this particular example Pareto consistency can be achieved for instance by letting the group probability be the *geometric* mean of individual probabilities, $f_k = f^{geo}$ for all k, where

$$f^{geo}\left(\left(p_i^k\right)\right) = \left(\prod_{i \in I} p_i^k\right)^{\frac{1}{n}}$$

Note that at the profile given in Table 1, this leads to group probabilities of 30% for each contingency, and thus a 9% probability for the investment to succeed. [3]

[3]A potential criticism of this aggregation rule is its asymmetric treatment of the positive and

But, as shown by the following result, serependipity cannot always save the day.

Proposition 1 *There exist events S and $\tau \in (0,1)$ such that no anonymous separable aggregation rule f is Pareto consistent with $\Phi_{(1_S - \tau)}$.*

Proof. Proposition 1 is verified by constructing an example with 6 marginal events, defining $S = (E_1 \cap E_2) \cup (E_3 \cap E_4) \cup (E_5 \cap E_6)$ and choosing $\alpha, \beta \in D$ with $\alpha > \beta$ and setting probability threshold $\tau \in (0,1)$ such that $\alpha^2 + 2\beta^2 > \tau > 2\alpha\beta + \beta^2$.

Agents will be assigned one of the following types of probability assessments.

	E_1	E_2	E_3	E_4	E_5	E_6	S	Decision
q_1	α	α	β	β	β	β	$\alpha^2 + 2\beta^2$	Yes
q_2	β	β	α	α	β	β	$\alpha^2 + 2\beta^2$	Yes
q_3	β	β	β	β	α	α	$\alpha^2 + 2\beta^2$	Yes
q_4	α	β	α	β	β	β	$2\alpha\beta + \beta^2$	No
q_5	β	α	β	α	β	β	$2\alpha\beta + \beta^2$	No
q_6	β	α	β	β	β	α	$2\alpha\beta + \beta^2$	No
q_7	β	β	β	α	α	β	$2\alpha\beta + \beta^2$	No

If the number of agents is even, define the profile P by assigning $\frac{n}{2}$ agents the belief q_1 and $\frac{n}{2}$ agents the belief q_2, and define the profile Q by assigning $\frac{n}{2}$ agents the belief q_4 and $\frac{n}{2}$ agents the belief q_5.

On the other hand, if the number of agents is odd, define the profile P by assigning $\frac{n-1}{2}$ agents the belief q_1, $\frac{n-1}{2}$ agents the belief q_2, and 1 agent the belief q_3. Likewise, define the profile Q by assigning $\frac{n-1}{2}$ agents the belief q_4, $\frac{n-3}{2}$ agents the belief q_5, 1 agent the belief q_6 and 1 agent the belief q_7.

The profiles P and Q have been constructed such that

negative realizations of the contingencies; for example, due to this asymmetry, this aggregation rule would fail to be Pareto consistent for decision problems of the form $\Phi_{(1_S - \tau)}$ if S is a disjunction rather than conjunction of two independent events.

We note that with only two independent events and an *odd* number of agents, the latter problem can be overcome in turn by using instead the event-wise median of the individual probabilities. This follows from results of Peters et al. (1992) . On the other hand, it is easy to see that in general the median is Pareto inconsistent if E is the conjunction of more than two events.

i) all agents in P assign probability $\alpha^2+2\beta^2$ to the event S and thus favor a positive outcome decision;

ii) all agents in Q assign probability $2\alpha\beta+\beta^2$ to the event S and thus favor a negative outcome decision;

iii) for each k, at P and Q the same number of agents assign a high/low probability to the event E_k.

By the anonymity and separability of f, the group probabilities $f_k(P)$ and $f_k(Q)$ on each marginal event E_k must be the same at both profiles in view of iii), and therefore the group decision must be the same as well. Yet since the agents agree on a different outcome decision at the two profiles in view of i) and ii), Pareto consistency must be violated at one of them. $\square$

We note that for an even number of agents, the above proof already works for the simpler event $S=(E_1\cap E_2)\cup(E_3\cap E_4)$. It fails, however, for an odd number of agents, since in that case Pareto consistency is achieved by letting $f_k=f^{med}$ for all k, where $f^{med}\left(\left(p_i^k\right)\right)$ is given as the median of the $\{p_i^k\}$.

Proposition 1 provokes the question of how to resolve the impossibility. In Nehring (2005), we have made a stab at answering, suggesting that in situations of "shared self-interest", the Pareto principle is compelling as a requirement of rational group choice, while it is defeasible in situations of "shared responsibility". However, this distinction is not intended as an exhaustive and clear-cut dichotomy, and much remains to be done to achieve a fully satisfactory resolution. Indeed, much of the interest of impossibility results such as Proposition 1 derives exactly from their challenge to inquire more deeply into the nature of the conflicting principles and their applicability to particular types of group decision problems.

3. RELATION TO THE LITERATURE

There is a sizeable literature on the aggregation of probability judgments only; see in particular the classic survey by Genest and Zidek (1985). In this literature, two aggregation rules play a dominant rule, the "linear" and the "logarithmic" "opinion pools". In the linear opinion pool, the group probability of each event is the (possibly weighted) *arithmetic* average of individual probabilities; by contrast, in the logarithmic opinion pool, the group probability of each *state* is proportional to the (possibly weighted) *geometric* average of individual probabilities. Neither rule has emerged as the dominant one. The linear opinion pool respects unanimous judgments of probability and expected utility by construction, and is therefore Pareto consistent. But it fails to be separable, and is thus deficient in terms of its "reason-basedness". This failure has motivated interest in the logarithmic opinion, which is separable. In the present setting, it yields, for any k, the group odds ratio $\frac{p_I^k}{1-p_I^k}$ as the geometric mean of the individual odds ratios $\frac{p_i^k}{1-p_i^k}$. In the example of Table 1, this yields the group probabilities $p_I^1 = p_I^2 = \frac{1}{2}$, leading to the failure of Pareto consistency described above.

The contribution of the present note vis-a-vis the Bayesian literature is to show that potential conflicts between reason-based judgment aggregation and the Pareto criterion arises under substantially weaker conditions than previously assumed. In particular, Proposition 1 shows that such conflicts are not tied to a specific, demanding model of decision-theoretic rationality at the group level such as the Bayesian one, but comes with the notion of reason-based group choice as such. In particular, such conflicts do not depend on the aggregation of an entire coherent probability measure, but arises already in the context of very simple binary decision problems. On the other hand, the impossibility result contained in the present note is non-trivial since the paring down of the aggregation requirements does make a difference, as it allows to tailor the aggregation rule to the decision function Φ_u at hand. For instance, while in Example 1 the logarithmic opinion pool violates Pareto consistency, Pareto consistency is achieved by eventwise aggregation according to f^{geo} or f^{med}, neither of which would yield a coherent probability measure if simultaneously applied to arbitrary events.

Probably more well-known among economists than the above literature are the potential conflicts between Bayesian group rationality and the Pareto axiom that arise from the simultaneous disagreement about probabilities and utilities, starting with the classic contribution of Hylland-Zeckhauser (1969) and including more recent contributions by Mongin (1995, 1998), Gilboa-Samet-Schmeidler (2004) and Chambers-Hayashi (2006). Of course, a disagreement on both dimensions makes it even easier to achieve failures of Pareto consistency, and even harder to overcome it. Assuming a separate aggregation of probabilities and utilities as in Hylland-Zeckhauser (1969), one could, for instance, easily demonstrate an analogue to Proposition 1 by means of an analogous proof.

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