Aumann Meets Spohn in the Tropics: Agreement and Speculation with Ranking Functions

Burkhard C. Schipper*

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Abstract

Ranking functions (Spohn, 1988, 1990, 2012; Goldszmidt and Pearl, 1992a, b, 1996) are used in artificial intelligence and formal epistemology to model degrees of disbelief. We prove two agreement theorems for ranking functions. The first extends Aumann's (1976) "No agreeing to disagree" theorem: if agents with a common prior ranking function commonly know their posterior ranks of an event, then these ranks coincide. The second replaces common knowledge by common belief with firmness f, the analog of Monderer and Samet's (1989) common p-belief. Unlike in the probabilistic case, agreement is then exact rather than approximate. Replacing sums by minima and products by sums, as in tropical mathematics, we define the expectation of a random variable with respect to a ranking function and prove a no-speculative-betting theorem. In contrast to the probabilistic case, its converse fails. The absence of commonly known acceptable bets is equivalent to the much weaker condition that every common knowledge component contains a state that no agent disbelieves. We show that the absence of commonly known acceptable bets from every common endowment characterizes a common prior, using a characterization of common priors as potentials of the agents' rank differences. We also show that agents who publicly announce their posterior ranks learn to agree in finitely many rounds. Finally, unlike their probabilistic counterparts, the agreement theorem and the no-betting theorem under a common prior extend to non-partitional information structures without any requirement other than that agents do not know false events.

Keywords: Ranking functions, tropical mathematics, agree-to-disagree, common knowledge, no-trade, common prior, non-partitional information structures.

JEL-Classification: C72, D82, D83.

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*Department of Economics, University of California, Davis, bcschipper@ucdavis.edu. I thank Gabriele Kern-Isberner for teaching me about ranking functions and comments on an earlier draft.

1 Introduction

The celebrated "No agreeing to disagree" theorem by Aumann (1976) shows in a multi-agent setting under the common prior assumption that if the agents' posterior beliefs in an event are common knowledge among the agents, then their posterior beliefs in the event must agree. That is, agents with the same background but potentially different information cannot agree to disagree about the likelihood of an event when rationally processing the information. Aumann's (1976) theorem is cast in a Bayesian setting in which agents form a common prior probabilistic belief and use rules of conditional probability for incorporating private information into their posterior probabilistic beliefs. Can an analogous result be obtained with non-probabilistic notions of (dis)belief?

Spohn (1988, 1990, 2012) presented a theory of revisions of disbeliefs represented by ordinal rankings that has been further developed in artificial intelligence (Goldszmidt and Pearl, 1992a, b, 1996). An event of rank zero is not disbelieved, events of higher ranks are more and more disbelieved, and events of rank ∞ are impossible (see Shackle, 1969, for a precursor in economics). Rankings are a familiar way of organizing beliefs. For instance, we can interpret a search engine's output as a ranking of events like "document x fits the query" by degree of disbelief. The top result is the event least disbelieved, and each further result is more disbelieved. Sometimes ranks are interpreted as orders of magnitude of infinitesimal probabilities (Spohn, 1988, Section 7; Goldszmidt and Pearl, 1996). An event of rank 1 is infinitely less likely than any event of rank 0. An event of rank 2 is in turn infinitely less likely than one of rank 1, and so on. A sizable literature has been developed in both formal philosophy and artificial intelligence exploring ranking functions (for surveys, see Huber, 2019, and Kern-Isberner et al., 2021). Yet, we are not aware of any application of ranking functions to multi-agent settings except the very recent development of game theory with ranking functions by Schipper and Kern-Isberner (2026). The feasibility of extending Aumann's agreement theorem to ranking functions presents a litmus test for coherent multi-agent reasoning with ranking functions.

In this paper, we first extend the "No agreeing to disagree" theorem to ranking functions. That is, in a multi-agent setting under a common prior ranking function, if the agents' conditional ranks of an event are common knowledge among the agents, then their conditional ranks of the event must agree. Using ranking functions rather than probabilities does not hamper agreement in a rational society in the face of differing individual information sources. We give three proofs. The first is self-contained and mirrors Aumann's original argument. The second and third reduce the theorem to the general agreement results of Cave (1983), Bacharach (1985), and Rubinstein and Wolinsky (1990) for union-consistent decision rules and to Samet's (2022) characterization of agreement by an extended sure-thing principle, respectively. Both locate the entire ranking-theoretic content of the theorem in a single property of conditional ranks. In probability theory, the posterior given a union of disjoint events is a weighted average of the posteriors given each event. In ranking theory, it is the corresponding min-plus or "tropical" combination, in which weights enter additively and the minimum replaces the sum.

Second, we generalize the agreement theorem from common knowledge to common belief with firmness f, the ranking-theoretic analog of Monderer and Samet's (1989) common p-belief. Since ranks combine by taking minima rather than sums, posterior ranks that are

commonly believed with firmness f agree *exactly* once they are truncated at f, whereas Monderer and Samet (1989) and Neeman (1996) obtain only approximate agreement.

Third, and economically most important, we consider speculative betting. Following the min-plus logic of ranking functions, we define the tropical expectation of a random variable by adding the rank of each state to the value of the random variable in that state and minimizing over states. Its decision-theoretic foundation is tropical expected utility theory (Schipper, 2026a; see also Giang and Shenoy, 2000, and Spohn, 2025, for related ranking-based decision theories). We show that, under a common prior ranking function, there is no bet such that it is common knowledge that all agents strictly prefer to take it. This is the ranking-theoretic analog of the no-trade theorems of Milgrom and Stokey (1982) and Sebenius and Geanakoplos (1983).

Fourth, we show that, in contrast to the probabilistic case, the converse of the no-trade theorem fails. In the probabilistic setting, a common prior exists if and only if there is no bet that is commonly known to be acceptable to all agents (Morris, 1994; Samet, 1998; Feinberg, 2000; Heifetz, 2006). With ranking functions, agents may have no common prior and may even agree to disagree, and yet no bet is acceptable to all of them. The reason is a clash between two kinds of linearity. Bets redistribute money, so they are linear in the ordinary sense, whereas tropical expectations are linear in the tropical sense. In probability theory, both are linear in the ordinary sense, which makes it without loss of generality to evaluate bets from a riskless position. With ranking functions, it is not. From a riskless position, an agent never values a bet above her payoff at a state she does not disbelieve, so that a single state that no agent disbelieves rules out every bet.

We pursue two routes to a characterization. The first route weakens the common prior. We show that there is no bet that is commonly known to be acceptable to all agents if and only if every common knowledge component contains a state that no agent disbelieves. This condition is much weaker than a common prior, and it does not suffice for agreement. The second route strengthens the absence of speculative betting by requiring that there is no bet that is commonly known to be acceptable to all agents when they are endowed with any common risky position, not only with a riskless one. A common endowment changes the states at which the agents' valuations are attained and thereby makes disagreements about disbelieved states tradable. We show that the absence of commonly known acceptable bets from every common endowment characterizes a common prior ranking function. To this end, we characterize common prior ranking functions as potentials of the agents' rank differences, combining two at first glance unrelated ideas, namely the cycle characterization of common priors by Rodrigues-Neto (2009) and the potential games of Monderer and Shapley (1996).

We also extend the dynamic agreement results of Geanakoplos and Polemarchakis (1982) and Cave (1983). Agents who publicly announce their posterior ranks, their beliefs with firmness f, or their tropical expectations reach agreement in finitely many rounds. As in the probabilistic case, learning to agree is not learning the truth. However, unlike in the probabilistic case, the announcements may lead agents not only to update but also to revise their beliefs, i.e., to condition on events they initially disbelieved. Such belief reversals can arise after long periods without observable change, as in market crashes without external shocks (Hart and Tauman, 2004). We also present a ranking version of the muddy children puzzle.

We then turn to non-partitional information structures, in which agents may fail to know

what they know or what they do not know. With probabilistic beliefs, Aumann's theorem then requires additional structure, such as positive introspection (Samet, 1990) or balancedness (Geanakoplos, 2021), the absence of commonly known speculation requires positive balancedness (Geanakoplos, 2021), and in general only approximate agreement obtains (Pacuit and Yang, 2026b). In contrast, with ranking functions, the agreement theorem, its version for beliefs, and the no-betting theorem under a common prior hold as soon as agents do not know false events. The reason is again tropical. A posterior rank or tropical expectation given a union of events lies between those given its parts, whether or not the parts overlap. Hence the properties on which these theorems rest are preserved under arbitrary unions, which Rubinstein and Wolinsky (1990) state as sufficient for agreement without partitions.

In the discussion, we show that, as in probability theory, an agent's priors form the tropical convex hull of her types, and explain why this does not lead to the probabilistic characterization of common priors: money is alien to the tropics. We also discuss the interpretation of common prior ranking functions in light of Morris (1995), explain what the existence of speculative betting reveals about where agents disagree, and show that our no-betting results require pessimism. Agents who value a random variable by its best rather than its worst outcome discounted by disbelief bet against each other even under a common prior and without private information.

The unifying theme of the paper is that, although ranks are not probabilities, ranking theory shares the algebraic structure of probability theory once ordinary arithmetic is replaced by the arithmetic of the tropical semiring (Speyer and Sturmfels, 2009), in which the sum of two numbers is their minimum and their product is their sum. In view of the interpretation by infinitesimal probabilities, this is no accident. The probability of a union of disjoint events is of the order of magnitude of its most likely part, and multiplying probabilities adds their orders of magnitude. Ranks of events combine by tropical addition, conditioning is tropical division, and expectations become tropical sums of tropical products. Each of our results is obtained by tropicalizing a classical argument, and the interesting differences from the probabilistic case arise precisely where tropical arithmetic differs from ordinary arithmetic: tropical addition is idempotent and has no inverse. As explained above, the most consequential instance concerns speculation. Money is alien to the tropics because money is linear in the ordinary sense, whereas tropical expectations are linear in the tropical sense. Schipper and Kern-Isberner (2026) use the same perspective to develop tropical game theory with ranking functions. Tropical mathematics has been recently applied to economics (Baldwin and Klemperer, 2019; Tran and Yu, 2019) and game theory (Akian, Gaubert, and Guterman, 2012; Allamigeon, Gaubert, and Meunier, 2023).

The paper is organized as follows. Section 2 introduces ranking functions and the tropical perspective. Sections 3 and 4 present the agreement theorems for common knowledge and for common belief with firmness f. Section 5 defines tropical expectations and proves the no-betting theorem under a common prior. Section 6 shows that its converse fails. Sections 7 and 8 present the two routes to a characterization. Section 9 studies learning to agree, and Section 10 turns to non-partitional information structures. Section 11 concludes with a discussion. The appendix contains the proofs of Theorem 6 and Lemma 7.

1.1 Related Literature

As Yang and Brandenburger (2026) nicely summarize, Aumann's (1976) theorem rests on (i) a common prior, (ii) information partitions, (iii) a positivity condition on possibility sets, and (iv) the knowledge modality. Each has been relaxed: the common prior has been localized to the true state and characterized by agreement properties (Bonanno and Nehring, 1999) and relaxed to an approximate common prior (Hellman, 2013); partitions by Geanakoplos (2021), Samet (1990, 2022), and Rubinstein and Wolinsky (1990); positivity by Nielsen (1984) and Brandenburger and Dekel (1987); and knowledge by Brandenburger and Dekel (1987) and Monderer and Samet (1989), who use (common) certainty and (common) p-belief, and by Heifetz, Meier, and Schipper (2013) and Galanis (2018), who allow for unawareness. Our paper changes a more basic ingredient, namely the representation of belief itself, from probabilities to ranking functions, and then revisits (i) to (iv).

Ranking functions are closely related to conditional probability systems, Popper measures, and lexicographic probability systems, which essentially coincide when the latter have disjoint supports, and for which agreement theorems have been obtained by Tsakas (2018), Bach and Perea (2013), Bach and Cabessa (2023), and Yang and Brandenburger (2026). Spohn (1986) shows that every σ-additive Popper measure is generated by a unique well-ordered family of probability measures, in the sense that the probability of an event conditional on B is computed with the first measure that assigns positive probability to B (see also Rényi, 1956). The index of that measure satisfies the law of disjunction so that conditional rankings as used in our paper can be viewed as qualitative analogs of conditional probability systems. The correspondence is not one-to-one, however. Ranks carry cardinal information, namely the size of the gaps between levels, which Spohn's conditioning rule preserves but conditioning a Popper measure does not; belief with firmness f in Section 4 and tropical expectations in Section 5 use this information. Conversely, common knowledge of posterior ranks is a weaker hypothesis than common knowledge of posterior conditional probabilities, so Theorem 1 does not follow from agreement theorems for conditional probabilities.

A further strand replaces posteriors by abstract decisions satisfying a sure-thing principle (Cave, 1983; Bacharach, 1985; Rubinstein and Wolinsky, 1990; Samet, 2010, 2022), or proves agreement theorems for qualitative plausibility orders, in which an agent believes an event if it holds at all most plausible states of her information set (Dégremont and Roy, 2012). Most closely related is Pacuit and Yang (2025), who use conditional plausibility measures (Friedman and Halpern, 1995; Halpern, 2001) to identify conditions under which models of belief satisfy analogs of the agreement results for common p-belief of Monderer and Samet (1989) and Neeman (1996).[1] Ranking functions can be considered as a special case of conditional plausibility measures with the order reversed. Pacuit and Yang (2026a, Theorem 4.7) also characterize Aumann's theorem for conditional plausibility measures and note in passing (Appendix A.6) that ranking functions satisfy the condition of this characterization. We prove the resulting agreement theorem explicitly. Pacuit and Yang also consider whether ranking functions admit an analog of the agreement results for common p-belief of Monderer and Samet (1989) and Neeman (1996). For each of the three ways of measuring how far

[1] I thank Leo Yang for bringing Pacuit and Yang (2025, 2026a, b) to my attention in summer 2026 after I shared my earlier draft with Theorem 1.

apart posterior ranks are that they consider, the resulting statement is either vacuous or false (Pacuit and Yang, 2026a, Appendix A.6). Yet, for ranking functions their notion of d-belief in an event means only that the event is not disbelieved beyond d. This is not belief in the event, since an event and its complement can both have rank 0. With Spohn's (2012) notion of belief with firmness f, which requires the complement to be disbelieved at least to degree f, our Theorem 2 provides a non-trivial analog of these results for ranking functions (see Section 4). Pacuit and Yang (2026b) study approximate agreement for probabilistic beliefs with non-partitional information. In Section 10, we show that, in contrast to the probabilistic case, agreement on posterior ranks and the absence of commonly known speculation under a common prior require only that agents do not know false events.

Finally, our no-betting results relate to the literature on speculative trade. Beyond the classic no-trade theorems (Milgrom and Stokey, 1982; Sebenius and Geanakoplos, 1983), Morris (1994) and Gizatulina and Hellman (2019) study trade with heterogeneous priors, and Heifetz, Meier, and Schipper (2013) and Galanis (2018) study speculation under unawareness. In the spirit of the weakenings of the common prior studied by Bonanno and Nehring (1999) and Hellman (2013), our Theorem 4 identifies the exact compatibility condition between posterior ranks that rules out speculative betting. It is related to the literature on trade under ambiguity (Billot et al., 2000; Rigotti, Shannon, and Strzalecki, 2008; Kajii and Ui, 2009; Martins-da-Rocha, 2010), which characterizes when a given allocation admits no trade. Our Theorems 5 and 7 characterize a common prior by a condition on posterior ranks and by the absence of speculative betting from any common endowment, respectively. The former result combines the idea of a cycle condition in the spirit of Rodrigues-Neto (2009), Hellman and Samet (2012), and Hellwig (2013) with ideas from potential games of Monderer and Shapley (1996). Our results on learning to agree build on Geanakoplos and Polemarchakis (1982) and Cave (1983), and our example of a market crash without external shocks adapts Hart and Tauman (2004).

2 Model

Let Ω be a finite set of states. We take the power set 2^Ω as the set of events. Denote the complement of E relative to Ω by $\neg E := \Omega \setminus E$.

Definition 1 (Ranking function)**.** *A ranking function* $\rho : 2^\Omega \longrightarrow \mathbb{N} \cup \{\infty\}$ *satisfies*

$$\begin{aligned} \rho(\Omega) &= 0, && (1)\\ \rho(\emptyset) &= \infty, && (2)\\ \rho(E \cup F) &= \min\{\rho(E), \rho(F)\} \text{ for any events } E, F \in 2^\Omega. && (3) \end{aligned}$$

For any event $E \in 2^\Omega$, $\rho(E)$ is the degree of disbelief in E. Property (3) is the law of disjunction for ranks (Spohn, 2012), also called finite minimitivity; it plays the role of finite additivity for probabilities. Similar to probabilities, a ranking function satisfies monotonicity in the sense that for any $E, F \in 2^\Omega$, $E \subseteq F$ implies $\rho(E) \geq \rho(F)$. This follows from property (3) since $E \subseteq F$ implies $F = E \cup F$, which implies $\rho(F) = \rho(E \cup F) = \min\{\rho(E), \rho(F)\} \leq \rho(E)$. Another consequence of property (3) is that ranking functions are completely determined by their values on singleton subsets of Ω. See Spohn (2012, Chapter 5.2) for an exposition of

ranking functions. Throughout, we use the conventions $\infty + x = \infty$ and $\infty - x = \infty$ for $x \in \mathbb{R}$.

Definition 2 (Conditional ranking function)**.** *Given a ranking function $\rho : 2^\Omega \longrightarrow \mathbb{N} \cup \{\infty\}$ and any $E \in 2^\Omega$ with $\rho(E) < \infty$, the conditional ranking function $\rho(\cdot \mid E) : 2^\Omega \longrightarrow \mathbb{N} \cup \{\infty\}$ is defined by, for all $F \in 2^\Omega$,*

$$\rho(F \mid E) \;:=\; \rho(F \cap E) - \rho(E).$$

We denote by $\mathcal{D}_\rho := \{E \in 2^\Omega : \rho(E) < \infty\}$ the set of events of finite rank, i.e., the events on which ρ can be conditioned. See Spohn (2012, Chapter 5.3) for an exposition of conditional ranking functions.

An agent with ranking function ρ *believes* an event F if she disbelieves its complement, i.e., if $\rho(\neg F) > 0$. Thus, the belief set of ρ is $\{F \in 2^\Omega : \rho(\neg F) > 0\}$. Note that by properties (1) and (3), $\min\{\rho(F), \rho(\neg F)\} = 0$. Thus, the agent never disbelieves both an event and its complement. Either she believes F, or she believes $\neg F$, or she is agnostic about F by believing neither when $\rho(F) = \rho(\neg F) = 0$. The latter case occurs when both F and $\neg F$ contain a state of rank 0. Note that she believes F if and only if F contains every state of rank 0. Hence beliefs are consistent, i.e., all believed events have a state in common, namely any state of rank 0, so that in particular the agent never believes both an event and its complement. Moreover, beliefs are closed under conjunction, i.e., if the agent believes F and G, then she also believes $F \cap G$. Beliefs held with probability one have the same two properties. Using the interpretation by infinitesimal probabilities sketched in the introduction, believing F means that $\neg F$ is infinitely less likely than F, so that F has probability one up to an infinitesimal. Yet, unlike probability-one beliefs, such beliefs are defeasible, i.e., revisable. If $\neg F$ has finite rank, the agent can condition on it or on any other event with finite rank. After learning E, she believes F if and only if $\rho(\neg F \mid E) > 0$, i.e., if and only if F contains every state of rank 0 in E. Conditioning thus plays the role of Bayesian conditioning and remains well defined when the agent learns something she previously disbelieved, in which case she revises her beliefs.

Ranks also measure how firmly an event is believed. If $\rho(\neg F) \geq f$, the agent believes F with firmness at least f (Spohn, 2012). Such a belief survives learning any event E with $\rho(E) < f$, since then $\rho(\neg F \mid E) \geq \rho(\neg F) - \rho(E) > 0$ by monotonicity. The larger f, the more surprising the information needed to overturn the belief. In this respect, firmness f plays a role similar to the threshold p in the p-belief of Monderer and Samet (1989). Sections 3 and 4 apply belief with firmness at least f to posterior ranking functions. Unlike p-belief for $p < 1$, however, belief with firmness at least f is closed under conjunction, since $\rho(\neg(F \cap G)) = \min\{\rho(\neg F), \rho(\neg G)\}$. If $\rho(\neg F) = \infty$, the agent is certain of F, and no information she can condition on overturns this belief, as with probability-one beliefs under Bayes' rule.

The parallels between ranks and probabilities drawn so far are not accidental. They reflect a common algebraic structure, which underlies all our results and which we now make explicit. The tropical semiring is $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with $x \oplus y := \min\{x, y\}$ and $x \odot y := x + y$. Its additive neutral element ("tropical zero") is ∞, and its multiplicative neutral element ("tropical one") is 0 (Speyer and Sturmfels, 2009). Ranks take values in the subsemiring $\mathbb{N} \cup \{\infty\}$. In this language, properties (1)–(3) say that a ranking function is a normalized

tropical measure, i.e., an idempotent probability measure in the sense of Maslov with values in $\mathbb{N} \cup \{\infty\}$ (see Kolokoltsov, 2001, and Del Moral and Doisy, 1999, for surveys): $\rho(\Omega)$ is the tropical one, $\rho(\emptyset)$ is the tropical zero, and $\rho(E \cup F) = \rho(E) \oplus \rho(F) = \min\{\rho(E), \rho(F)\}$, so that $\rho(E) = \bigoplus_{\omega \in E} \rho(\{\omega\}) = \min_{\omega \in E} \rho(\{\omega\})$, just as $p(E) = \sum_{\omega \in E} p(\omega)$. Conditioning is defined by $\rho(F \mid E) \odot \rho(E) = \rho(F \mid E) + \rho(E) = \rho(F \cap E)$, the tropical analog of $p(F \mid E) \cdot p(E) = p(F \cap E)$; tropical division by a finite element is ordinary subtraction. Spohn (1988, pp. 116 and 126; 2009, Section 3.1) describes the same translation from probability theory to ranking theory, and the interpretation of ranks as orders of magnitude of infinitesimal probabilities makes it precise: if $p_\varepsilon(\omega) \propto \varepsilon^{\rho(\{\omega\})}$, then $\log_\varepsilon$ turns sums of probabilities into minima of ranks and products into sums as $\varepsilon \to 0$. While the min-plus structure of ranking functions is well known (e.g., Spohn, 1988; Halpern, 2003), we are not aware of work other than Schipper and Kern-Isberner (2026) that connects ranking theory to tropical mathematics and idempotent analysis and uses this connection to derive and explain results.

There are two features of tropical arithmetic that drive the differences between our results and their probabilistic counterparts. First, tropical addition, the minimum, is an idempotent operation, i.e., $x \oplus x = \min\{x, x\} = x$. A ranking function is called an idempotent measure because it is additive with respect to this idempotent operation. I.e., by the law of disjunction, property (3), $\rho(E \cup F) = \rho(E) \oplus \rho(F) = \min\{\rho(E), \rho(F)\}$. Since $\oplus$ is idempotent, this holds for any events E and F, even if they overlap. There is no need to correct for double counting as in the probabilistic case, where $p(E \cup F) = p(E) + p(F) - p(E \cap F)$. This matters for how posteriors combine. Consider two disjoint events P and Q of finite rank. In the probabilistic case, the posterior given $P \cup Q$ is a weighted average of the posteriors given P and given Q with weights $p(P \mid P \cup Q)$ and $p(Q \mid P \cup Q)$. Thus, it lies between them. For ranks, we have

$$\rho(E \mid P \cup Q) = \min\{\rho(E \mid P) + \rho(P \mid P \cup Q),\ \rho(E \mid Q) + \rho(Q \mid P \cup Q)\}.$$

That is, the posterior rank given $P \cup Q$ is a tropical weighted average of the posterior ranks given P and given Q with nonnegative weights, at least one of which is 0. It also lies between $\rho(E \mid P)$ and $\rho(E \mid Q)$. Yet, due to idempotency, this holds even if P and Q overlap. For probabilities, it may fail for overlapping events.[2] In particular, if the posterior ranks given P and given Q coincide, then so does the posterior rank given $P \cup Q$. This property is at the heart of our agreement results in Section 3 and their extension to non-partitional information structures in Section 10. Second, tropical addition has no inverse. That is, there is no tropical subtraction (Speyer and Sturmfels, 2009). Notions that are defined in probability theory via differences of probabilities have no direct tropical analog. For instance, Monderer and Samet (1989) measure approximate agreement by the distance between posteriors. With ranks, such notions must be rebuilt, as we discuss in Section 4, where agreement is measured by truncating posterior ranks at f.

So far, we considered the ranking function of a single agent. To study agreement, we need multiple agents who share the same background but receive different private information. We follow Aumann (1976) and model information by partitions of the state space. That is,

[2]For instance, let $\Omega = \{a, b, c\}$ with equally likely states, $P = \{a, b\}$, $Q = \{b, c\}$, and $E = \{a, c\}$. Then $p(E \mid P) = p(E \mid Q) = 1/2$, but $p(E \mid P \cup Q) = 2/3$.

at each state, an agent knows only that the true state lies in a certain subset of states, her possibility set at that state.

Definition 3 (Aumann structure)**.** *A finite Aumann structure* $\langle I, \Omega, (\Pi_i)_{i\in I}\rangle$ *is defined by*

- *a nonempty finite set of agents* $I = \{1, \ldots, n\}$*,*
- *a nonempty finite set of states* Ω*,*

and for each agent $i \in I$*, a possibility correspondence* $\Pi_i : \Omega \longrightarrow 2^\Omega$ *that forms a partition of* Ω *by satisfying for all* $\omega \in \Omega$*,*

(i) Reflexivity: $\omega \in \Pi_i(\omega)$*,*

(ii) Stationarity: $\omega' \in \Pi_i(\omega)$ *implies* $\Pi_i(\omega') = \Pi_i(\omega)$*.*

An Aumann structure is equivalent to a Kripke frame with a partitional accessibility relation for each agent (i.e., S5 knowledge). We call a set $\Pi_i(\omega)$ agent i's possibility set at ω.

As in Aumann (1999), define knowledge, mutual knowledge, and common knowledge as follows:

Definition 4. *For each agent* $i \in I$*, the knowledge operator* $K_i : 2^\Omega \longrightarrow 2^\Omega$ *is defined by, for any event* $E \in 2^\Omega$*,*

$$K_i(E) := \{\omega \in \Omega : \Pi_i(\omega) \subseteq E\}.$$

The mutual knowledge (or everybody knows) operator is defined by

$$K(E) := \bigcap_{i\in I} K_i(E).$$

The common knowledge operator is defined by

$$CK(E) := \bigcap_{\ell=1}^{\infty} K^\ell(E).$$

Let $\Pi_* : \Omega \longrightarrow 2^\Omega$ be the partitional correspondence that forms the meet (i.e., the finest common coarsening) of the partitions induced by Π_i for all $i \in I$. Formally, Π_* satisfies reflexivity and stationarity as in Definition 3, and

(a) $\Pi_i(\omega) \subseteq \Pi_*(\omega)$ for all $i \in I$ and $\omega \in \Omega$, and

(b) for every correspondence $\Pi' : \Omega \longrightarrow 2^\Omega$ that satisfies reflexivity and stationarity and $\Pi_i(\omega) \subseteq \Pi'(\omega)$ for all $i \in I$ and $\omega \in \Omega$, we have $\Pi_*(\omega) \subseteq \Pi'(\omega)$ for all $\omega \in \Omega$.

That is, Π_* is coarser than each Π_i by (a), and it is the finest such partitional correspondence by (b). Equivalently, $\omega' \in \Pi_*(\omega)$ if and only if there is a finite sequence $\omega = \omega_0, \omega_1, \ldots, \omega_m = \omega'$ such that for each $t = 1, \ldots, m$, $\omega_t \in \Pi_{i_t}(\omega_{t-1})$ for some agent $i_t \in I$. The *join* of the agents' partitions is the partition whose cells are the sets $\Pi_1(\omega) \cap \cdots \cap \Pi_n(\omega)$ with $\omega \in \Omega$. Aumann (1976) proved that for any event $E \in 2^\Omega$,

$$CK(E) = \{\omega \in \Omega : \Pi_*(\omega) \subseteq E\}.$$

The following assumption is analogous to Aumann's common prior assumption:

Assumption 1. *Assume a common prior disbelief given by a ranking function* $\rho : 2^\Omega \longrightarrow \mathbb{N} \cup \{\infty\}$. *Moreover, require that no possibility set is ex ante maximally common prior disbelieved, i.e., for all* $i \in I$ *and all* $\omega \in \Omega$, $\rho(\Pi_i(\omega)) < \infty$.

The second part of the assumption is the analog of the requirement in Aumann (1976) that every possibility set has positive prior probability. Note that it is weaker than requiring that every state has finite rank; individual states may be maximally disbelieved. By property (3), it implies $\rho(\Pi_*(\omega)) < \infty$ for all $\omega \in \Omega$.

For an event $E \in 2^\Omega$, agents $i \in I$, and numbers $\kappa_i(E) \in \mathbb{N} \cup \{\infty\}$ for $i \in I$, define events

$$[\kappa_i(E)] := \{\omega \in \Omega : \rho(E \mid \Pi_i(\omega)) = \kappa_i(E)\}.$$

This is the event in which agent i's interim disbelief rank of event E is $\kappa_i(E)$. Note that $[\kappa_i(E)]$ is a union of possibility sets of agent i.

3 Agreement

The following theorem is the extension of Aumann's (1976) "No agreeing to disagree" theorem to ranking functions:

Theorem 1. *Assumption 1 implies that if* $CK\left(\bigcap_{i\in I}[\kappa_i(E)]\right) \neq \emptyset$, *then* $\kappa_1(E) = \kappa_2(E) = \cdots = \kappa_n(E)$.

We give three proofs. The first is self-contained and mirrors Aumann's (1976) original argument. The second reduces the theorem to the general agreement results of Cave (1983) and Bacharach (1985), which shows that the only ranking-theoretic ingredient is a union-consistency property of conditional ranks. The third uses Samet's (2022) characterization of the impossibility of agreeing to disagree by an extended sure-thing principle, which avoids decision functions on events that are not an agent's information. The proofs serve different audiences. The first is suited to courses on ranking functions or applications of tropical mathematics: it uses only the axioms of ranking functions and Aumann's characterization of common knowledge, and it shows how the classical argument is tropicalized. The second is suited to a game-theoretic lecture on agreement theorems: it subsumes the result under the approach of Cave, Bacharach, and Rubinstein and Wolinsky, with Lemma 1 as the key step. The third addresses the same audience and is formulated purely in terms of the agents' knowledge, in line with the critique of the Cave-Bacharach approach by Moses and Nachum (1990) and Samet (2010, 2022).

Proof 1 of Theorem 1 (self-contained). Since $CK\left(\bigcap_{i\in I}[\kappa_i(E)]\right) \neq \emptyset$, there exists $\omega \in CK\left(\bigcap_{i\in I}[\kappa_i(E)]\right)$. By Aumann's (1976) characterization of common knowledge, $CK(G) = \{\omega' \in \Omega : \Pi_*(\omega') \subseteq G\}$ for every event G. Hence $\Pi_*(\omega) \subseteq \bigcap_{i\in I}[\kappa_i(E)]$.

Since Π_* represents the meet of the partitions represented by Π_i, $i \in I$, for each agent $i \in I$,

$$\Pi_*(\omega) \;=\; \bigcup_{\omega' \in \Pi_*(\omega)} \Pi_i(\omega'). \tag{4}$$

Indeed, let $\omega' \in \Pi_*(\omega)$. By stationarity of Π_*, $\Pi_*(\omega') = \Pi_*(\omega)$. By property (a) of the meet of the agents' partitions, $\Pi_i(\omega') \subseteq \Pi_*(\omega') = \Pi_*(\omega)$. Hence the right-hand side of equation (4) is contained in the left-hand side. Conversely, every $\omega' \in \Pi_*(\omega)$ lies in $\Pi_i(\omega')$ by reflexivity. Hence the left-hand side is contained in the right-hand side. Since Ω is finite, the union in equation (4) is a union of finitely many distinct possibility sets of agent i.

Fix $i \in I$ and $\omega' \in \Pi_*(\omega)$. Since $\omega' \in [\kappa_i(E)]$, we have $\rho(E \mid \Pi_i(\omega')) = \kappa_i(E)$. By Assumption 1, $\rho(\Pi_i(\omega')) < \infty$, so Definition 2 yields $\rho(E \cap \Pi_i(\omega')) - \rho(\Pi_i(\omega')) = \kappa_i(E)$. Adding the finite number $\rho(\Pi_i(\omega'))$ to both sides yields

$$\rho(E \cap \Pi_i(\omega')) \;=\; \kappa_i(E) + \rho(\Pi_i(\omega')).$$

If $\kappa_i(E) = \infty$, this equation also holds. In this case $\rho(E \cap \Pi_i(\omega')) = \infty$, since otherwise the difference $\rho(E \cap \Pi_i(\omega')) - \rho(\Pi_i(\omega'))$ would be finite, and both sides equal ∞. Taking the minimum over the possibility sets $\{\Pi_i(\omega')\}_{\omega' \in \Pi_*(\omega)}$ (i.e., "tropical addition") yields

$$\begin{aligned}
\min_{\omega' \in \Pi_*(\omega)} \rho(E \cap \Pi_i(\omega')) &= \kappa_i(E) + \min_{\omega' \in \Pi_*(\omega)} \rho(\Pi_i(\omega')) \\
\rho\Big(E \cap \bigcup_{\omega' \in \Pi_*(\omega)} \Pi_i(\omega')\Big) &= \kappa_i(E) + \rho\Big(\bigcup_{\omega' \in \Pi_*(\omega)} \Pi_i(\omega')\Big) \\
\rho(E \cap \Pi_*(\omega)) &= \kappa_i(E) + \rho(\Pi_*(\omega)) \\
\kappa_i(E) &= \rho(E \mid \Pi_*(\omega)). \qquad (5)
\end{aligned}$$

The first line holds since adding the same number $\kappa_i(E)$ to every term does not change which term is smallest. If $\kappa_i(E) = \infty$, both sides of the first line equal ∞. The second line follows from $E \cap \bigcup_{\omega'} \Pi_i(\omega') = \bigcup_{\omega'} (E \cap \Pi_i(\omega'))$ and from property (3), applied finitely often, which says that the rank of a finite union is the minimum of the ranks. A possibility set that occurs for several ω' changes neither the union nor the minimum. The third line follows from equation (4). For the last line, note that $\Pi_i(\omega) \subseteq \Pi_*(\omega)$ by property (a), so monotonicity and Assumption 1 yield $\rho(\Pi_*(\omega)) \leq \rho(\Pi_i(\omega)) < \infty$. Subtracting the finite number $\rho(\Pi_*(\omega))$ from both sides of the third line and using Definition 2 yields the last line. The right-hand side of equation (5) does not depend on i. Since i was arbitrary, $\kappa_i(E) = \rho(E \mid \Pi_*(\omega)) = \kappa_j(E)$ for all $i, j \in I$. □

For the second proof, we use the notion of union consistency from the Cave-Bacharach approach. Recall that $\mathcal{D}_\rho$ is the set of events of finite rank, which is closed under finite unions by property (3). A *decision function* is a function $\delta : \mathcal{D}_\rho \longrightarrow X$ into some set X of decisions. If an agent uses δ, then her decision at state ω is $\delta(\Pi_i(\omega))$. That is, her decision depends on the state only through her information. A decision function δ is *union consistent* if for all disjoint $P, Q \in \mathcal{D}_\rho$, $\delta(P) = \delta(Q) = x$ implies $\delta(P \cup Q) = x$. Cave (1983) and Bacharach (1985) show that if all agents use the same union-consistent decision function ("like-mindedness") and their decisions are common knowledge, then their decisions agree. Rubinstein and Wolinsky (1990, Proposition 1) generalize this result by replacing decisions with properties of events that may differ across agents: if each agent's property is preserved under disjoint unions and no event has the properties of two agents simultaneously, then it cannot be common knowledge that each agent's information has her property.[3]

[3]Union consistency is Cave's (1983) term. Bacharach (1985) imposes the corresponding condition for

Lemma 1. *Let ρ be a ranking function. For every $E \in 2^\Omega$, the function $\delta_E : \mathcal{D}_\rho \longrightarrow \mathbb{N} \cup \{\infty\}$ defined by $\delta_E(P) := \rho(E \mid P)$ is union consistent.*

PROOF. Let $P, Q \in \mathcal{D}_\rho$ be disjoint, and suppose that $\rho(E \mid P) = \rho(E \mid Q) = \kappa$ for some $\kappa \in \mathbb{N} \cup \{\infty\}$. We show that $\rho(E \mid P \cup Q) = \kappa$. Note that

$$\begin{aligned}
\rho(E \cap (P \cup Q)) &= \rho((E \cap P) \cup (E \cap Q)) \\
&= \min\{\rho(E \cap P), \rho(E \cap Q)\} \\
&= \min\{\rho(E \mid P) + \rho(P), \rho(E \mid Q) + \rho(Q)\} \\
&= \min\{\kappa + \rho(P), \kappa + \rho(Q)\} \\
&= \kappa + \min\{\rho(P), \rho(Q)\} \\
&= \kappa + \rho(P \cup Q),
\end{aligned}$$

where the first equality holds since intersection distributes over union, and the second and the last equalities follow from property (3). The third equality follows from Definition 2 since $P, Q \in \mathcal{D}_\rho$. Indeed, $\rho(E \cap P) - \rho(P) = \rho(E \mid P)$, and adding the finite number $\rho(P)$ to both sides yields $\rho(E \cap P) = \rho(E \mid P) + \rho(P)$. If $\rho(E \mid P) = \infty$, then $\rho(E \cap P) = \infty$, and both sides equal ∞. The same holds for Q. The fourth equality follows from the supposition. The fifth equality holds since adding the same number κ to $\rho(P)$ and to $\rho(Q)$ does not change which of the two is smaller. If $\kappa = \infty$, both sides of the fifth equality equal ∞. Since $P, Q \in \mathcal{D}_\rho$, property (3) implies $\rho(P \cup Q) = \min\{\rho(P), \rho(Q)\} < \infty$. Hence Definition 2 yields $\rho(E \mid P \cup Q) = \rho(E \cap (P \cup Q)) - \rho(P \cup Q) = \kappa + \rho(P \cup Q) - \rho(P \cup Q) = \kappa$, where for $\kappa = \infty$ we use the convention $\infty - n = \infty$. □

PROOF 2 OF THEOREM 1 (VIA CAVE, 1983, BACHARACH, 1985, AND RUBINSTEIN AND WOLINSKY, 1990). Consider the decision function δ_E of Lemma 1. By Assumption 1, $\Pi_i(\omega) \in \mathcal{D}_\rho$ for all $i \in I$ and $\omega \in \Omega$. If agent i uses δ_E, then her decision at ω is $\delta_E(\Pi_i(\omega)) = \rho(E \mid \Pi_i(\omega))$, i.e., her posterior rank of E at ω. Since all agents use the same decision function, they are like-minded in the sense of Bacharach (1985). By Lemma 1, δ_E is union consistent. By the definition of $[\kappa_i(E)]$, the event $\bigcap_{i \in I}[\kappa_i(E)]$ is the event that each agent i's decision equals $\kappa_i(E)$. The agreement theorem of Cave (1983) and Bacharach (1985) then implies $\kappa_1(E) = \cdots = \kappa_n(E)$ whenever this event is common knowledge at some state. Equivalently, in terms of Rubinstein and Wolinsky (1990), the properties "$\rho(E \mid F) = \kappa_i(E)$" of events F are preserved under disjoint unions by Lemma 1. Moreover, no event has two of them with different values, since $\rho(E \mid X)$ is a single number. While Cave (1983) and Bacharach (1985) require a decision function to be defined on all nonempty events, our δ_E is defined only on $\mathcal{D}_\rho$. To apply their result literally, extend δ_E by $\delta_E(F) := \spadesuit$ for $F \notin \mathcal{D}_\rho$, where $\spadesuit$ is a symbol distinct from all ranks. The extension remains union consistent. To see this, let P and Q be disjoint nonempty events with $\delta_E(P) = \delta_E(Q) = x$. There are

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arbitrary partitions of an event and calls it the sure thing condition, a form of Savage's sure-thing principle. For critiques of the interpretation of this condition, see Moses and Nachum (1990) and Samet (2010, 2022). Their objection is that union consistency requires the decision function to be defined on unions of possibility sets that are not themselves anybody's possibility set. Here, $\delta_E(P) = \rho(E \mid P)$ is defined directly from the common prior for every $P \in \mathcal{D}_\rho$, so the objection loses its force, just as in Aumann's (1976) probabilistic setting.

three cases. First, if $P, Q \in \mathcal{D}_\rho$, then $\delta_E(P \cup Q) = x$ by Lemma 1. Second, if $P, Q \notin \mathcal{D}_\rho$, then $x = \spadesuit$. By property (3), $\rho(P \cup Q) = \min\{\rho(P), \rho(Q)\} = \infty$, so $P \cup Q \notin \mathcal{D}_\rho$ and $\delta_E(P \cup Q) = \spadesuit = x$. Third, if exactly one of P and Q lies in $\mathcal{D}_\rho$, then one of $\delta_E(P)$ and $\delta_E(Q)$ is a rank and the other is $\spadesuit$. Then $\delta_E(P) \neq \delta_E(Q)$, so this case does not occur. By Assumption 1, δ_E does not take the value $\spadesuit$ at any possibility set, so the agents' decisions are unaffected. □

Proof 3 of Theorem 1 (via Samet, 2022). Samet (2022) considers decision functions $d_i : \Omega \longrightarrow X$ that depend on the state only through agent i's possibility set. For two agents, he shows that the impossibility of agreeing to disagree, i.e., $CK([d_i = x_i] \cap [d_j = x_j]) \neq \emptyset$ implies $x_i = x_j$, is equivalent to his extended possibility sure-thing principle: for every nonempty event G that is a union of possibility sets of agent i and a union of possibility sets of agent j, if $d_i \equiv x_i$ and $d_j \equiv x_j$ on G, then $x_i = x_j$ (Samet, 2022, Theorem 2). Let $d_i(\omega) := \rho(E \mid \Pi_i(\omega))$ for all $i \in I$ and $\omega \in \Omega$. By stationarity, d_i depends on the state only through $\Pi_i(\omega)$. Moreover, $[d_i = \kappa_i(E)] = [\kappa_i(E)]$ by the definition of $[\kappa_i(E)]$.

We verify the principle. Fix $i \neq j$ and let G be as in the principle, so that $d_i(\omega) = x_i$ and $d_j(\omega) = x_j$ for all $\omega \in G$. We show that $\rho(E \mid G) = x_i$. Since Ω is finite and G is a nonempty union of possibility sets of agent i, we can write $G = C_1 \cup \cdots \cup C_k$ with $k \geq 1$ and distinct possibility sets $C_1, \ldots, C_k$ of agent i. These sets are pairwise disjoint, since Π_i is partitional, and each has finite rank by Assumption 1. For each ℓ, pick $\omega \in C_\ell$. By stationarity, $\Pi_i(\omega) = C_\ell$, so $\rho(E \mid C_\ell) = d_i(\omega) = x_i$. We show by induction on ℓ that $\rho(E \mid C_1 \cup \cdots \cup C_\ell) = x_i$. For $\ell = 1$, this was just shown. Suppose it holds for some $\ell < k$. By property (3), the rank of $C_1 \cup \cdots \cup C_\ell$ is the minimum of finitely many finite ranks, so this event lies in $\mathcal{D}_\rho$. It is disjoint from $C_{\ell+1}$. Lemma 1 applied to $P := C_1 \cup \cdots \cup C_\ell$ and $Q := C_{\ell+1}$ yields $\rho(E \mid C_1 \cup \cdots \cup C_{\ell+1}) = x_i$. For $\ell = k$, we obtain $\rho(E \mid G) = x_i$. By the same argument for agent j, $\rho(E \mid G) = x_j$. Hence $x_i = x_j$. So the principle holds for every pair of agents, and by Samet's theorem the impossibility of agreeing to disagree holds for every pair.

Finally, suppose $\omega \in CK\left(\bigcap_{k \in I}[\kappa_k(E)]\right)$, and fix $i, j \in I$. By Aumann's (1976) characterization of common knowledge, $\Pi_*(\omega) \subseteq \bigcap_{k \in I}[\kappa_k(E)] \subseteq [\kappa_i(E)] \cap [\kappa_j(E)]$. Let Π_{ij} be the meet of Π_i and Π_j alone. The correspondence Π_* satisfies reflexivity and stationarity, and $\Pi_i(\omega') \subseteq \Pi_*(\omega')$ and $\Pi_j(\omega') \subseteq \Pi_*(\omega')$ for all $\omega' \in \Omega$ by property (a). Hence property (b) of the meet Π_{ij} yields $\Pi_{ij}(\omega) \subseteq \Pi_*(\omega)$. That is, the meet of Π_i and Π_j is finer than the meet of all partitions. Thus $\Pi_{ij}(\omega) \subseteq [\kappa_i(E)] \cap [\kappa_j(E)]$. By Aumann's characterization applied to the two agents i and j, the event $[\kappa_i(E)] \cap [\kappa_j(E)] = [d_i = \kappa_i(E)] \cap [d_j = \kappa_j(E)]$ is common knowledge between i and j at ω. By the impossibility of agreeing to disagree for the pair i, j, $\kappa_i(E) = \kappa_j(E)$. Since i and j were arbitrary, $\kappa_i(E) = \kappa_j(E)$ for all $i, j \in I$. □

By the same argument, posterior ranks also satisfy the interpersonal sure-thing principle of Samet (2010) and its expandability: if agent i knows that agent j is at least as knowledgeable, then $\Pi_i(\omega)$ is a union of possibility sets of agent j, and Lemma 1 transfers j's posterior rank to i. Hence Theorem 1 also follows from Samet's (2010) agreement theorem, just as Aumann's (1976) theorem does (Samet, 2010, Section 3.4).

The three proofs are closely related. The chain of equalities in Proof 1 is precisely a repeated application of Lemma 1 to the possibility sets of agent i that make up $\Pi_*(\omega)$, and

Proofs 2 and 3 apply Lemma 1 in the same way. Thus, the ranking-theoretic content of the theorem lies entirely in Lemma 1, the tropical counterpart of the fact that the posterior given a disjoint union is a weighted average of the posteriors given each part (see Section 2).

Remark 1 (Agreement is not information sharing)**.** *Equation (5) shows more than agreement. The commonly known posterior rank equals $\rho(E \mid \Pi_*(\omega))$, the rank conditional on what is commonly known at ω. That is, agents agree on the rank they would assign if they used only their common knowledge. This need not be the rank they would assign if they pooled their private information. For instance, let $\Omega = \{1,2,3,4\}$ with $\rho(\{\omega\}) = 0$ for all $\omega \in \Omega$, let Π_1 induce $\{\{1,2\},\{3,4\}\}$ and Π_2 induce $\{\{1,3\},\{2,4\}\}$, and let $E = \{2,3\}$. Both agents assign rank 0 to E at every state. This is common knowledge since the meet of the agents' partitions is $\{\Omega\}$, and indeed $\rho(E \mid \Omega) = 0$. Yet at state 1, the pooled information $\Pi_1(1) \cap \Pi_2(1) = \{1\}$ reveals that E is false.*

Conditional ranks satisfy a stronger property than union consistency. We say that a ranking function ρ is *conglomerable*[4] if for all $E \in 2^\Omega$ and all $P, Q \in \mathcal{D}_\rho$,

$$\min\{\rho(E \mid P), \rho(E \mid Q)\} \;\leq\; \rho(E \mid P \cup Q) \;\leq\; \max\{\rho(E \mid P), \rho(E \mid Q)\}.$$

Unlike in the probabilistic case, P and Q need not be disjoint.

Lemma 2 (Conglomerability)**.** *Every ranking function is conglomerable.*

Proof. Let $E \in 2^\Omega$ and $P, Q \in \mathcal{D}_\rho$. Let $a := \rho(E \mid P)$, $b := \rho(E \mid Q)$, $p := \rho(P)$, and $q := \rho(Q)$. Since the claim is symmetric in P and Q, we may assume without loss of generality that $p \leq q$. Since $P, Q \in \mathcal{D}_\rho$, both p and q are finite. By property (3), $\rho(P \cup Q) = \min\{p, q\} = p < \infty$. Moreover,

$$\begin{aligned} \rho(E \cap (P \cup Q)) &= \rho((E \cap P) \cup (E \cap Q)) \\ &= \min\{\rho(E \cap P), \rho(E \cap Q)\} \\ &= \min\{a + p, b + q\}, \end{aligned}$$

where the first equality holds since intersection distributes over union and the second equality follows from property (3). The third equality follows from Definition 2 since $P, Q \in \mathcal{D}_\rho$. Indeed, $\rho(E \cap P) - p = a$, and adding the finite number p to both sides yields $\rho(E \cap P) = a + p$. If $a = \infty$, then $\rho(E \cap P) = \infty$ and both sides equal ∞. In the same way, $\rho(E \cap Q) = b + q$. Hence, by Definition 2,

$$\begin{aligned} \rho(E \mid P \cup Q) &= \rho(E \cap (P \cup Q)) - \rho(P \cup Q) \\ &= \min\{a + p, b + q\} - p \\ &= \min\{a, b + (q - p)\}. \end{aligned}$$

[4]Following de Finetti (1972), a probability measure p is conglomerable with respect to a partition $\{B_k\}_k$ of an event B if $p(E \mid B_k) \in [a, b]$ for all k implies $p(E \mid B) \in [a, b]$. Since $p(E \mid B) = \sum_k p(E \mid B_k)\, p(B_k \mid B)$ is a weighted average of the $p(E \mid B_k)$, every probability measure is conglomerable with respect to finite partitions. For ranking functions, $\rho(E \mid P \cup Q)$ is a tropical weighted average of $\rho(E \mid P)$ and $\rho(E \mid Q)$ (see Section 2), and therefore lies between them.

The second equality uses $\rho(P \cup Q) = p$. The third equality holds since subtracting the finite number p from both terms does not change which term is smaller, and $(a + p) - p = a$ and $(b + q) - p = b + (q - p)$. If a or b is ∞, the corresponding terms equal ∞ on both sides. Since $p \leq q < \infty$, $q - p$ is a nonnegative integer. We now show the two inequalities. First, $b + (q - p) \geq b \geq \min\{a, b\}$ and $a \geq \min\{a, b\}$. So both terms in $\min\{a, b + (q - p)\}$ are at least $\min\{a, b\}$, and thus $\min\{a, b + (q - p)\} \geq \min\{a, b\}$. Second, the minimum of two numbers is weakly smaller than each of them, so $\min\{a, b + (q - p)\} \leq a \leq \max\{a, b\}$. These comparisons hold in $\mathbb{N} \cup \{\infty\}$, where ∞ is larger than every integer. □

Conglomerability implies union consistency of posterior ranks, i.e., Lemma 1: if $\rho(E \mid P) = \rho(E \mid Q) = x$, then $x \leq \rho(E \mid P \cup Q) \leq x$, so $\rho(E \mid P \cup Q) = x$. In words, if the posterior ranks given P and given Q coincide, then so does the posterior rank given $P \cup Q$. The converse does not hold for general functions: a union-consistent function need not take a value on $P \cup Q$ between its values on P and on Q. For instance, let $\Omega = \{1, 2\}$ and $\rho(\{\omega\}) = 0$ for all $\omega \in \Omega$, so that $\mathcal{D}_\rho$ consists of all nonempty events, and let $\delta(\{1\}) = 0$, $\delta(\{2\}) = 1$, and $\delta(\{1, 2\}) = 5$. Then δ is union consistent, since no two disjoint events have the same value, but $\delta(\{1, 2\})$ does not lie between $\delta(\{1\})$ and $\delta(\{2\})$.

By induction, Lemma 2 extends to finite unions: $\rho(E \mid \bigcup_k P_k)$ lies between $\min_k \rho(E \mid P_k)$ and $\max_k \rho(E \mid P_k)$. Moreover, by Lemma 2, union consistency, even for unions that are not disjoint, holds not only for the posterior rank itself but also for any function that records only which of finitely many given intervals the posterior rank lies in, since an interval contains every number between two of its points. For instance, belief with firmness at least f is preserved under unions in the following sense.

Lemma 3. *Let ρ be a ranking function, $E \in 2^\Omega$, $f \in \{1, 2, \ldots\} \cup \{\infty\}$, and $P, Q \in \mathcal{D}_\rho$. If $\rho(\neg E \mid P) \geq f$ and $\rho(\neg E \mid Q) \geq f$, then $\rho(\neg E \mid P \cup Q) \geq f$. If $\rho(\neg E \mid P) < f$ and $\rho(\neg E \mid Q) < f$, then $\rho(\neg E \mid P \cup Q) < f$.*

Proof. By Lemma 2, $\rho(\neg E \mid P \cup Q)$ lies between $\rho(\neg E \mid P)$ and $\rho(\neg E \mid Q)$. □

This yields an agreement result on qualitative beliefs. We apply the notion from Section 2 to posterior ranks.

Definition 5 (Belief with firmness at least f)**.** *Agent i* believes E with firmness at least f *at ω if $\rho(\neg E \mid \Pi_i(\omega)) \geq f$.*

Corollary 1 (No agreeing to disagree on beliefs)**.** *Under Assumption 1, for any $E \in 2^\Omega$, $f \in \{1, 2, \ldots\} \cup \{\infty\}$, and $i, j \in I$, it cannot be common knowledge that agent i believes E with firmness at least f while agent j does not. In particular ($f = 1$), agents cannot agree to disagree on whether E is believed.*

Proof. For each agent $k \in I$, define $F_k := \{\omega' \in \Omega : \rho(\neg E \mid \Pi_k(\omega')) \geq f\}$, the event that agent k believes E with firmness at least f. We show that $CK(F_i \cap \neg F_j) = \emptyset$. Suppose to the contrary that there is $\omega \in CK(F_i \cap \neg F_j)$, and let $M := \Pi_*(\omega)$. Then $M \subseteq F_i \cap \neg F_j$ by Aumann's (1976) characterization of common knowledge. By Assumption 1, every possibility set has finite rank. Since $\Pi_i(\omega) \subseteq M$ by property (a), monotonicity yields $\rho(M) \leq \rho(\Pi_i(\omega)) < \infty$.

First, consider agent i. By equation (4), $M = C_1 \cup \cdots \cup C_m$, where $C_1, \ldots, C_m$ are the distinct sets $\Pi_i(\omega')$ with $\omega' \in M$. There are finitely many such sets since Ω is finite. Each C_ℓ equals $\Pi_i(\omega')$ for some $\omega' \in M \subseteq F_i$, so $\rho(\neg E \mid C_\ell) \geq f$. We show by induction on ℓ that $\rho(\neg E \mid C_1 \cup \cdots \cup C_\ell) \geq f$ for $\ell = 1, \ldots, m$. For $\ell = 1$, this was just shown. Suppose that it holds for some $\ell < m$, and let $P := C_1 \cup \cdots \cup C_\ell$ and $Q := C_{\ell+1}$. Then P has finite rank by property (3), P and Q are disjoint since distinct possibility sets of agent i are disjoint, $\rho(\neg E \mid P) \geq f$ by the induction hypothesis, and $\rho(\neg E \mid Q) \geq f$. By the first part of Lemma 3, $\rho(\neg E \mid P \cup Q) \geq f$, which completes the induction. For $\ell = m$, we obtain $\rho(\neg E \mid M) \geq f$.

Second, consider agent j. By equation (4), $M = D_1 \cup \cdots \cup D_{m'}$, where $D_1, \ldots, D_{m'}$ are the distinct sets $\Pi_j(\omega')$ with $\omega' \in M$. Each D_ℓ equals $\Pi_j(\omega')$ for some $\omega' \in M \subseteq \neg F_j$, so $\rho(\neg E \mid D_\ell) < f$. The same induction, now with the second part of Lemma 3, yields $\rho(\neg E \mid D_1 \cup \cdots \cup D_\ell) < f$ for $\ell = 1, \ldots, m'$, and in particular $\rho(\neg E \mid M) < f$.

Thus $\rho(\neg E \mid M) \geq f$ and $\rho(\neg E \mid M) < f$, a contradiction. □

In probability theory, the analogous statement, namely that it cannot be common knowledge that one agent assigns probability at least p to E while another assigns probability less than p, is a no-betting statement: the first agent weakly prefers to buy a ticket paying 1 if E occurs at price p, while the second strictly prefers to sell it. With ranking functions, this equivalence breaks down because the tropical expectation of $-X$ is not the negative of the tropical expectation of X. Indeed, Corollary 1 relies on a common prior, whereas, as we will see in Section 7, the absence of commonly known acceptable bets does not.

For $f = 1$, Corollary 1 is related to the agreement theorem of Dégremont and Roy (2012) for epistemic plausibility models. In these models, each agent has an information partition and a total plausibility preorder on states, and she believes an event if it holds at all most plausible states of her information cell. Dégremont and Roy (2012, Corollary 3.2) show that if all agents share the same well-founded plausibility order, it cannot be common knowledge that one agent believes an event while another does not. A common prior ranking function induces such an order, namely $\omega \leq \omega'$ if and only if $\rho(\{\omega\}) \leq \rho(\{\omega'\})$, which is well-founded since ranks are natural numbers, and belief with firmness at least 1 coincides with their notion of belief. Hence the case $f = 1$ of Corollary 1 follows from their result. The cases $f > 1$ use the cardinal content of ranks, which a plausibility order does not capture.

4 Common Belief with Firmness f

Monderer and Samet (1989) replace common knowledge by common p-belief and show that posteriors that are common p-belief are approximately equal. We now show that the ranking-theoretic analog delivers exact agreement.

Inspired by Spohn's notion of firmness, we define the following belief operators.

Definition 6 (Firmness-f belief and common belief with firmness f)**.** *For $f \in \{1, 2, \ldots\} \cup \{\infty\}$, $i \in I$, and $E \in 2^\Omega$, the* firmness-f belief operator *is defined by*

$$B_i^f(E) := \{\omega \in \Omega : \rho(\neg E \mid \Pi_i(\omega)) \geq f\}.$$

The mutual (or everybody) belief with firmness f *is defined by* $B^f(E) := \bigcap_{i\in I} B_i^f(E)$, *and* common belief with firmness f *is defined by*

$$CB^f(E) := \bigcap_{\ell=1}^{\infty} (B^f)^\ell(E).$$

That is, $B_i^f(E)$ is the event that agent i believes E with firmness at least f.[5] We exclude $f = 0$, since $\rho(\neg E \mid \Pi_i(\omega)) \geq 0$ always holds, so that $B_i^0(E) = \Omega$ for every event E, even for $E = \emptyset$.

The following lemma collects properties of the firmness-f belief operator. Several of them are not used below but help to compare it with knowledge and with probabilistic belief.

Lemma 4. *Under Assumption 1, for all* $i \in I$, $f \in \{1, 2, \ldots\} \cup \{\infty\}$, *and* $E, E' \in 2^\Omega$*:*

(i) *Necessitation:* $B_i^f(\Omega) = \Omega$.

(ii) *Consistency:* $B_i^f(\emptyset) = \emptyset$, *and* $B_i^f(E) \subseteq \neg B_i^s(\neg E)$ *for all* $s \in \{1, 2, \ldots\} \cup \{\infty\}$.

(iii) *Conjunction:* $B_i^f(E \cap E') = B_i^f(E) \cap B_i^f(E')$.

(iv) *Monotonicity:* $E \subseteq E'$ *implies* $B_i^f(E) \subseteq B_i^f(E')$.

(v) *Knowledge implies belief:* $K_i(E) \subseteq B_i^f(E)$.

(vi) *If* E *is a union of possibility sets of agent* i, *then* $B_i^f(E) = E$.

(vii) *Positive and negative introspection:* $B_i^f(E) = B_i^s(B_i^f(E)) = K_i(B_i^f(E))$ *and* $\neg B_i^f(E) = B_i^s(\neg B_i^f(E)) = K_i(\neg B_i^f(E))$ *for all* $s \in \{1, 2, \ldots\} \cup \{\infty\}$.

(viii) *Monotonicity in firmness: if* $f \leq f'$, *then* $B_i^{f'}(E) \subseteq B_i^f(E)$.

(ix) $CB^f(E) \subseteq B_i^f(E)$ *and* $CB^f(E) \subseteq B_i^f(CB^f(E))$.

Proof. Fix $\omega \in \Omega$ and let $C := \Pi_i(\omega)$. By Assumption 1, $\rho(C) < \infty$, so $\rho(\cdot \mid C)$ is defined by Definition 2. It is a ranking function. Indeed, $\rho(\Omega \mid C) = \rho(C) - \rho(C) = 0$, and $\rho(\emptyset \mid C) = \rho(\emptyset) - \rho(C) = \infty$ by property (2). Moreover, for all events G, G', property (3) yields

$$\begin{aligned}\rho(G \cup G' \mid C) &= \rho((G \cap C) \cup (G' \cap C)) - \rho(C) = \min\{\rho(G \cap C), \rho(G' \cap C)\} - \rho(C) \\ &= \min\{\rho(G \mid C), \rho(G' \mid C)\},\end{aligned}$$

since subtracting the finite number $\rho(C)$ from both terms does not change which term is smaller. By definition, $\omega \in B_i^f(E)$ if and only if $\rho(\neg E \mid C) \geq f$. In each part, we show the claim at the arbitrary state ω.

[5]The operator B_i^f depends only on agent i's conditional ranks. It can be defined analogously with agent-specific priors or, more generally, with the ranking-function types of Section 7, and Lemma 4 continues to hold.

(i) Since $\neg\Omega = \emptyset$, $\rho(\neg\Omega \mid C) = \rho(\emptyset \mid C) = \infty \geq f$, also for $f = \infty$. So $\omega \in B_i^f(\Omega)$. Since ω is arbitrary, $B_i^f(\Omega) = \Omega$.

(ii) Since $\neg\emptyset = \Omega$, $\rho(\neg\emptyset \mid C) = \rho(\Omega \mid C) = 0 < f$, because $f \geq 1$. So $\omega \notin B_i^f(\emptyset)$. Since ω is arbitrary, $B_i^f(\emptyset) = \emptyset$. Moreover, since $\neg E \cup E = \Omega$, property (3) for $\rho(\cdot \mid C)$ yields $\min\{\rho(\neg E \mid C), \rho(E \mid C)\} = \rho(\Omega \mid C) = 0$. Hence $\rho(\neg E \mid C) = 0$ or $\rho(E \mid C) = 0$. In the first case, $\rho(\neg E \mid C) = 0 < f$, so $\omega \notin B_i^f(E)$. In the second case, $\rho(\neg\neg E \mid C) = \rho(E \mid C) = 0 < s$, because $s \geq 1$, so $\omega \notin B_i^s(\neg E)$. Thus ω cannot belong to both $B_i^f(E)$ and $B_i^s(\neg E)$, i.e., $B_i^f(E) \subseteq \neg B_i^s(\neg E)$.

(iii) Since $\neg(E \cap E') = \neg E \cup \neg E'$, property (3) for $\rho(\cdot \mid C)$ yields $\rho(\neg(E \cap E') \mid C) = \min\{\rho(\neg E \mid C), \rho(\neg E' \mid C)\}$. The minimum of two numbers is at least f if and only if both numbers are at least f. Hence $\omega \in B_i^f(E \cap E')$ if and only if $\omega \in B_i^f(E)$ and $\omega \in B_i^f(E')$.

(iv) If $E \subseteq E'$, then $E = E \cap E'$. By (iii), $B_i^f(E) = B_i^f(E \cap E') = B_i^f(E) \cap B_i^f(E') \subseteq B_i^f(E')$.

(v) Let $\omega \in K_i(E)$, i.e., $C \subseteq E$. Then $\neg E \cap C = \emptyset$. By Definition 2 and property (2), $\rho(\neg E \mid C) = \rho(\neg E \cap C) - \rho(C) = \rho(\emptyset) - \rho(C) = \infty \geq f$. So $\omega \in B_i^f(E)$.

(vi) Let E be a union of possibility sets of agent i. We first show that if $C \cap E \neq \emptyset$, then $C \subseteq E$. Let $\omega' \in C \cap E$. Since E is a union of possibility sets of agent i, there is ω'' with $\omega' \in \Pi_i(\omega'') \subseteq E$. By stationarity, $\Pi_i(\omega') = \Pi_i(\omega'')$. Since $\omega' \in C = \Pi_i(\omega)$, stationarity also yields $\Pi_i(\omega') = C$. Hence $C = \Pi_i(\omega'') \subseteq E$. Now, if $\omega \in E$, then $\omega \in C \cap E$ by reflexivity, so $C \subseteq E$. Then $\omega \in K_i(E)$, and $\omega \in B_i^f(E)$ by (v). If $\omega \notin E$, then $C \not\subseteq E$, so $C \cap E = \emptyset$, i.e., $C \subseteq \neg E$. Then $\rho(\neg E \mid C) = \rho(\neg E \cap C) - \rho(C) = \rho(C) - \rho(C) = 0 < f$, since $\rho(C)$ is finite and $f \geq 1$. So $\omega \notin B_i^f(E)$. Hence $B_i^f(E) = E$.

(vii) We first show that $B_i^f(E)$ is a union of possibility sets of agent i. Let $\omega' \in B_i^f(E)$ and $\omega'' \in \Pi_i(\omega')$. By stationarity, $\Pi_i(\omega'') = \Pi_i(\omega')$, so $\rho(\neg E \mid \Pi_i(\omega'')) = \rho(\neg E \mid \Pi_i(\omega')) \geq f$, i.e., $\omega'' \in B_i^f(E)$. Thus $\Pi_i(\omega') \subseteq B_i^f(E)$ for every $\omega' \in B_i^f(E)$. Together with reflexivity, $B_i^f(E) = \bigcup_{\omega' \in B_i^f(E)} \Pi_i(\omega')$. The same argument with "$< f$" in place of "$\geq f$" shows that $\neg B_i^f(E)$ is a union of possibility sets of agent i. Next, $K_i(G) = G$ for every union G of possibility sets of agent i. Indeed, if $\omega' \in K_i(G)$, then $\omega' \in \Pi_i(\omega') \subseteq G$ by reflexivity. Conversely, if $\omega' \in G$, then $\Pi_i(\omega') \subseteq G$ by the first part of the proof of (vi), applied with ω' in place of ω and G in place of E, so $\omega' \in K_i(G)$. Applying (vi) with firmness s and this fact to $G = B_i^f(E)$ and to $G = \neg B_i^f(E)$ yields the claims.

(viii) If $\omega \in B_i^{f'}(E)$, then $\rho(\neg E \mid C) \geq f' \geq f$, so $\omega \in B_i^f(E)$.

(ix) For the first inclusion, by definition, $CB^f(E) = \bigcap_{\ell \geq 1} (B^f)^\ell(E) \subseteq (B^f)^1(E) = B^f(E) = \bigcap_{j \in I} B_j^f(E) \subseteq B_i^f(E)$. For the second inclusion, note that the sets $(B^f)^\ell(E)$, $\ell \geq 1$, are subsets of the finite set Ω. So there are only finitely many distinct ones. Let $\mathcal{G}$ be the finite, nonempty collection of these distinct sets. Then $CB^f(E) = \bigcap_{G \in \mathcal{G}} G$. Applying (iii) finitely often yields $B_i^f(CB^f(E)) = \bigcap_{G \in \mathcal{G}} B_i^f(G)$. Each $G \in \mathcal{G}$ equals $(B^f)^\ell(E)$ for some $\ell \geq 1$. For this ℓ, $B_i^f(G) \supseteq B^f(G) = (B^f)^{\ell+1}(E) \supseteq CB^f(E)$. Hence

$$B_i^f(CB^f(E)) = \bigcap_{G \in \mathcal{G}} B_i^f(G) \supseteq \bigcap_{G \in \mathcal{G}} B^f(G) \supseteq CB^f(E).$$

□

Unlike knowledge, firmness-f belief does not satisfy the truth axiom: an agent may believe an event that is false. In the terminology of modal logic, B_i^f satisfies the axioms of the

logic KD45 of belief. The firmness-f belief operator thus shares necessitation, monotonicity, introspection, and monotonicity in the threshold with the p-belief operator of Monderer and Samet (1989), $B_i^p(E) := \{\omega \in \Omega : p(E \mid \Pi_i(\omega)) \geq p\}$. Yet, q-belief in E excludes p-belief in $\neg E$ only if $p + q > 1$, whereas by (ii) belief in E with any positive firmness excludes belief in $\neg E$ with any positive firmness. Moreover, for $p < 1$, p-belief satisfies only one inclusion of conjunction: $B_i^p(E \cap E') \subseteq B_i^p(E) \cap B_i^p(E')$, but two events that are each believed with probability at least p need not have an intersection that is believed with probability at least p.

The following result is exact rather than approximate because ranks combine by taking minima. If an agent believes an event F with firmness at least f, the part of her possibility set outside F has rank at least f higher than the possibility set itself. It therefore either does not affect her posterior rank of E at all, or affects it only at ranks at least f. With probabilities, the part outside F generally shifts the posterior, so that only approximate agreement obtains. The result concerns truncated posterior ranks.

Definition 7 (Truncated rank)**.** *For $f \in \{1, 2, \ldots\} \cup \{\infty\}$ and $k \in \mathbb{N} \cup \{\infty\}$, the rank k* truncated at f *is* $\min\{k, f\}$.

That is, ranks strictly below f are kept as they are, while all ranks at least f are identified with f, meaning "at least f". For $f = \infty$, truncation has no effect. Agreement on posterior ranks truncated at f thus means that if one agent's posterior rank of E is strictly below f, then all agents have exactly this posterior rank, and if one agent's posterior rank is at least f, then so are all others' posterior ranks, which may differ.

Theorem 2. *Let Assumption 1 hold and let $f \in \{1, 2, \ldots\} \cup \{\infty\}$. If $F := CB^f\left(\bigcap_{j \in I}[\kappa_j(E)]\right) \neq \emptyset$, then for all $i \in I$,*

$$\min\{\kappa_i(E), f\} = \min\{\rho(E \mid F), f\}.$$

In particular, $\min\{\kappa_i(E), f\} = \min\{\kappa_j(E), f\}$ for all $i, j \in I$. That is, either (i) all agents have the same posterior rank of E, and it is strictly below f, or (ii) all agents' posterior ranks of E are at least f and may differ.

The proof of Theorem 2 follows the idea of Proof 1 of Theorem 1, with the commonly believed event F in place of the cell of the meet of the agents' partitions. The difficulty is that F need not be a union of possibility sets of an agent. Step 1 shows that each agent's posterior rank of E is constant on F. Step 2 shows that, for each agent i, the union G_i of her possibility sets that intersect F differs from F only by states whose rank exceeds the rank of F by at least f. Step 3 computes $\rho(E \cap G_i)$ in two ways, once as the minimum over agent i's possibility sets and once by splitting G_i into F and the rest. Comparing the two expressions shows that agent i's posterior rank of E coincides with $\rho(E \mid F)$ up to truncation at f. Since $\rho(E \mid F)$ does not depend on i, agreement follows.

Proof of Theorem 2. *Step 1: To show $F \subseteq [\kappa_i(E)]$ for all $i \in I$.* Fix $i \in I$ and let $H := \bigcap_{j \in I}[\kappa_j(E)]$, so that $F = CB^f(H)$. First, by Lemma 4 (ix), $F \subseteq B_i^f(H)$. That is, at every state in F, agent i believes H with firmness at least f. Second, since $H \subseteq [\kappa_i(E)]$, monotonicity (Lemma 4 (iv)) yields $B_i^f(H) \subseteq B_i^f([\kappa_i(E)])$. Third, $[\kappa_i(E)]$ is a union of possibility sets of agent i. Indeed, if $\omega \in [\kappa_i(E)]$ and $\omega' \in \Pi_i(\omega)$, then $\Pi_i(\omega') = \Pi_i(\omega)$

by stationarity, so $\rho(E \mid \Pi_i(\omega')) = \rho(E \mid \Pi_i(\omega)) = \kappa_i(E)$ and $\omega' \in [\kappa_i(E)]$. Together with reflexivity, this yields $[\kappa_i(E)] = \bigcup_{\omega \in [\kappa_i(E)]} \Pi_i(\omega)$. Hence, $B_i^f([\kappa_i(E)]) = [\kappa_i(E)]$ by Lemma 4 (vi). Combining the three steps, $F \subseteq B_i^f(H) \subseteq B_i^f([\kappa_i(E)]) = [\kappa_i(E)]$. That is, at every state in F, agent i's posterior rank of E is $\kappa_i(E)$.

Step 2: Fix $i \in I$. Let $\mathcal{C}_i$ be the set of possibility sets of agent i that intersect F, and let $G_i := \bigcup_{C \in \mathcal{C}_i} C$. Since Ω is finite, $\mathcal{C}_i$ is finite. Every $\omega \in F$ lies in $\Pi_i(\omega)$ by reflexivity, and $\Pi_i(\omega) \in \mathcal{C}_i$ since it contains $\omega \in F$. Hence $F \subseteq G_i$. Moreover, $\mathcal{C}_i$ is nonempty since $F \neq \emptyset$. Let $C \in \mathcal{C}_i$ and pick $\omega \in C \cap F$. Since $F = CB^f(H)$, the second inclusion of Lemma 4 (ix), applied to the event H, yields $F \subseteq B_i^f(F)$. So $\omega \in B_i^f(F)$. Since $\omega \in C$, stationarity yields $\Pi_i(\omega) = C$. By the definition of B_i^f and Definition 2, this means $\rho(\neg F \mid C) = \rho(\neg F \cap C) - \rho(C) = \rho(C \setminus F) - \rho(C) \geq f$. Since $\rho(C) < \infty$ by Assumption 1, adding $\rho(C)$ to both sides yields

$$\rho(C \setminus F) \geq f + \rho(C).$$

Since $C = (C \cap F) \cup (C \setminus F)$, property (3) yields $\rho(C) = \min\{\rho(C \cap F), \rho(C \setminus F)\}$. Since $f \geq 1$ and $\rho(C) < \infty$, the inequality above yields $\rho(C \setminus F) > \rho(C)$. So the minimum is not $\rho(C \setminus F)$, and hence $\rho(C) = \rho(C \cap F)$. Since $F \subseteq G_i$, we have $F = \bigcup_{C \in \mathcal{C}_i}(C \cap F)$. By property (3), applied finitely often,

$$\rho(F) = \min_{C \in \mathcal{C}_i} \rho(C \cap F) = \min_{C \in \mathcal{C}_i} \rho(C) = \rho(G_i) < \infty,$$

where the last equality again follows from property (3), and $\rho(F) < \infty$ since $\mathcal{C}_i$ is finite and nonempty and each $\rho(C)$ is finite. Moreover, $G_i \setminus F = \bigcup_{C \in \mathcal{C}_i}(C \setminus F)$. By property (3), the inequality $\rho(C \setminus F) \geq f + \rho(C)$ shown above, and $\rho(C) \geq \rho(F)$ for all $C \in \mathcal{C}_i$,

$$\rho(G_i \setminus F) = \min_{C \in \mathcal{C}_i} \rho(C \setminus F) \geq \min_{C \in \mathcal{C}_i} (f + \rho(C)) = f + \rho(F).$$

Step 3: Let i and $\mathcal{C}_i$ be as in Step 2. We compute $\rho(E \cap G_i)$ in two ways. First, let $C \in \mathcal{C}_i$ and pick $\omega \in C \cap F$. By Step 1, $\omega \in [\kappa_i(E)]$, i.e., $\rho(E \mid \Pi_i(\omega)) = \kappa_i(E)$. By stationarity, $\Pi_i(\omega) = C$, so $\rho(E \mid C) = \kappa_i(E)$. By Definition 2, $\rho(E \mid C) = \rho(E \cap C) - \rho(C)$. Since $\rho(C) < \infty$ by Assumption 1, adding $\rho(C)$ to both sides yields $\rho(E \cap C) = \kappa_i(E) + \rho(C)$. If $\kappa_i(E) = \infty$, then $\rho(E \cap C) = \infty$ and both sides equal ∞. Since $E \cap G_i = \bigcup_{C \in \mathcal{C}_i}(E \cap C)$, property (3), applied finitely often, yields

$$\begin{aligned} \rho(E \cap G_i) &= \min_{C \in \mathcal{C}_i} \rho(E \cap C) = \min_{C \in \mathcal{C}_i} (\kappa_i(E) + \rho(C)) \\ &= \kappa_i(E) + \min_{C \in \mathcal{C}_i} \rho(C) = \kappa_i(E) + \rho(F). \end{aligned}$$

The third equality holds since adding the same number $\kappa_i(E)$ to every term does not change which term is smallest, and the last equality holds by Step 2.

Second, since $F \subseteq G_i$ by Step 2, we have $E \cap G_i = (E \cap F) \cup (E \cap (G_i \setminus F))$. Since $\rho(F) < \infty$ by Step 2, $y := \rho(E \cap (G_i \setminus F)) - \rho(F)$ is well defined, and $\rho(E \cap (G_i \setminus F)) = \rho(F) + y$. Also since $\rho(F) < \infty$, Definition 2 yields $\rho(E \cap F) = \rho(F) + \rho(E \mid F)$. Hence property (3) yields

$$\rho(E \cap G_i) = \min\{\rho(E \cap F), \rho(E \cap (G_i \setminus F))\}$$

$$= \min\{\rho(F) + \rho(E \mid F), \rho(F) + y\} = \rho(F) + \min\{\rho(E \mid F), y\}.$$

Since $E \cap (G_i \setminus F) \subseteq G_i \setminus F$, monotonicity and Step 2 yield $\rho(E \cap (G_i \setminus F)) \geq \rho(G_i \setminus F) \geq f + \rho(F)$. Hence $y \geq f$.

Comparing the two expressions for $\rho(E \cap G_i)$ yields $\kappa_i(E) + \rho(F) = \rho(F) + \min\{\rho(E \mid F), y\}$. Subtracting the finite number $\rho(F)$ from both sides yields $\kappa_i(E) = \min\{\rho(E \mid F), y\}$. Therefore,

$$\min\{\kappa_i(E), f\} = \min\{\rho(E \mid F), y, f\} = \min\{\rho(E \mid F), f\},$$

where the last equality holds since $y \geq f$. The right-hand side does not depend on i. Hence $\min\{\kappa_i(E), f\} = \min\{\kappa_j(E), f\}$ for all $i, j \in I$. If $\min\{\rho(E \mid F), f\} < f$, then $\kappa_i(E) = \rho(E \mid F) < f$ for all $i \in I$, which is case (i). Otherwise, $\min\{\kappa_i(E), f\} = f$, i.e., $\kappa_i(E) \geq f$ for all $i \in I$, which is case (ii). □

How should Theorem 2 be interpreted? By (i), agents agree exactly whenever some agent's posterior rank of E is strictly below f. By (ii), agents can disagree only if all their posterior ranks are at least f, i.e., only if all of them believe $\neg E$ with firmness at least f. The result is exact within an approximation. Interpreting ranks as orders of magnitude of infinitesimal probabilities, disagreement is confined to events to which all agents assign probability of order ε^f or smaller, which vanishes as f grows. This is reminiscent of the approximate agreement results for common p-belief of Monderer and Samet (1989) and Neeman (1996). However, the approximation is of a different kind than theirs. Consider the question "What is your posterior rank of E?", whose possible answers are $0, 1, 2, \ldots, \infty$. Theorem 2 does not say that agents give the same answer to this question. It says that they give the same answer to the question "Which of $0, 1, \ldots, f-1$, or 'at least f' is your posterior rank of E?" This question is coarser than the first: its answer can be derived from the answer to the first question, but not conversely. With probabilities and $p < 1$, Monderer and Samet (1989) only bound the distance between commonly p-believed posteriors. Their result thus does not imply that agents give the same answer to any such coarser question, not even to the question whether the posterior exceeds a given threshold.

Remark 2 (Special cases)**.** *For $f = 1$, Theorem 2 says that if the agents' posterior ranks of E are commonly believed, then they agree on whether E is disbelieved at all, i.e., whether $\kappa_i(E) = 0$. In this case, common belief with firmness f coincides with common belief in epistemic plausibility models, and Dégremont and Roy (2012, Theorem 3.1) show for a common and well-founded plausibility order that it cannot be common belief that one agent believes an event while another does not. For $f = \infty$, Theorem 1 follows from Theorem 2. Indeed, $K_i(H) \subseteq B_i^\infty(H)$ for every event H by Lemma 4 (v), and B^∞ is monotone, so $CK(H) \subseteq CB^\infty(H)$. If every state has finite prior rank, then $B_i^\infty = K_i$ and $CB^\infty = CK$, so that Theorem 1 is exactly the special case $f = \infty$ of Theorem 2.*

Pacuit and Yang (2025, 2026a) study agreement theorems for a general class of representations of uncertainty, which they call conditional plausibility measures. A conditional plausibility measure assigns to each event E, conditional on each event C, a plausibility value $Pl(E \mid C)$. Plausibility values are only partially ordered. The event conditioned on receives the top value, the empty event the bottom value, and larger events receive weakly larger values. Probabilities are the leading example. Agent i is said to d-believe E if her

plausibility of E is at least d. Pacuit and Yang (2025, Theorem 5.5; 2026a, Theorem 5.6 and Remark 5.8) characterize when an analog of the approximate agreement results of Monderer and Samet (1989) and Neeman (1996) holds. That is, when does common d-belief in the agents' posteriors imply that the posteriors are close? Their characterization requires one to specify what "close" means. For lexicographic probability systems, they obtain approximate agreement on first-level posteriors (Pacuit and Yang, 2026a, Theorem A.10).

In an appendix, Pacuit and Yang (2026a, Appendix A.6) consider ranking functions as one example. They take the plausibility of an event to be its rank, with the order reversed, so that ranks are a special case with a total order. Then d-belief in E means $\rho(E \mid \cdot) \leq d$. That is, E is not disbelieved beyond d. This is not belief in E. For instance, for $d = 0$, both E and $\neg E$ may be "0-believed" at the same time. To measure how far apart posterior ranks are, they consider ordinary addition, ordinary subtraction, and the minimum. For each of these, the resulting approximate agreement statement is either vacuous or false. Our Theorem 2 uses Spohn's notion of belief, i.e., E is believed with firmness f if $\neg E$ is disbelieved at least to degree f, $\rho(\neg E \mid \cdot) \geq f$. With this notion of belief, and with "close" meaning equal once truncated at f, we are able to obtain our non-trivial result.

5 Tropical Expectations and Speculative Betting

A random variable is a function $X : \Omega \longrightarrow \mathbb{R}$. Recall that $\mathcal{D}_\rho$ is the set of events of finite rank w.r.t. ρ.

Definition 8 (Tropical conditional expectation)**.** *For a random variable X and $F \in \mathcal{D}_\rho$, the* tropical conditional expectation *of X given F is*

$$\mathbb{E}_\rho[X \mid F] := \min_{\omega \in F} \{X(\omega) + \rho(\{\omega\} \mid F)\}.$$

By property (3), $\rho(F) = \min_{\omega \in F} \rho(\{\omega\})$, and $\rho(F) < \infty$. So some $\omega \in F$ has $\rho(\{\omega\}) = \rho(F)$, i.e., $\rho(\{\omega\} \mid F) = 0$ by Definition 2, and the minimum is finite. In tropical notation, $\mathbb{E}_\rho[X \mid F] = \bigoplus_{\omega \in F} X(\omega) \odot \rho(\{\omega\} \mid F)$, the tropical analog of the (standard) conditional expectation of X w.r.t. a probability measure p, $\sum_{\omega \in F} X(\omega) \cdot p(\omega \mid F)$. The tropical expectation evaluates X by its worst outcome discounted by disbelief: the payoff in each state is raised by the state's rank, so that a bad outcome counts less the more it is disbelieved, and X is valued by the lowest of these adjusted payoffs. Schipper (2026a) provides a decision-theoretic foundation. An agent who ranks random variables by $\mathbb{E}_\rho[X \mid F]$ is a pessimistic tropical expected utility maximizer with linear utility. Her preferences are characterized by a weak order, monotonicity, strong tropical independence, i.e., if X is weakly preferred to Y, then $\min\{X + c, Z\}$ is weakly preferred to $\min\{Y + c, Z\}$ for every random variable Z and constant c, and solvability, i.e., every random variable has a certainty equivalent (Schipper, 2026a, Corollary 1). Other ranking-based criteria in the literature are optimistic rather than pessimistic. Throughout, agents are pessimistic. We compare the criteria in Section 11.5, where it turns out that our no-betting results require pessimism.

Tropical expectation is also the order-of-magnitude limit of ordinary expectation, in line with the interpretation of ranks as orders of magnitude of infinitesimal probabilities

(Goldszmidt and Pearl, 1996; Spohn, 2009, Section 3.1): if $p_\varepsilon(\omega) \propto \varepsilon^{\rho(\{\omega\}|F)}$ on F, then $\lim_{\varepsilon\to 0} \log_\varepsilon \sum_{\omega\in F} p_\varepsilon(\omega)\varepsilon^{X(\omega)} = \mathbb{E}_\rho[X \mid F]$.

The standard probabilistic no-trade theorems rest on two properties of conditional expectations: The law of total expectation and linearity. Tropical expectations satisfy the law of total expectation, which is a form of tropical linearity in beliefs, and they are tropically linear in random variables. Yet, our tropical no-trade theorem will not rely on tropical linearity in random variables but rather on a property that has no probabilistic counterpart: A single state of conditional rank 0 bounds the conditional tropical expectations of *all* random variables at once.

Lemma 5. *Let $F \in \mathcal{D}_\rho$.*

(i) Law of total expectation: For every random variable X, if $\mathcal{P}$ is a partition of F with $\rho(C) < \infty$ for all $C \in \mathcal{P}$, then

$$\mathbb{E}_\rho[X \mid F] = \min_{C\in\mathcal{P}} \{\mathbb{E}_\rho[X \mid C] + \rho(C \mid F)\}.$$

(ii) There exists $\omega_0 \in F$ such that for all random variables X, we have $\mathbb{E}_\rho[X \mid F] \le X(\omega_0)$.

(iii) Conglomerability: For every random variable X and all $P, Q \in \mathcal{D}_\rho$,

$$\min\{\mathbb{E}_\rho[X \mid P], \mathbb{E}_\rho[X \mid Q]\} \;\le\; \mathbb{E}_\rho[X \mid P \cup Q] \;\le\; \max\{\mathbb{E}_\rho[X \mid P], \mathbb{E}_\rho[X \mid Q]\}.$$

In particular, if $\mathbb{E}_\rho[X \mid P] = \mathbb{E}_\rho[X \mid Q] = x$, then $\mathbb{E}_\rho[X \mid P \cup Q] = x$ (union consistency), and if $\mathbb{E}_\rho[X \mid P] > 0$ and $\mathbb{E}_\rho[X \mid Q] > 0$, then $\mathbb{E}_\rho[X \mid P \cup Q] > 0$.

Proof. (i) For $\omega \in C \in \mathcal{P}$,

$$\rho(\{\omega\} \mid F) = \rho(\{\omega\}) - \rho(F) = \big(\rho(\{\omega\}) - \rho(C)\big) + \big(\rho(C) - \rho(F)\big) = \rho(\{\omega\} \mid C) + \rho(C \mid F) \quad (6)$$

by Definition 2, where all conditional ranks are defined since $\rho(C) < \infty$ and $\rho(F) < \infty$ by the assumption on $\mathcal{P}$ and the fact that $F \in \mathcal{D}_\rho$, respectively. Hence

$$\begin{aligned}
\mathbb{E}_\rho[X \mid F] &= \min_{\omega\in F}\{X(\omega) + \rho(\{\omega\} \mid F)\} \\
&= \min_{C\in\mathcal{P}} \min_{\omega\in C}\{X(\omega) + \rho(\{\omega\} \mid F)\} \\
&= \min_{C\in\mathcal{P}} \min_{\omega\in C}\{X(\omega) + \rho(\{\omega\} \mid C) + \rho(C \mid F)\} \\
&= \min_{C\in\mathcal{P}} \Big\{ \min_{\omega\in C}\{X(\omega) + \rho(\{\omega\} \mid C)\} + \rho(C \mid F)\Big\} \\
&= \min_{C\in\mathcal{P}} \big\{\mathbb{E}_\rho[X \mid C] + \rho(C \mid F)\big\}.
\end{aligned}$$

The first equality is the definition of the tropical conditional expectation. The second holds because F is the union of the cells of $\mathcal{P}$, so the minimum over F is the minimum over the cells of the minima over each cell. The third follows from (6). The fourth holds because $\rho(C \mid F)$ does not depend on $\omega \in C$. The fifth is again the definition of the tropical conditional expectation, which applies since $\rho(C) < \infty$.

(ii) Since $F \in \mathcal{D}_\rho$, $\rho(F) < \infty$. By property (3), $\rho(F) = \min_{\omega \in F} \rho(\{\omega\})$, and since Ω is finite, the minimum is attained at some $\omega_0 \in F$. Thus $\rho(\{\omega_0\}) = \rho(F)$, i.e., $\rho(\{\omega_0\} \mid F) = \rho(\{\omega_0\}) - \rho(F) = 0$ by Definition 2. Now let X be any random variable. Since $\omega_0 \in F$, the minimum defining $\mathbb{E}_\rho[X \mid F]$ is weakly smaller than the term for ω_0, so

$$\mathbb{E}_\rho[X \mid F] = \min_{\omega \in F} \{X(\omega) + \rho(\{\omega\} \mid F)\} \leq X(\omega_0) + \rho(\{\omega_0\} \mid F) = X(\omega_0).$$

The state ω_0 depends only on ρ and F, not on X, so the same ω_0 works for all random variables.

(iii) Since $\mathcal{D}_\rho$ is closed under finite unions, $P \cup Q \in \mathcal{D}_\rho$. Let $S \in \{P, Q\}$. Since $S \subseteq P \cup Q$, Definition 2 yields $\rho(S \mid P \cup Q) = \rho(S) - \rho(P \cup Q)$, which is a natural number by monotonicity. Hence

$$\begin{aligned}
\mathbb{E}_\rho[X \mid P \cup Q] &= \min_{\omega \in P \cup Q} \{X(\omega) + \rho(\{\omega\}) - \rho(P \cup Q)\} \\
&= \min \Big\{ \min_{\omega \in P} \{X(\omega) + \rho(\{\omega\}) - \rho(P \cup Q)\}, \\
&\qquad \min_{\omega \in Q} \{X(\omega) + \rho(\{\omega\}) - \rho(P \cup Q)\} \Big\} \\
&= \min \Big\{ \min_{\omega \in P} \{X(\omega) + \rho(\{\omega\}) - \rho(P)\} + \rho(P \mid P \cup Q), \\
&\qquad \min_{\omega \in Q} \{X(\omega) + \rho(\{\omega\}) - \rho(Q)\} + \rho(Q \mid P \cup Q) \Big\} \\
&= \min \big\{ \mathbb{E}_\rho[X \mid P] + \rho(P \mid P \cup Q), \mathbb{E}_\rho[X \mid Q] + \rho(Q \mid P \cup Q) \big\}.
\end{aligned}$$

The first equality is the definition of the tropical conditional expectation together with Definition 2. The second holds because the minimum over $P \cup Q$ is the smaller of the minima over P and over Q. A state in $P \cap Q$ appears in both inner minima, which does not change the overall minimum. The third holds because $-\rho(P \cup Q) = -\rho(S) + \big(\rho(S) - \rho(P \cup Q)\big) = -\rho(S) + \rho(S \mid P \cup Q)$ for $S \in \{P, Q\}$. Terms with $\rho(\{\omega\}) = \infty$ equal ∞ on both sides. The fourth is again the definition of the tropical conditional expectation, which applies since $P, Q \in \mathcal{D}_\rho$.

We now show the two inequalities. First, since $\rho(P \mid P \cup Q) \geq 0$ and $\rho(Q \mid P \cup Q) \geq 0$, each of the two terms in the last line is at least the corresponding conditional expectation, and hence at least $\min\{\mathbb{E}_\rho[X \mid P], \mathbb{E}_\rho[X \mid Q]\}$. So is their minimum. Second, by property (3), $\rho(P \cup Q) = \min\{\rho(P), \rho(Q)\}$, so $\rho(P \mid P \cup Q) = 0$ or $\rho(Q \mid P \cup Q) = 0$. If $\rho(P \mid P \cup Q) = 0$, then the minimum in the last line is weakly smaller than its first term, so $\mathbb{E}_\rho[X \mid P \cup Q] \leq \mathbb{E}_\rho[X \mid P] \leq \max\{\mathbb{E}_\rho[X \mid P], \mathbb{E}_\rho[X \mid Q]\}$. The case $\rho(Q \mid P \cup Q) = 0$ is analogous. Finally, if both conditional expectations equal x, then the two inequalities yield $x \leq \mathbb{E}_\rho[X \mid P \cup Q] \leq x$. If both are positive, then the first inequality yields $\mathbb{E}_\rho[X \mid P \cup Q] \geq \min\{\mathbb{E}_\rho[X \mid P], \mathbb{E}_\rho[X \mid Q]\} > 0$. □

Part (iii) is the analog of Lemma 2 for tropical expectations. As there, P and Q need not be disjoint. Note that $\rho(C \mid F) \geq 0$ in (i), so if $\mathbb{E}_\rho[X \mid C] > 0$ for all $C \in \mathcal{P}$, then $\mathbb{E}_\rho[X \mid F] > 0$.

Remark 3 (Possible rejection of both sides of a bet)**.** *Applying part (ii) of Lemma 5 to X and to $-X$, with the same ω_0, yields*

$$\mathbb{E}_\rho[X \mid F] + \mathbb{E}_\rho[-X \mid F] \leq X(\omega_0) - X(\omega_0) = 0.$$

With probabilistic expectations, the analog to the left-hand side equals 0*. With tropical expectations, it can be strictly negative. This implies that a "tropical" agent may reject both sides of a bet like a risk-averse expected utility maximizer while a risk-neutral expected utility maximizer weakly prefers at least one side of a bet. This just emphasizes that risk aversion is baked into the pessimistic tropical expectation.*

Definition 9 (Bet)**.** *A* bet *is a profile of random variables $(X_i)_{i\in I}$ with $\sum_{i\in I} X_i(\omega) = 0$ for all $\omega \in \Omega$.*

We interpret $X_i(\omega)$ to be the gain or loss to agent i in state ω depending on whether it is positive or negative, respectively. The condition $\sum_{i\in I} X_i(\omega) = 0$ says that the bet only redistributes money among the agents.

Definition 10 (Acceptance)**.** *Let Assumption 1 hold. Agent i* accepts *the bet at ω if $\mathbb{E}_\rho[X_i \mid \Pi_i(\omega)] > 0$. Let*

$$☺_i(X_i) := \{\omega \in \Omega : \mathbb{E}_\rho[X_i \mid \Pi_i(\omega)] > 0\}$$

be the event that agent i accepts the bet.

The following theorem is the ranking-theoretic analog of the no-trade theorems of Milgrom and Stokey (1982) and Sebenius and Geanakoplos (1983):

Theorem 3 (No speculative betting under a common prior)**.** *Under Assumption 1, for every bet $(X_i)_{i\in I}$, $CK\left(\bigcap_{i\in I} ☺_i(X_i)\right) = \emptyset$.*

Proof. Suppose to the contrary that there is $\omega \in CK\left(\bigcap_{i\in I} ☺_i(X_i)\right)$, and let $M := \Pi_*(\omega)$. By Aumann's (1976) characterization of common knowledge, $M \subseteq \bigcap_{i\in I} ☺_i(X_i)$, so $M \subseteq ☺_i(X_i)$ for all $i \in I$. Since $\Pi_i(\omega) \subseteq M$ by property (a), monotonicity and Assumption 1 yield $\rho(M) \leq \rho(\Pi_i(\omega)) < \infty$, so $M \in \mathcal{D}_\rho$.

Fix $i \in I$. By equation (4), M is the union of the possibility sets $\Pi_i(\omega')$, $\omega' \in M$. Two distinct possibility sets of agent i are disjoint, since Π_i is partitional. Hence the distinct sets among them form a partition $\mathcal{P}_i$ of M, which is finite since Ω is finite. By Assumption 1, $\rho(C) < \infty$ for all $C \in \mathcal{P}_i$. Let $C \in \mathcal{P}_i$ and pick $\omega' \in C$. By stationarity, $\Pi_i(\omega') = C$. Since $\omega' \in M \subseteq ☺_i(X_i)$, we have $\mathbb{E}_\rho[X_i \mid C] = \mathbb{E}_\rho[X_i \mid \Pi_i(\omega')] > 0$. Moreover, since $C \subseteq M$, monotonicity yields $\rho(C) \geq \rho(M)$. So, by Definition 2, $\rho(C \mid M) = \rho(C \cap M) - \rho(M) = \rho(C) - \rho(M) \geq 0$. By Lemma 5 (i),

$$\mathbb{E}_\rho[X_i \mid M] = \min_{C\in\mathcal{P}_i} \{\mathbb{E}_\rho[X_i \mid C] + \rho(C \mid M)\} > 0. \tag{7}$$

By Lemma 5 (ii), there exists $\omega_0 \in M$ such that for all $i \in I$, $\mathbb{E}_\rho[X_i \mid M] \leq X_i(\omega_0)$. Note that ω_0 depends only on ρ and M but not $i \in I$. Hence, $X_i(\omega_0) \geq \mathbb{E}_\rho[X_i \mid M] > 0$ for all $i \in I$, where the last inequality follows from inequality (7). Hence $\sum_{i\in I} X_i(\omega_0) > 0$, contradicting $\sum_{i\in I} X_i(\omega_0) = 0$. □

The proof mirrors the proof of the probabilistic no-trade theorems (e.g., Milgrom and Stokey, 1982; Sebenius and Geanakoplos, 1983), making use of the law of total expectation. However, in place of linearity in random variables, it uses Lemma 5 (ii). Yet, as we will show in Section 7, the theorem can also be proved without the law of total expectation.

Absence of common knowledge of strict preferences to trade as in Theorem 3 corresponds to the no-trade notion used by Samet (1998, Corollary), Feinberg (2000, Theorems 2 and 7), Heifetz (2006, Propositions 1 and 3), and Heifetz, Meier, and Schipper (2013). Our result also extends to slightly different notions of no-trade used throughout the literature. For instance, Morris (1994) requires only that all agents weakly prefer to trade and some agent strictly prefers it. This notion is covered by the next remark.

Remark 4 (Weak and strict acceptance)**.** *Theorem 3 extends to weak acceptance, provided some agent accepts strictly: It cannot be common knowledge that all agents weakly accept the bet and at least one agent strictly accepts it. This follows from Lemma 6 and the 'if' part of the proof of Theorem 4 in Section 7.*

Remark 5 (Common knowledge of weak acceptance)**.** *With a common prior probability, common knowledge of weak acceptance forces indifference. If every agent's conditional expected gain is nonnegative throughout a cell M of the meet of the agents' partitions, then her expected gain given M is nonnegative. These expected gains sum to* 0*, so each is* 0*, and hence every agent's conditional expected gain is* 0 *on every possibility set of positive probability in M. With ranking functions, common knowledge of weak acceptance forces indifference only at states of minimal rank in M. This follows from arguments in the proof of Lemma 6 and in the 'if' part of the proof of Theorem 4. Elsewhere, agents may strictly accept, even all at the same state. For example, let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with $(\rho(\{\omega_1\}), \rho(\{\omega_2\}), \rho(\{\omega_3\}), \rho(\{\omega_4\})) = (0, 1, 1, 2)$, let agent 1's partition be $\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and agent 2's be $\{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$, and consider the bet with $(X_1(\omega_1), X_1(\omega_2), X_1(\omega_3), X_1(\omega_4)) = (0, -1, 1, 0)$ and $X_2 = -X_1$. Agent 1's tropical expectation is* 0 *on $\{\omega_1, \omega_2\}$ and* 1 *on $\{\omega_3, \omega_4\}$, and agent 2's is* 0 *on $\{\omega_1, \omega_3\}$ and* 1 *on $\{\omega_2, \omega_4\}$. Both agents weakly accept at every state, which is common knowledge since the meet of the agents' partitions is $\{\Omega\}$, and both strictly accept at state ω_4. However, Theorem 3 excludes that the latter fact is common knowledge. Each agent ignores her loss of* 1 *in a state of rank* 1*, because it is offset by her disbelief in that state.*

How is our no-trade theorem for ranking functions related to other non-probabilistic no-trade theorems in the literature? Bacharach (1985, Proposition 9) shows the following result: If like-minded agents choose net trades by a common decision rule that satisfies his sure thing condition, such as a maximin rule, and the chosen trades are common knowledge and balance, then no trade takes place. A rule that selects the unique trade maximizing the tropical expectation satisfies the sure thing condition by the law of total expectation in Lemma 5 (i). I.e., if a trade is the unique maximizer conditional on each cell of a partition of F, then it is the unique maximizer conditional on F. Hence a no-trade result in Bacharach's sense also holds for ranking-theoretic agents. In our setting, the bet is given, and agents only accept or reject it. Since agents hold opposite sides of the bet, their decision problems differ so that they are not like-minded in Bacharach's sense. However, Rubinstein and Wolinsky (1990, Proposition 1, Conclusion 2, and the Remark on p. 190) allow different set functions for different agents and derive the no-trade theorem for any decision theory in

which a strict preference conditional on each of two disjoint events persists conditional on their union. Theorem 3 is an instance of their result, extended to n agents: the set functions $F \mapsto [\mathbb{E}_\rho[X_i \mid F] > 0]$ are preserved under unions by Lemma 5 (i), and by Lemma 5 (ii) no event satisfies all of them for a bet $(X_i)_{i \in I}$.

6 Failure of the Converse

In probability theory, the absence of a bet whose acceptance by all agents is common knowledge implies the existence of a common prior (Morris, 1994, Lemma A2 (iv); Samet, 1998, Corollary; Feinberg, 2000, Theorems 2 and 7; Heifetz, 2006, Propositions 1 and 3). This is the converse of the no-trade theorem. However, with ranking functions the converse fails. In fact, the following example demonstrates that it even fails in the case of agents having no information.

Example 1. *Let $I = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and $\Pi_1(\omega) = \Pi_2(\omega) = \Omega$ for all ω, so that there is no private information. Suppose agent 1 has the ranking function ρ_1 with $(\rho_1(\{\omega_1\}), \rho_1(\{\omega_2\}), \rho_1(\{\omega_3\})) = (0, 1, 2)$ and agent 2 has the ranking function ρ_2 with $(\rho_2(\{\omega_1\}), \rho_2(\{\omega_2\}), \rho_2(\{\omega_3\})) = (0, 2, 1)$. Since $\rho_1 \neq \rho_2$, there is no common prior. Yet there is no bet that both agents accept. Indeed, neither agent disbelieves ω_1, so $\mathbb{E}_{\rho_1}[X_1 \mid \Omega] \leq X_1(\omega_1)$ and $\mathbb{E}_{\rho_2}[X_2 \mid \Omega] \leq X_2(\omega_1)$ for every bet (X_1, X_2). If both expectations were positive, then $X_1(\omega_1) + X_2(\omega_1) > 0$, contradicting $X_1 + X_2 = 0$. Moreover, the agents commonly know that agent 1 assigns rank 1 to $\{\omega_2\}$ while agent 2 assigns rank 2. That is, they agree to disagree. With $(\rho_1(\{\omega_1\}), \rho_1(\{\omega_2\}), \rho_1(\{\omega_3\})) = (0, 0, 1)$ and $(\rho_2(\{\omega_1\}), \rho_2(\{\omega_2\}), \rho_2(\{\omega_3\})) = (0, 1, 0)$, again no bet is accepted by both, but agent 1 does not disbelieve $\{\omega_2\}$ while agent 2 does, so the agents do not even agree on whether $\{\omega_2\}$ is disbelieved.*

Example 1 illustrates why the absence of speculative betting and agreement come apart in ranking theory. Whether there is a bet acceptable to all agents depends only on the states that the agents do not disbelieve. In Example 1, the state ω_1 of rank 0 for both agents rules out any such bet, whatever the ranks of ω_2 and ω_3. This observation motivates our characterization theorem in the next section. In contrast, posterior ranks depend on the entire ranking function so that agents may agree to disagree about events they disbelieve. In probability theory, every state of positive probability matters both for betting and for posteriors, and both are tied to the common prior.

The failure of the converse reflects a clash between two kinds of linearity. Bets redistribute money, so the condition $\sum_{i \in I} X_i = 0$ is linear in the ordinary sense, whereas tropical expectations are linear in the tropical sense. In probability theory, both are linear in the ordinary sense. Then $E[-X] = -E[X]$, so that an agent's price for buying a random variable equals her price for selling it, and two agents who disagree about the expectation of a random variable can bet on it. With ranking functions, an agent's price for buying X, $\mathbb{E}_\rho[X \mid F]$, is generally lower than her price for selling it, $-\mathbb{E}_\rho[-X \mid F]$, by Remark 3. In Example 1, let $X := 3 \cdot \mathbf{1}_{\Omega \setminus \{\omega_2\}}$. Then $\mathbb{E}_{\rho_1}[X \mid \Omega] = 1 < 2 = \mathbb{E}_{\rho_2}[X \mid \Omega]$, so agent 2 values X more than agent 1 does. Yet agent 1's price for selling X is $-\mathbb{E}_{\rho_1}[-X \mid \Omega] = 3$, which exceeds agent 2's price for buying it. Since neither agent disbelieves ω_1, every agent's price

for buying any random variable X is weakly smaller than $X(\omega_1)$, and her price for selling it is weakly greater than $X(\omega_1)$. We return to this clash of linearities in Section 11.2.

7 A First Route: Weakening the Common Prior

To identify what weakening of the common prior assumption the absence of speculative betting characterizes, we do not assume a common prior from the outset but take the agents' interim ranking functions as primitives. This is analogous to type spaces with partitions in the probabilistic literature (e.g., Samet, 1998). Let $R(\Omega)$ denote the set of ranking functions on Ω.

Definition 11 (Ranking type structure)**.** *A* ranking type structure $\langle I, \Omega, (\Pi_i)_{i\in I}, (r_i)_{i\in I}\rangle$ *consists of a finite Aumann structure and, for each $i \in I$, a ranking-type function $r_i : \Omega \longrightarrow R(\Omega)$ such that for all $\omega \in \Omega$,*

(a) Certainty of own information: $r_i(\omega)(\Omega \setminus \Pi_i(\omega)) = \infty$*, and*

(b) Knowledge of own type: $\omega' \in \Pi_i(\omega)$ *implies* $r_i(\omega') = r_i(\omega)$*.*

Here $r_i(\omega)$ is agent i's ranking-function type at ω. By (a), she has maximal disbelief in states she knows have not occurred. Since $r_i(\omega)(\Omega) = 0$ by property (1) and $r_i(\omega)(\Omega) = \min\{r_i(\omega)(\Pi_i(\omega)), r_i(\omega)(\Omega \setminus \Pi_i(\omega))\}$ by property (3), (a) implies $r_i(\omega)(\Pi_i(\omega)) = 0$. That is, she does not disbelieve her own information. Condition (b) says that the ranking-type function is measurable w.r.t. the agent's partition implying that her ranking function only depends on her own information and she always knows her own ranking function. Her rank of an event E at ω is $r_i(\omega)(E)$, and her tropical interim expectation of a random variable X at ω is

$$\mathbb{E}_i[X](\omega) := \min_{\omega'\in\Omega}\{X(\omega') + r_i(\omega)(\{\omega'\})\}.$$

By (a), the minimum can equivalently be taken over $\Pi_i(\omega)$.

Definition 12 (Prior and common prior)**.** *In a ranking type structure, a ranking function ρ_i is a* prior *for agent i if $\rho_i(\Pi_i(\omega)) < \infty$ and $\rho_i(\cdot \mid \Pi_i(\omega)) = r_i(\omega)$ for all $\omega \in \Omega$. A ranking function ρ is a* common prior *if it is a prior for every agent.*

As in probability theory, an agent's prior is not uniquely determined by her interim rankings. A prior ρ_i consists of the ranks $\rho_i(C)$ of the cells C of agent i's partition and the conditional ranking functions $\rho_i(\cdot \mid C)$. Two priors that differ only in the ranks they assign to the cells induce the same conditional ranking functions. In fact, every prior ρ_i of agent i satisfies

$$\rho_i(\{\omega\}) = \bigoplus_C \rho_i(C) \odot r_i(\omega_C)(\{\omega\}) = \min_C\{\rho_i(C) + r_i(\omega_C)(\{\omega\})\} \quad \text{for all } \omega \in \Omega,$$

where C ranges over the cells of agent i's partition and $\omega_C \in C$. By (b), $r_i(\omega_C)$ does not depend on the choice of ω_C. That is, every prior of agent i is a tropical convex combination of her ranking-function types, weighted by the prior's ranks of the cells, the analog of

Samet's (1998) observation that the priors of an agent form the convex hull of her types (see Section 11.1). Since agents accept or reject bets after learning their information, only their ranking-function types matter. If ρ is a common prior, then the agents' interim ranking functions are the same as if every agent had a prior ranking function ρ.

In this section, we do not impose Assumption 1. If it holds, then $r_i(\omega) := \rho(\cdot \mid \Pi_i(\omega))$ defines a ranking type structure with common prior ρ, and $\mathbb{E}_i[X](\omega) = \mathbb{E}_\rho[X \mid \Pi_i(\omega)]$.

In a ranking type structure, a bet and the events $☺_i(X_i)$ are defined as before, with $\mathbb{E}_i[X_i](\omega)$ in place of $\mathbb{E}_\rho[X_i \mid \Pi_i(\omega)]$.

Definition 13 (Common unsurprising state)**.** *A cell M of the meet of the agents' partitions contains a* common unsurprising state *if there is $\omega_0 \in M$ with $r_i(\omega_0)(\{\omega_0\}) = 0$ for all $i \in I$.*

That is, no agent disbelieves ω_0 given her information at ω_0, i.e., no agent would be surprised by ω_0. In the terminology of Shackle (1969), ω_0 carries zero potential surprise for every agent.

Theorem 4 (Characterization of no speculative betting)**.** *$CK\left(\bigcap_{i\in I} ☺_i(X_i)\right) = \emptyset$ for every bet $(X_i)_{i\in I}$ if and only if every cell of the meet of the agents' partitions contains a common unsurprising state.*

Proof. If: Suppose that every cell of the meet of the agents' partitions contains a common unsurprising state, and suppose to the contrary that there are a bet $(X_i)_{i\in I}$ and $\omega \in CK\left(\bigcap_{i\in I} ☺_i(X_i)\right)$. Let $M := \Pi_*(\omega)$ be the cell of the meet of the agents' partitions that contains ω. By Aumann's (1976) characterization of common knowledge, $M \subseteq \bigcap_{i\in I} ☺_i(X_i)$, so $M \subseteq ☺_i(X_i)$ for all i. Let $\omega_0 \in M$ be a common unsurprising state. Fix $i \in I$. Then

$$X_i(\omega_0) = X_i(\omega_0) + r_i(\omega_0)(\{\omega_0\}) \geq \min_{\omega'\in\Omega}\{X_i(\omega') + r_i(\omega_0)(\{\omega'\})\} = \mathbb{E}_i[X_i](\omega_0) > 0.$$

The first equality holds since $r_i(\omega_0)(\{\omega_0\}) = 0$ by the definition of a common unsurprising state. The first inequality holds since a minimum is weakly smaller than each of its terms, in particular the term for $\omega' = \omega_0$. The second equality is the definition of $\mathbb{E}_i[X_i](\omega_0)$. The last inequality holds since $\omega_0 \in M \subseteq ☺_i(X_i)$. Summing over $i \in I$ yields $\sum_i X_i(\omega_0) > 0$, contradicting $\sum_i X_i(\omega_0) = 0$.

Only if: We prove the contrapositive. Suppose that some cell M of the meet of the agents' partitions contains no common unsurprising state. We construct a bet $(X_i)_{i\in I}$ with $M \subseteq CK\left(\bigcap_{i\in I} ☺_i(X_i)\right)$. Since $M \neq \emptyset$ by reflexivity of Π_*, this shows $CK\left(\bigcap_{i\in I} ☺_i(X_i)\right) \neq \emptyset$. For $\omega \in M$, define the "disbelief budget" at ω as the "number of dollars", one for each agent who disbelieves ω,

$$b(\omega) := \left|\{j \in I : r_j(\omega)(\{\omega\}) \geq 1\}\right|.$$

Note that $b(\omega) \geq 1$ because ω is not a common unsurprising state. Define a bet by distributing the disbelief budget among agents by setting, for all $i \in I$,

$$X_i(\omega) := \begin{cases} b(\omega)/n & \text{if } r_i(\omega)(\{\omega\}) = 0, \\ -(1 - b(\omega)/n) & \text{if } r_i(\omega)(\{\omega\}) \geq 1. \end{cases}$$

That is, each agent who disbelieves ω pays $1-b(\omega)/n$, and each agent who does not disbelieve ω receives $b(\omega)/n$. Then

$$\sum_{i\in I} X_i(\omega) = (n - b(\omega)) \cdot \frac{b(\omega)}{n} - b(\omega) \cdot \left(1 - \frac{b(\omega)}{n}\right) = 0.$$

Furthermore, for all $i \in I$,

$$X_i(\omega) + r_i(\omega)(\{\omega\}) \geq \frac{b(\omega)}{n} \geq \frac{1}{n}.$$

The first inequality holds with equality if $r_i(\omega)(\{\omega\}) = 0$, by the first case in the definition of $X_i(\omega)$. If $r_i(\omega)(\{\omega\}) \geq 1$, then $X_i(\omega) + r_i(\omega)(\{\omega\}) \geq -(1 - b(\omega)/n) + 1 = b(\omega)/n$ by the second case, where the left-hand side is ∞ if $r_i(\omega)(\{\omega\}) = \infty$. The second inequality holds because $b(\omega) \geq 1$.

For $\omega \notin M$, let $X_i(\omega) := 0$ for all i, so that $\sum_i X_i(\omega) = 0$. Thus $(X_i)_{i\in I}$ is a bet, and $X_i(\omega) + r_i(\omega)(\{\omega\}) \geq 1/n$ for all $i \in I$ and $\omega \in M$.

To show acceptance of the bet, we show that for all $\omega \in M$ and $i \in I$,

$$\mathbb{E}_i[X_i](\omega) = \min_{\omega'\in\Omega}\{X_i(\omega') + r_i(\omega)(\{\omega'\})\} \geq \frac{1}{n}.$$

Fix $\omega \in M$, $i \in I$, and $\omega' \in \Omega$. If $\omega' \notin \Pi_i(\omega)$, then $r_i(\omega)(\{\omega'\}) = \infty$ by condition (a) of Definition 11, so the term for ω' is ∞. If $\omega' \in \Pi_i(\omega)$, then the term for ω' satisfies $X_i(\omega')+r_i(\omega)(\{\omega'\}) = X_i(\omega')+r_i(\omega')(\{\omega'\}) \geq 1/n$. The equality holds because $r_i(\omega) = r_i(\omega')$ by condition (b) of Definition 11. The inequality follows from the construction above, since $\omega' \in M$ because $\Pi_i(\omega) \subseteq \Pi_*(\omega) = M$ by property (a) of the meet of the agents' partitions. So every term in the minimum is at least $1/n$, and thus $\mathbb{E}_i[X_i](\omega) \geq 1/n > 0$, i.e., $\omega \in \text{☺}_i(X_i)$.

The last step is to show that acceptance of the bet is common knowledge on M. Since $\omega \in M$ and $i \in I$ were arbitrary, $M \subseteq \bigcap_{i\in I} \text{☺}_i(X_i)$. Since M is a cell of the meet of the agents' partitions, $\Pi_*(\omega) = M \subseteq \bigcap_{i\in I} \text{☺}_i(X_i)$. By Aumann's (1976) characterization of common knowledge, $\omega \in CK\left(\bigcap_{i\in I} \text{☺}_i(X_i)\right)$. Since $\omega \in M$ was arbitrary, $M \subseteq CK\left(\bigcap_{i\in I} \text{☺}_i(X_i)\right)$. □

As announced in Remark 4, the same argument as in the 'if' part of the proof shows that it cannot be common knowledge that all agents weakly accept the bet and at least one agent strictly accepts it. A common prior guarantees a common unsurprising state.

Lemma 6. *In a ranking type structure with a common prior, every cell of the meet of the agents' partitions contains a common unsurprising state.*

PROOF. Let ρ be a common prior, M a cell of the meet of the agents' partitions, and $\omega_0 \in M$ a state with $\rho(\{\omega_0\}) = \rho(M)$, which exists by property (3). Fix $i \in I$. Then

$$0 \leq r_i(\omega_0)(\{\omega_0\}) = \rho(\{\omega_0\} \mid \Pi_i(\omega_0)) = \rho(\{\omega_0\}) - \rho(\Pi_i(\omega_0)) \leq \rho(\{\omega_0\}) - \rho(M) = 0.$$

The first inequality holds since ranks are nonnegative. The first equality holds since ρ is a prior for agent i. The second equality follows from Definition 2, since $\omega_0 \in \Pi_i(\omega_0)$ by reflexivity and $\rho(\Pi_i(\omega_0)) < \infty$ by the definition of a prior. The second inequality follows

from monotonicity, since $\Pi_i(\omega_0) \subseteq \Pi_*(\omega_0) = M$ by property (a) of the meet of the agents' partitions and stationarity of Π_*. The last equality holds by the choice of ω_0, where $\rho(M) \leq \rho(\Pi_i(\omega_0)) < \infty$. Hence $r_i(\omega_0)(\{\omega_0\}) = 0$ for all $i \in I$, i.e., ω_0 is a common unsurprising state in M. □

As noted after Definition 12, under Assumption 1 the conditional ranks define a ranking type structure with common prior ρ. Thus Theorem 3 also follows from Lemma 6 and Theorem 4. In Example 1, state ω_1 is a common unsurprising state, which is why no bet is accepted by both agents.

Why is the condition in Theorem 4 so much weaker than a common prior? Formally, since a linear function on the set of probability measures $\Delta(F)$ attains its minimum at a point mass, the tropical expectation can be written as a minimum of expected values,

$$\mathbb{E}_\rho[X \mid F] = \min_{\pi \in \Delta(F)} \Big\{ \sum_{\omega \in F} \pi(\omega) X(\omega) + c(\pi) \Big\} \quad \text{with } c(\pi) := \sum_{\omega \in F} \pi(\omega) \rho(\{\omega\} \mid F),$$

where states of infinite rank are excluded from the support. This is a mathematical identity, not a foundation, since ranking-theoretic agents hold no probabilities. Yet, it shows that tropical expectations have the form of the variational preferences of Maccheroni, Marinacci, and Rustichini (2006), $\min_\pi \{\mathbb{E}_\pi[X] + c(\pi)\}$, with a cost function c that is linear in π and grounded, $\min_\pi c(\pi) = 0$. This relates our result to the literature on trade under ambiguity. In a ranking type structure, the probabilities at which agent i's cost function is zero at ω are those supported on the states $\omega' \in \Pi_i(\omega)$ with $r_i(\omega)(\{\omega'\}) = 0$. Theorem 4 says that betting is impossible if and only if each cell of the meet of the agents' partitions contains a state ω_0 with $r_i(\omega_0)(\{\omega_0\}) = 0$ for all $i \in I$, i.e., if and only if the point mass on some state of each cell is a zero-cost belief for every agent. This parallels results on trade under ambiguity, where no betting is equivalent to the intersection of the agents' sets of priors being nonempty rather than to a common prior (Billot et al., 2000, for maxmin expected utility; Rigotti, Shannon, and Strzalecki, 2008, for variational preferences and ex ante trade; Kajii and Ui, 2009, and Martins-da-Rocha, 2010, for interim trade; see also Dominiak and Lefort, 2015, for agreement under Choquet expected utility). Ranking-theoretic agents, who evaluate bets by worst cases discounted by disbelief, behave like ambiguity-averse agents in this respect. I.e., they refuse to bet against each other as long as there is a state that none of them disbelieves. Unlike ambiguity-averse agents, however, they do not exhibit Ellsberg-type behavior (see Schipper, 2026a, Example 3).

8 A Second Route: Strengthening No Betting

In this section, we aim for a characterization of the common prior by strengthening the absence of speculative betting. First, we analyze to what extent common priors can be characterized in terms of the agents' posterior ranks. Second, we use this analysis to show that the absence of speculative betting from *any common endowment* characterizes a common prior.

8.1 Common Priors as Potentials

We seek to characterize common priors in terms of the agents' posterior ranks. This characterization is of interest in its own right, and it is the key to the characterization by bets in Section 8.2.

Definition 14 (Full support)**.** *A ranking type structure has* full support *if* $r_i(\omega)(\{\omega'\}) < \infty$ *for all* $i \in I$*,* $\omega \in \Omega$*, and* $\omega' \in \Pi_i(\omega)$*.*

This is the analog of the full-support assumption in the probabilistic literature (e.g., Milgrom and Stokey, 1982; Morris, 1994; Rodrigues-Neto, 2009; Hellman and Samet, 2012).

We make use of two at first glance unrelated ideas by Rodrigues-Neto (2009) and Monderer and Shapley (1996), respectively.

Definition 15 (Path and cycle)**.** *A* path *from* ω *to* ω' *is a sequence* $\omega = \omega_0, \omega_1, \ldots, \omega_m = \omega'$ *together with agents* $i_1, \ldots, i_m$ *such that* $\omega_t \in \Pi_{i_t}(\omega_{t-1})$ *for all* $t = 1, \ldots, m$*, where we allow* $m = 0$*. Its* sum *is*

$$\sum_{t=1}^{m} \left(r_{i_t}(\omega_{t-1})(\{\omega_t\}) - r_{i_t}(\omega_{t-1})(\{\omega_{t-1}\}) \right).$$

A cycle *is a path from a state to itself.*

Under full support, the sum of a path is a finite integer. By the characterization of the meet of the agents' partitions, two states are connected by a path if and only if they lie in the same cell of the meet of the agents' partitions. The difference $r_i(\omega)(\{\omega'\}) - r_i(\omega)(\{\omega\})$ for $\omega' \in \Pi_i(\omega)$ measures how much more agent i disbelieves ω' than ω given her information $\Pi_i(\omega)$. Rodrigues-Neto (2009) characterizes common priors in probabilistic type spaces by requiring that products of the agents' posterior likelihood ratios along cycles be equal to one. As we will see below, its analog for ranking functions requires sums of rank differences along cycles to equal zero.

Next comes the idea from potential games by Monderer and Shapley (1996):

Definition 16 (Potential)**.** *A function* $P : \Omega \longrightarrow \mathbb{Z}$ *is a* potential *if for all* $i \in I$*,* $\omega \in \Omega$*, and* $\omega' \in \Pi_i(\omega)$*,*

$$r_i(\omega)(\{\omega'\}) - r_i(\omega)(\{\omega\}) = P(\omega') - P(\omega). \tag{8}$$

A potential places all agents' comparisons on a single scale. How much more one state is disbelieved than another does not depend on which agent compares the two states or on her information. In Monderer and Shapley (1996), an (exact) potential is a single function whose differences equal each player's payoff differences under unilateral deviations.

Theorem 5 (Common priors as potentials)**.** *Consider a ranking type structure with full support. The following are equivalent:*

(i) There is a common prior.

(ii) There is a potential.

(iii) Every cycle has sum zero, i.e., for every cycle,

$$\sum_{t=1}^{m} \left(r_{i_t}(\omega_{t-1})(\{\omega_t\}) - r_{i_t}(\omega_{t-1})(\{\omega_{t-1}\}) \right) = 0. \tag{9}$$

Moreover, a ranking function ρ is a common prior if and only if $\rho(\{\cdot\})$ is a potential. Furthermore, any two potentials differ only by a constant on each cell of the meet of the agents' partitions.

Theorem 5 says that a common prior is a potential, normalized to be a ranking function. Since potentials are unique up to a constant on each cell of the meet of the agents' partitions, so are common priors. These constants must keep all ranks nonnegative, and some cell must retain a minimum rank of 0. Equation (9) is necessary, because under a potential the rank differences along a cycle add up to $P(\omega_0) - P(\omega_0) = 0$. If the sum along some cycle were let's say positive, then chaining the agents' comparisons along the cycle would rank ω_0 above itself, and no common ranking function could represent all these comparisons. Theorem 5 shows that this is the only obstacle to a common prior.

Proof of Theorem 5. We show that a common prior is a potential (Step 1) and that a ranking function that is a potential is a common prior (Step 2). This proves the 'moreover' part and that (i) implies (ii). Then we show that (ii) implies (iii) (Step 3), (iii) implies (ii) (Step 4), and (ii) implies (i) (Step 5). Step 6 shows uniqueness.

Step 1: A common prior is a potential. Let ρ be a common prior, $\omega' \in \Pi_i(\omega)$, and $C := \Pi_i(\omega)$. Then

$$\begin{aligned} r_i(\omega)(\{\omega'\}) - r_i(\omega)(\{\omega\}) &= \big(\rho(\{\omega'\}) - \rho(C)\big) - \big(\rho(\{\omega\}) - \rho(C)\big) \\ &= \rho(\{\omega'\}) - \rho(\{\omega\}). \end{aligned}$$

The first equality holds because $\rho(C) < \infty$ and $r_i(\omega) = \rho(\cdot \mid C)$ since ρ is a prior for agent i, and Definition 2 applies to $\omega, \omega' \in C$. The second holds because $\rho(C)$ cancels. Since $\rho(C) < \infty$ and $r_i(\omega)(\{\omega'\}) < \infty$, $\rho(\{\omega'\})$ is a nonnegative integer. Every state lies in one of its own possibility sets, so $\rho(\{\cdot\})$ maps Ω to $\mathbb{Z}$ and is a potential.

Step 2: A ranking function that is a potential is a common prior. Let ρ be a ranking function such that $\rho(\{\cdot\})$ is a potential. Let C be a possibility set of agent i and $\omega' \in C$, so that $C = \Pi_i(\omega')$ by stationarity. Then $\rho(C) < \infty$ since $\rho(\{\cdot\})$ takes values in $\mathbb{Z}$, and

$$\begin{aligned} \rho(\{\omega'\} \mid C) &= \rho(\{\omega'\}) - \min_{\omega'' \in C} \rho(\{\omega''\}) \\ &= \max_{\omega'' \in C} \left(\rho(\{\omega'\}) - \rho(\{\omega''\}) \right) \\ &= \max_{\omega'' \in C} \left(r_i(\omega'')(\{\omega'\}) - r_i(\omega'')(\{\omega''\}) \right) \\ &= \max_{\omega'' \in C} \left(r_i(\omega')(\{\omega'\}) - r_i(\omega')(\{\omega''\}) \right) \\ &= r_i(\omega')(\{\omega'\}) - r_i(\omega')(C) \\ &= r_i(\omega')(\{\omega'\}). \end{aligned}$$

The first equality follows from Definition 2 and property (3). The second rewrites the subtraction of a minimum as a maximum of differences. The third follows from equation (8), since $\omega' \in \Pi_i(\omega'') = C$. The fourth holds because $r_i(\omega'') = r_i(\omega')$ by condition (b) of Definition 11. The fifth follows from property (3). The sixth holds because $r_i(\omega')(C) = 0$ by condition (a) of Definition 11 and $r_i(\omega')(\Omega) = 0$. Since $r_i(\omega'') = r_i(\omega')$ for all $\omega'' \in C = \Pi_i(\omega')$, this shows $\rho(\{\omega''\} \mid C) = r_i(\omega')(\{\omega''\})$ for all $\omega'' \in C$. Both $\rho(\cdot \mid C)$ and $r_i(\omega')$ assign rank ∞ to states outside C, by Definition 2 and by condition (a) of Definition 11, respectively. Hence $\rho(\cdot \mid \Pi_i(\omega')) = r_i(\omega')$. Since $\rho(\Pi_i(\omega')) < \infty$, ρ is a prior for every agent, i.e., a common prior.

Step 3: (ii) implies (iii). Let P be a potential, and consider a cycle. Then

$$\begin{aligned}\sum_{t=1}^{m} \left(r_{i_t}(\omega_{t-1})(\{\omega_t\}) - r_{i_t}(\omega_{t-1})(\{\omega_{t-1}\})\right) &= \sum_{t=1}^{m} \left(P(\omega_t) - P(\omega_{t-1})\right) \\ &= P(\omega_m) - P(\omega_0) = 0.\end{aligned}$$

The first equality follows from equation (8), since $\omega_t \in \Pi_{i_t}(\omega_{t-1})$. The second holds because the sum telescopes, and the third because $\omega_m = \omega_0$.

Step 4: (iii) implies (ii). The idea is to accumulate the agents' rank differences along paths. By equation (9), the accumulated rank differences depend only on the endpoints of a path. Thus, they assign each state a single value, which is a potential.

First, the sum of a path depends only on its endpoints. Each step of a path can be reversed by the same agent. If $\omega_t \in \Pi_{i_t}(\omega_{t-1})$, then $\omega_{t-1} \in \Pi_{i_t}(\omega_{t-1}) = \Pi_{i_t}(\omega_t)$ by reflexivity and stationarity. Since $r_{i_t}(\omega_t) = r_{i_t}(\omega_{t-1})$ by condition (b) of Definition 11, the reversed step contributes the negative of the original term. Following one path from ω to ω' and then the reverse of another gives a cycle whose sum is the difference of the two sums. By equation (9), this difference is 0.

For each cell M of the meet of the agents' partitions, fix $\bar{\omega}_M \in M$. We define $P(\omega)$ as the sum of any path from $\bar{\omega}_M$ to ω, where M is the cell containing ω. Let $\omega' \in \Pi_i(\omega)$. Then ω' lies in the same cell M as ω by property (a) of the meet. Concatenating a path from $\bar{\omega}_M$ to ω with the one-step path from ω to ω' by agent i yields a path from $\bar{\omega}_M$ to ω'. Its sum is $P(\omega')$ by the definition of P, and it is the sum of the path to ω plus the contribution of the last step. Hence

$$P(\omega') = P(\omega) + r_i(\omega)(\{\omega'\}) - r_i(\omega)(\{\omega\}),$$

i.e., P satisfies equation (8).

Step 5: (ii) implies (i). Let P be a potential. For $\omega \in \Omega$, let M be the cell of the meet of the agents' partitions containing ω, and let

$$\rho(\{\omega\}) := P(\omega) - \min_{\omega'' \in M} P(\omega''),$$

$\rho(E) := \min_{\omega \in E} \rho(\{\omega\})$ for $E \neq \emptyset$, and $\rho(\emptyset) := \infty$. Since two consecutive states of a path lie in the same cell, subtracting a constant on each cell preserves equation (8), so $\rho(\{\cdot\})$ is a potential. Its values are nonnegative integers, and it attains 0 in every cell, so $\rho(\Omega) = 0$. Property (2) holds by definition, and property (3) holds because the minimum over $E \cup F$ is the smaller of the minima over E and over F. So ρ is a ranking function, and by Step 2 it is a common prior.

Step 6: Uniqueness. Let P and P' be potentials, and $D := P - P'$. By equation (8), $D(\omega') = D(\omega)$ whenever $\omega' \in \Pi_i(\omega)$ for some $i \in I$. So D is constant along every path, hence on every cell of the meet of the agents' partitions. □

The equivalence of (i) and (iii) in Theorem 5 is the min-plus analog of the characterizations of common priors by Rodrigues-Neto (2009) and Hellman and Samet (2012, Proposition 4), in which products of likelihood ratios along cycles must equal one, or, equivalently, products along chains connecting two states must not depend on the chain. The uniqueness part of Theorem 5 is the analog of Hellman and Samet (2012, Proposition 3). The equivalence of (ii) and (iii) in Theorem 5 is the analog of the characterization of exact potential games by Monderer and Shapley (1996, Theorem 2.8), with states in the role of strategy profiles, possibility sets in the role of the sets of profiles a player can reach by unilateral deviations, and rank differences in the role of payoff differences. Step 4 of the proof follows their construction of the potential, and the uniqueness part corresponds to Monderer and Shapley (1996, Lemma 2.7). The cycle condition can be checked efficiently by computing potentials along a spanning tree of each cell of the meet of the agents' partitions. Moreover, Monderer and Shapley (1996, Theorem 2.8) show that it suffices to check closed paths of length 4, which is the analog of Hellwig (2013, Proposition 2), who shows that if every possibility set of each agent intersects every possibility set of every other agent, it suffices to verify the cycle equations by Rodrigues-Neto (2009) for cycles of length at most 4. Hellwig's argument uses only that the contributions of edges add up along a cycle in logarithms and change sign when an edge is reversed, which holds for the rank differences in equation (9). So, under the same condition, it suffices to verify equation (9) for cycles of length at most 4.

8.2 No Betting from Common Endowments

In Sections 5 to 7, we assumed that agents evaluate bets from a riskless position. In probability theory, this is without loss of generality, since expected values are additive, $E[Y + X] = E[Y] + E[X]$, so that $E[Y + X] - E[Y] = E[X]$ for every random variable Y that an agent may already be endowed with. Tropical expected values are not additive. In general, $\mathbb{E}_\rho[Y + X \mid F] \neq \mathbb{E}_\rho[Y \mid F] + \mathbb{E}_\rho[X \mid F]$, and equality holds for all X only if Y is constant on the states of finite rank in F. In tropical terms, adding Y is tropical multiplication by Y, and tropical expectations are tropically linear only with respect to constants (Section 11.2). An agent who is endowed with Y evaluates a bet X by

$$\mathbb{E}_\rho[Y + X \mid F] = \min_{\omega \in F} \big\{ X(\omega) + \big(Y(\omega) + \rho(\{\omega\} \mid F) \big) \big\},$$

i.e., as if her ranks were shifted by Y. A risky position thus acts like a tropical change of measure. It changes the states at which the agent's valuation is attained: with a riskless position, these are the states that she does not disbelieve, while with a risky position, they are the states at which Y plus disbelief is lowest, which may be disbelieved states at which Y takes a low value. Bets that pay at these states can then exploit disagreements about disbelieved states that a riskless position hides.

Example 1 (continued). *Recall that in Example 1, there is no private information, and the agents' ranks of* $\omega_1, \omega_2, \omega_3$ *are* $(0, 1, 2)$ *for agent 1 and* $(0, 2, 1)$ *for agent 2. Neither*

agent disbelieves ω_1, but they disagree about the disbelieved states: agent 1 disbelieves ω_2 less than ω_3, while agent 2 disbelieves ω_3 less than ω_2. There is no common prior, yet no bet is acceptable to both agents from a riskless position, since ω_1 is a common unsurprising state.

Now let both agents be endowed with Y with $(Y(\omega_1), Y(\omega_2), Y(\omega_3)) = (10, 0, 0)$. Then

$$\mathbb{E}_{\rho_1}[Y \mid \Omega] = \min\{10+0, 0+1, 0+2\} = 1 \quad \textit{and} \quad \mathbb{E}_{\rho_2}[Y \mid \Omega] = \min\{10+0, 0+2, 0+1\} = 1.$$

Since Y pays 10 at ω_1, the agents' valuations are no longer attained at ω_1. Agent 1's valuation is attained only at ω_2, and agent 2's only at ω_3. Consider the bet with $(X_1(\omega_1), X_1(\omega_2), X_1(\omega_3)) = (0, 1/2, -1/2)$ and $X_2 = -X_1$. Each agent gains $1/2$ at the state at which her valuation is attained and pays $1/2$ at the other disbelieved state. Then

$$\mathbb{E}_{\rho_1}[Y + X_1 \mid \Omega] = \min\{10, 1/2+1, -1/2+2\} = 3/2 > 1,$$
$$\mathbb{E}_{\rho_2}[Y + X_2 \mid \Omega] = \min\{10, -1/2+2, 1/2+1\} = 3/2 > 1.$$

Each agent's loss falls on the state that she disbelieves more, at which Y plus her disbelief exceeds her valuation by 1. Since the loss of $1/2$ is smaller, it does not lower her valuation, while the gain raises it. So both agents accept the bet, and since there is no private information, this is common knowledge. From a riskless position, the same bet is not acceptable to agent 1, since $\mathbb{E}_{\rho_1}[X_1 \mid \Omega] = \min\{0, 1/2+1, -1/2+2\} = 0$ is attained at ω_1, at which the bet pays nothing. The common endowment thus makes the agents' disagreement about the disbelieved states ω_2 and ω_3 tradable, which the common unsurprising state ω_1 hides from a riskless position.

The example suggests strengthening the absence of speculative betting by requiring it for every common endowment.

Definition 17 (Common endowment)**.** *A* common endowment *is a random variable Y with which every agent is endowed.*

Definition 18 (Acceptance with common endowment)**.** *Agent i with common endowment Y* accepts *X_i at ω if $\mathbb{E}_i[Y + X_i](\omega) > \mathbb{E}_i[Y](\omega)$. Let*

$$\text{☺}_i^Y(X_i) := \{\omega \in \Omega : \mathbb{E}_i[Y + X_i](\omega) > \mathbb{E}_i[Y](\omega)\}$$

be the event that agent i with common endowment Y accepts X_i.

Since $\mathbb{E}_i[0] = 0$, we have $\text{☺}_i^0(X_i) = \text{☺}_i(X_i)$.

Definition 19 (Common bleakest state)**.** *A state ω is a* common bleakest state for Y *if for all $i \in I$,*

$$\omega \in \arg\min_{\omega' \in \Omega}\{Y(\omega') + r_i(\omega)(\{\omega'\})\},$$

i.e., $Y(\omega) + r_i(\omega)(\{\omega\}) = \mathbb{E}_i[Y](\omega)$.

That is, every agent's valuation of Y is attained at ω itself. The name reflects the intuition that, for every agent, ω minimizes the common endowment plus disbelief. The outlook at ω is bleak because the endowment is low there and the agent does not disbelieve

ω enough to discount it. For $Y = 0$, this intuition does not apply, since the endowment is the same at every state. A common bleakest state for 0 is then simply a state that no agent disbelieves, for which we already have a name, namely a common unsurprising state. This suggests the following generalization of Theorem 4 from a riskless position to arbitrary common endowments.

Theorem 6 (Characterization of no speculative betting from a common endowment)**.** *Let Y be a common endowment. Then $CK\left(\bigcap_{i\in I} \text{☺}_i^Y(X_i)\right) = \emptyset$ for every bet $(X_i)_{i\in I}$ if and only if every cell of the meet of the agents' partitions contains a common bleakest state for Y.*

For $Y = 0$, Theorem 6 reduces to Theorem 4. The proof of Theorem 6 is analogous to the proof of Theorem 4. In the 'if' part, the common unsurprising state is replaced by a common bleakest state for Y. In the 'only if' part, the disbelief budget at a state ω counts the agents i for whom ω is not bleakest, i.e., with $Y(\omega) + r_i(\omega)(\{\omega\}) > \mathbb{E}_i[Y](\omega)$, rather than the agents who disbelieve ω. In the proof of Theorem 4, every positive rank is at least 1, which makes the constructed bet acceptable. Since Y need not take integer values, the excess of $Y(\omega) + r_i(\omega)(\{\omega\})$ over $\mathbb{E}_i[Y](\omega)$ may be smaller than 1, so the bet is scaled down below the smallest positive such excess. Moreover, $\mathbb{E}_i[Y]$ is constant on each possibility set of agent i by condition (b) of Definition 11, which plays the role of $\mathbb{E}_i[0] = 0$. For completeness, we include the proof in Appendix A.

Theorem 6 is used for the next theorem showing that requiring the absence of commonly known acceptable bets for *every* common endowment characterizes a common prior.

Theorem 7 (No speculative betting from common endowments)**.** *Consider a ranking type structure with full support. The following are equivalent:*

(i) There is a common prior.

(ii) For every common endowment Y, every cell of the meet of the agents' partitions contains a common bleakest state for Y.

(iii) For every common endowment Y and every bet $(X_i)_{i\in I}$, $CK\left(\bigcap_{i\in I} \text{☺}_i^Y(X_i)\right) = \emptyset$.

That (i) implies (iii) extends Theorem 3 from a riskless position to all common endowments, and it does not require full support. That (iii) implies (i) is its converse. In probability theory, acceptance from a common endowment does not depend on the endowment, since $E[Y + X_i] > E[Y]$ if and only if $E[X_i] > 0$. So the probabilistic analogs of both implications are the no-trade theorem (Milgrom and Stokey, 1982; Sebenius and Geanakoplos, 1983) and its converse (Morris, 1994; Samet, 1998; Feinberg, 2000; Heifetz, 2006), which consider bets from a riskless position only.

Proof of Theorem 7. By Theorem 6, applied to each common endowment Y, (ii) and (iii) are equivalent. It remains to show that (i) and (ii) are equivalent.

(i) implies (ii): Let ρ be a common prior, Y a common endowment, and M a cell of the meet of the agents' partitions. Choose $\omega^* \in \arg\min_{\omega\in M}\{Y(\omega) + \rho(\{\omega\})\}$. The minimum is finite, since $\rho(M) \leq \rho(\Pi_i(\omega)) < \infty$ for $\omega \in M$. Fix $i \in I$ and let $C := \Pi_i(\omega^*)$. For every $\omega' \in C$,

$$Y(\omega^*) + r_i(\omega^*)(\{\omega^*\}) = Y(\omega^*) + \rho(\{\omega^*\}) - \rho(C)$$

$$\begin{aligned} &\leq Y(\omega') + \rho(\{\omega'\}) - \rho(C) \\ &= Y(\omega') + r_i(\omega^*)(\{\omega'\}). \end{aligned} \tag{10}$$

The first equality holds because $r_i(\omega^*) = \rho(\cdot \mid C)$ and $\rho(C) < \infty$ since ρ is a prior for agent i, and Definition 2 applies to $\omega^* \in C$. The inequality holds by the choice of ω^*, since $C \subseteq M$ by property (a) of the meet of the agents' partitions. The second equality holds for the same reasons as the first, applied to $\omega' \in C$. Since states outside C have rank ∞ under $r_i(\omega^*)$ by condition (a) of Definition 11, (10) shows that $Y(\omega^*) + r_i(\omega^*)(\{\omega^*\}) = \mathbb{E}_i[Y](\omega^*)$. Since $i \in I$ was arbitrary, ω^* is a common bleakest state for Y.

(ii) implies (i): We prove the contrapositive. Suppose that there is no common prior. We construct a common endowment Y and a cell of the meet of the agents' partitions without a common bleakest state for Y. The idea is that a cycle with nonzero sum allows the endowment to make every state along the cycle a state that is not bleakest for some agent, so that the agents' bleakest states "chase each other around" the cycle.

Step 1: A cycle with negative sum and distinct states. By Theorem 5, some cycle has nonzero sum. For $\omega' \in \Pi_i(\omega)$, write $d_i(\omega, \omega') := r_i(\omega)(\{\omega'\}) - r_i(\omega)(\{\omega\})$. As shown in Step 4 of the proof of Theorem 5, reversing a step negates its contribution, i.e., $d_i(\omega', \omega) = -d_i(\omega, \omega')$. So reversing a cycle with positive sum yields a cycle with negative sum. If a cycle with negative sum visits some state twice, then cutting it at that state splits it into two shorter cycles whose sums add up to the original sum, so that one of them has a negative sum. Repeating this yields a cycle $\omega_0, \ldots, \omega_m = \omega_0$ with agents $i_1, \ldots, i_m$, distinct states $\omega_0, \ldots, \omega_{m-1}$, and sum $s < 0$. All its states lie in one cell M of the meet of the agents' partitions.

Step 2: The common endowment on the cycle. Write $d_t := d_{i_t}(\omega_{t-1}, \omega_t)$ for $t = 1, \ldots, m$, and define $\varepsilon := -s/m > 0$, the average decrease in disbelief per step along the cycle. Set $Y(\omega_0) := 0$ and let $Y(\omega_t) := Y(\omega_{t-1}) - d_t - \varepsilon$ for $t = 1, \ldots, m-1$. Then

$$\begin{aligned} Y(\omega_{m-1}) - d_m - \varepsilon &= -\sum_{t=1}^{m} d_t - m\varepsilon \\ &= -s - m\varepsilon \\ &= 0 = Y(\omega_m). \end{aligned} \tag{11}$$

The first equality holds by the recursive definition of Y. The second holds by the definition of s, the third by the definition of ε, and the last because $\omega_m = \omega_0$ and $Y(\omega_0) = 0$. So the recursion $Y(\omega_t) = Y(\omega_{t-1}) - d_t - \varepsilon$ also holds for $t = m$. Hence, for all $t = 1, \ldots, m$,

$$\begin{aligned} Y(\omega_t) + r_{i_t}(\omega_{t-1})(\{\omega_t\}) &= Y(\omega_t) + d_t + r_{i_t}(\omega_{t-1})(\{\omega_{t-1}\}) \\ &= Y(\omega_{t-1}) - \varepsilon + r_{i_t}(\omega_{t-1})(\{\omega_{t-1}\}) \\ &< Y(\omega_{t-1}) + r_{i_t}(\omega_{t-1})(\{\omega_{t-1}\}). \end{aligned} \tag{12}$$

The first equality holds by the definition of d_t. The second holds by the recursion and (11). The inequality holds because $\varepsilon > 0$. Since $\omega_t \in \Pi_{i_t}(\omega_{t-1})$, $\mathbb{E}_{i_t}[Y](\omega_{t-1})$ is weakly smaller than the left-hand side of (12), so $Y(\omega_{t-1}) + r_{i_t}(\omega_{t-1})(\{\omega_{t-1}\}) > \mathbb{E}_{i_t}[Y](\omega_{t-1})$, i.e., ω_{t-1} is not bleakest for agent i_t. Since this holds for any $t = 1, \ldots, m$, no state of the cycle is a common bleakest state for Y.

Step 3: The common endowment off the cycle. Let $S := \{\omega_0, \ldots, \omega_{m-1}\}$ be the set of states of the cycle. Since M is a cell of the meet of the agents' partitions, every state of M is connected to S by a path. For $\omega \in M$, let $k(\omega)$ be the length of a shortest path from ω to S, so that $k(\omega) = 0$ if and only if $\omega \in S$. For $\omega \in M \setminus S$, choose an agent j and a state $\omega' \in \Pi_j(\omega)$ with $k(\omega') = k(\omega) - 1$. Define Y in the order of increasing k by

$$Y(\omega) := Y(\omega') + d_j(\omega, \omega') + 1,$$

and set $Y(\omega) := 0$ for $\omega \notin M$. Then

$$\begin{aligned} Y(\omega') + r_j(\omega)(\{\omega'\}) &= Y(\omega) - d_j(\omega, \omega') - 1 + r_j(\omega)(\{\omega'\}) \\ &= Y(\omega) - 1 + r_j(\omega)(\{\omega\}) \\ &< Y(\omega) + r_j(\omega)(\{\omega\}). \end{aligned} \tag{13}$$

The first equality holds by the definition of $Y(\omega)$, the second by the definition of d_j, and the inequality is immediate. As in Step 2, $Y(\omega) + r_j(\omega)(\{\omega\}) > \mathbb{E}_j[Y](\omega)$.

By Steps 2 and 3, we conclude that no state of M is a common bleakest state for Y, so (ii) fails. □

The endowment must be common to all agents. With different endowments, agents may trade to share risk even under a common prior. For instance, let $\Omega = \{\omega_1, \omega_2\}$, let there be no private information, let $\rho(\{\omega_1\}) = \rho(\{\omega_2\}) = 0$, and let agent 1 be endowed with Y_1 and agent 2 with Y_2, where $(Y_1(\omega_1), Y_1(\omega_2)) = (1, 0)$ and $(Y_2(\omega_1), Y_2(\omega_2)) = (0, 1)$. Both agents value their endowments at 0. Consider the bet with $(X_1(\omega_1), X_1(\omega_2)) = (-1/2, 1/2)$ and $X_2 = -X_1$. Then $Y_1 + X_1$ and $Y_2 + X_2$ both pay $1/2$ in each state, and both agents value them at $1/2$.

Unlike Theorem 5, Theorem 7 characterizes a common prior by behavior, namely the acceptance of bets. Like every such characterization in the probabilistic literature, it involves acceptance at possibility sets other than the actual one, since common knowledge concerns all possibility sets in a cell of the meet of the agents' partitions. Whether choices at possibility sets other than the actual one are meaningful depends on whether the states describe contingencies considered possible at an actual prior stage (see Gul, 1998, pp. 923–925, for a discussion).

The following example shows that full support is necessary for Theorem 7 because otherwise (ii) and (iii) do not imply (i).

Example 2. *Let $I = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2\}$, $\Pi_1(\omega) = \Omega$ and $\Pi_2(\omega) = \{\omega\}$ for all ω, and let $r_1(\omega)(\{\omega_1\}) = 0$ and $r_1(\omega)(\{\omega_2\}) = \infty$ for all ω. Agent 2's types are determined by condition (a) of Definition 11, i.e., $r_2(\omega_1)(\{\omega_1\}) = r_2(\omega_2)(\{\omega_2\}) = 0$ and $r_2(\omega_1)(\{\omega_2\}) = r_2(\omega_2)(\{\omega_1\}) = \infty$. Observe that there is no common prior. A prior ρ for agent 2 satisfies $\rho(\{\omega_2\}) = \rho(\Pi_2(\omega_2)) < \infty$, whereas a prior ρ for agent 1 satisfies $\rho(\{\omega_2\}) - \rho(\Omega) = r_1(\omega_2)(\{\omega_2\}) = \infty$. Yet ω_1 is a common bleakest state for every common endowment Y, since $\mathbb{E}_1[Y](\omega_1) = Y(\omega_1)$ and $\mathbb{E}_2[Y](\omega_1) = Y(\omega_1)$. By Theorem 6, there is no bet whose acceptance by all agents is common knowledge from any common endowment.*

9 Learning to Agree

Geanakoplos and Polemarchakis (1982) show that two agents who alternately announce their posterior probabilities of an event reach agreement in finitely many rounds, even if their posteriors are not common knowledge initially, a dynamic and communicational version of Aumann's agreement theorem. Cave (1983) generalizes this to the simultaneous public announcement of actions chosen by union-consistent action rules. We show that the dynamic communicational agreement theorem also holds for ranking functions.

The process works as follows. Fix an event E. At each round, every agent publicly and truthfully announces her rank of E conditional on her private information and everything announced before. Each announcement rules out the states at which the agent would have announced something else. So the set of states that are consistent with all announcements, and hence publicly known to be possible, shrinks. Once a round rules out no further state, every agent's announcement is the same at all states that are still publicly possible, so the announcements are common knowledge. The argument of Theorem 1 then applies to this set and yields agreement. In fact, we show that not only announcements of conditional ranks (see (a) in Proposition 1) lead to agreement but also coarser announcements like whether E is believed with firmness at least f (see (b) in Proposition 1) or announcements of (tropical) expectations of a random variable (see (c) in Proposition 1).

Proposition 1. *Let Assumption 1 hold, and suppose that $\rho(C) < \infty$ for every cell C of the join of the agents' partitions. Let g be one of the following functions, defined on events G of finite rank:*

(a) $g(G) := \rho(E \mid G)$ for a fixed event E;

(b) $g(G) := 1$ if $\rho(\neg E \mid G) \geq f$ and $g(G) := 0$ otherwise, for a fixed event E and $f \in \{1, 2, \ldots\} \cup \{\infty\}$;

(c) $g(G) := \mathbb{E}_\rho[X \mid G]$ for a fixed random variable X.

Fix the true state ω, and let $H_0 := \Pi_*(\omega)$. At each round $t = 1, 2, \ldots$, every agent i publicly and truthfully announces $g(\Pi_i(\omega) \cap H_{t-1})$, and the set of states consistent with all announcements so far becomes

$$H_t := \big\{\omega' \in H_{t-1} : g(\Pi_i(\omega') \cap H_{t-1}) = g(\Pi_i(\omega) \cap H_{t-1}) \text{ for all } i \in I\big\}.$$

We call the states in H_t publicly possible *after round t. Then there is a round $T \leq |\Pi_*(\omega)| - 1$ such that $H_t = H_T$ for all rounds $t \geq T$, and from round $T + 1$ on, all agents announce the same value, namely $g(H_T)$.*

Proof. *Step 1: Convergence in finitely many rounds.* By definition, $H_t \subseteq H_{t-1}$, and H_t depends only on H_{t-1}. Hence, if $H_t = H_{t-1}$ for some round t, then $H_{t'} = H_{t-1}$ for all rounds $t' \geq t$. Otherwise, H_t is a proper subset of H_{t-1}, i.e., round t removes at least one state. Since $\omega \in H_t$ for all rounds t and $H_0 = \Pi_*(\omega)$ is finite because Ω is finite, at most $|\Pi_*(\omega)| - 1$ rounds remove a state. Thus, there exists a round $T \leq |\Pi_*(\omega)| - 1$ such that $H_t = H_T =: H$ for all rounds $t \geq T$.

Step 2: Each agent's announcement is constant on H. Since $H_{T+1} = H$, the definition of H_{T+1} yields

$$g(\Pi_i(\omega') \cap H) = g(\Pi_i(\omega) \cap H) =: x_i \quad \text{for all } \omega' \in H \text{ and } i \in I.$$

That is, whatever her private information within H, agent i would announce x_i.

Step 3: Agreement. Fix $i \in I$, and let $C_1, \ldots, C_m$ be the distinct sets $\Pi_i(\omega') \cap H$ with $\omega' \in H$. They partition H, and there are finitely many since Ω is finite. Each of them is a union of cells of the join of the agents' partitions, since H is determined by the agents' announcements, which depend only on the cell of the join, and so each has finite rank. By Step 2, $g(C_\ell) = x_i$ for $\ell = 1, \ldots, m$. The function g is union consistent, for (a) by Lemma 1, for (b) by Lemma 3, and for (c) by Lemma 5 (iii). We show by induction on ℓ that $g(C_1 \cup \cdots \cup C_\ell) = x_i$ for $\ell = 1, \ldots, m$. For $\ell = 1$, this holds by Step 2. Suppose that it holds for some $\ell < m$, and let $P := C_1 \cup \cdots \cup C_\ell$ and $Q := C_{\ell+1}$. Then P and Q are disjoint, have finite rank, and satisfy $g(P) = g(Q) = x_i$, so that $g(P \cup Q) = x_i$ by union consistency. For $\ell = m$, we obtain $g(H) = x_i$. Since $i \in I$ was arbitrary, $x_i = g(H)$ for all $i \in I$. □

The following examples illustrate Proposition 1. The first shows how announcements reveal private information and that truthful communication and learning to agree do not necessarily lead to learning the truth. In the standard probabilistic setting, Geanakoplos and Polemarchakis (1982, Proposition 3) show that agents may reach agreement before their private information is pooled, and Bonanno and Nehring (1997, Section 2) give an example in which communication even moves the agents further from the truth. The same can happen with ranking functions.

Example 3 (Learning to agree is not learning the truth)**.** *Let $\Omega = \{\omega_1, \ldots, \omega_5\}$, and let the common prior ρ be given by $(\rho(\{\omega_1\}), \ldots, \rho(\{\omega_5\})) = (0, 1, 1, 0, 1)$. Let agent 1's partition be $\{\{\omega_1, \omega_4\}, \{\omega_2, \omega_5\}, \{\omega_3\}\}$ and agent 2's be $\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5\}\}$. Let $E = \{\omega_2, \omega_3\}$, let the agents announce their conditional ranks of E as in Proposition 1 (a), and let the true state be ω_5, so that E does not obtain.*

In round 1, agent 1's possibility set is $\{\omega_2, \omega_5\}$, and she announces $\rho(E \mid \{\omega_2, \omega_5\}) = 1 - 1 = 0$. Had her possibility set been $\{\omega_1, \omega_4\}$, she would have announced ∞, since this set does not intersect E. So her announcement publicly rules out ω_1 and ω_4. Agent 2's possibility set is $\{\omega_3, \omega_4, \omega_5\}$, and he announces $\rho(E \mid \{\omega_3, \omega_4, \omega_5\}) = 1 - 0 = 1$. He would have announced 1 also had his possibility set been $\{\omega_1, \omega_2\}$, so his announcement rules out nothing. Thus, the agents disagree in round 1, and the set of publicly possible states becomes $H_1 = \{\omega_2, \omega_3, \omega_5\}$.

In round 2, agent 1 still announces $\rho(E \mid \{\omega_2, \omega_5\}) = 0$, while agent 2 now conditions on $\{\omega_3, \omega_4, \omega_5\} \cap H_1 = \{\omega_3, \omega_5\}$ and announces $\rho(E \mid \{\omega_3, \omega_5\}) = 1 - 1 = 0$. Both agents announce 0 at every remaining state, so the process stops with agreement, and neither agent disbelieves E.

Yet the agents reach agreement before their private information is pooled. Pooling it at ω_5 would yield $\{\omega_2, \omega_5\} \cap \{\omega_3, \omega_4, \omega_5\} = \{\omega_5\}$, and $\rho(E \mid \{\omega_5\}) = \infty$. So the agents agree not to disbelieve an event that their pooled information rules out. Moreover, communication moves agent 2 away from the truth. Initially, he correctly disbelieves E, but only because ω_4, a state outside E, is the least disbelieved state in his possibility set. Agent 1's announcement

reveals that ω_4 did not occur. Among the remaining states, ω_3, which lies in E, is as little disbelieved as ω_5, so agent 2 no longer disbelieves E. Learning to agree thus brings agent 2 further from the truth!

In Example 3, agent 2's conditioning event in round 2, $\{\omega_3, \omega_5\}$, consists only of states that he initially disbelieves. The announcements thus lead to a genuine revision of his beliefs, not merely to a refinement. Unlike Bayesian updating, which is undefined after events of probability zero, conditioning on disbelieved events poses no difficulty with ranking functions. Since each set $\Pi_i(\omega') \cap H_{t-1}$ is a union of cells of the join, which have finite rank by the hypothesis of Proposition 1, the process never requires conditioning on an event of rank ∞.

Agreement may also be reached only after many rounds in which nothing observable changes. The next example applies Proposition 1 (b) to trading.

Example 4 (A market crash without external shocks)**.** *The following example adapts Hart and Tauman (2004) to ranking functions. Let $\Omega = \{\omega_1, \ldots, \omega_9\}$, and let the common prior ρ be given by $(\rho(\{\omega_1\}), \ldots, \rho(\{\omega_9\})) = (2, 2, 2, 1, 1, 1, 0, 0, 0)$. Let agent 1's partition be $\{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \{\omega_7, \omega_8, \omega_9\}\}$ and agent 2's be $\{\{\omega_1, \ldots, \omega_4\}, \{\omega_5, \ldots, \omega_8\}, \{\omega_9\}\}$. Let $E = \{\omega_1, \omega_5, \omega_9\}$ be a bad outcome, such as a decline in a company's earnings. Each day, each agent buys if she disbelieves E, i.e., $\rho(E \mid \cdot) \geq 1$, and sells otherwise. This is rule (b) of Proposition 1 for the event $\neg E$ and $f = 1$. Let the true state be ω_1. The following table shows, for each day, the publicly possible states before that day, each agent's conditional rank of E given each of her possibility sets intersected with these states, where a dash indicates that the intersection is empty, so that the possibility set has been ruled out and the conditional rank is not defined, and each agent's action at ω_1.*

		Agent 1				*Agent 2*			
		Conditional rank of E				*Conditional rank of E*			
Day	*Publicly possible*	ω_1–ω_3	ω_4–ω_6	ω_7–ω_9	*Action*	ω_1–ω_4	ω_5–ω_8	ω_9	*Action*
1	$\omega_1, \ldots, \omega_9$	0	0	0	*sell*	1	1	0	*buy*
2	$\omega_1, \ldots, \omega_8$	0	0	∞	*sell*	1	1	–	*buy*
3	$\omega_1, \ldots, \omega_6$	0	0	–	*sell*	1	0	–	*buy*
4	$\omega_1, \ldots, \omega_4$	0	∞	–	*sell*	1	–	–	*buy*
5	$\omega_1, \omega_2, \omega_3$	0	–	–	*sell*	0	–	–	*sell*

On days 1 to 4, agent 1 sells and agent 2 buys, so nothing observable changes and the market appears "stable". Yet each day publicly rules out one possibility set, at which one of the agents would have acted differently: first $\{\omega_9\}$, then $\{\omega_7, \omega_8\}$, $\{\omega_5, \omega_6\}$, and $\{\omega_4\}$. Agent 2 disbelieves E on days 1 to 4 only because ω_4, a good state, is less disbelieved than ω_1, i.e., $\rho(E \mid \{\omega_1, \ldots, \omega_4\}) = 2 - 1 = 1$. Once ω_4 is publicly ruled out on day 4, her least disbelieved states are $\omega_1, \omega_2, \omega_3$, among them the bad state ω_1. She no longer disbelieves E and sells as well. This "crash" is the agreement of Proposition 1, reached after four days on which nothing observable changes.

While learning to agree may not lead to learning the truth as Example 3 demonstrated, learning the truth may follow a similar mechanism as learning to agree. We illustrate this point with a ranking function version of the famous "Muddy Children Puzzle" (Littlewood, 1953; Moses, Dolev, and Halpern, 1986), which we may dub "Spohn's Muddy Children".

Example 5 (Spohn's muddy children)**.** *There are n children, some of whom have mud on their foreheads. A state is the set $S \subseteq \{1, \ldots, n\}$ of muddy children, so $\Omega = 2^{\{1,\ldots,n\}}$. Each child sees the others but not herself, so child i's possibility set at S is $\{S \setminus \{i\}, S \cup \{i\}\}$. Let the common prior ρ be given by $\rho(\{S\}) := |S|$, so that each muddy child adds one degree of disbelief. At every state, child i's posterior rank of being muddy is $(|S\setminus\{i\}|+1)-|S\setminus\{i\}| = 1$. That is, every child believes with firmness 1 that she is clean. For the muddy children, this belief is false. The father publicly announces that at least one child is muddy, which rules out state $\emptyset$. Then, in each round, every child announces whether she believes that she is muddy. Since the two states in a child's possibility set differ by one degree of disbelief, she believes that she is clean as long as both are publicly possible, and she believes that she is muddy only when the clean alternative has been publicly ruled out, in which case she knows it. Hence the announcements are the same as in the classical puzzle (see Geanakoplos and Polemarchakis, 1982, fn. 6). With $k \geq 1$ muddy children, every child announces in rounds $1, \ldots, k-1$ that she believes she is clean. In round k, the muddy children abruptly revise their firm but false belief and announce that they are muddy, while the clean children believe correctly throughout. Since knowledge corresponds to belief with firmness ∞, the classical puzzle is the knowledge case of this belief dynamics. Unlike in Proposition 1, each child's announcement concerns a different event, namely whether she herself is muddy, so that the process ends in learning rather than agreement, as in Cave's (1983) process with different action rules.*

Geanakoplos and Polemarchakis (1982, Proposition 4) also show that, for a randomly chosen event, posterior probabilities generically differ across cells, so that the first announcements reveal all private information and the process converges in one step to the outcome of pooling information. Their argument does not carry over to ranking functions. A posterior rank,

$$\rho(E \mid C) = \min_{\omega \in E \cap C} \rho(\{\omega\}) - \min_{\omega \in C} \rho(\{\omega\}),$$

depends only on the minimal ranks in $E \cap C$ and in C and takes values in $\mathbb{N} \cup \{\infty\}$. So distinct cells may share the same posterior rank of E, as for agent 2 in round 1 of Example 3. With ranking functions, slow convergence and divergence from the outcome of pooling information are not non-generic, although this does not make them generic.

10 Non-Partitional Structures

Partitional information requires that agents know only true events, know what they know, and know what they do not know. Geanakoplos (2021), Samet (1990), and Rubinstein and Wolinsky (1990) study agreement and speculation when these requirements are relaxed, and Pacuit and Yang (2026b) study approximate agreement in this setting. With probabilistic beliefs, Aumann's theorem survives the loss of negative introspection (Samet, 1990), but both agreement and the absence of speculation require additional structure. We now show that with ranking functions, the agreement theorem, its qualitative version for beliefs, and the no-betting theorem under a common prior hold as soon as agents do not know false events.

As before, we consider for each agent $i \in I$ a possibility correspondence $\Pi_i : \Omega \longrightarrow 2^\Omega$ but different from Definition 3 we now drop stationarity. We only assume that each agent is *nondeluded* (Geanakoplos, 2021), i.e., Π_i is reflexive, $\omega \in \Pi_i(\omega)$ for all $\omega \in \Omega$. We do not require that Π_i satisfy positive introspection, negative introspection, or any other structure, so the sets $\Pi_i(\omega)$ may overlap without being nested. Knowledge, mutual knowledge, and common knowledge are defined as in Section 2. We say that an event G is *public* if $\Pi_i(\omega) \subseteq G$ for all $\omega \in G$ and all $i \in I$. The following characterization of common knowledge by public events is well known for partitional structures (Aumann, 1976; Monderer and Samet, 1989) and holds under nondelusion alone (see Geanakoplos, 2021, Section 7, p. 385, and Rubinstein and Wolinsky, 1990, Section 2, as well as, for abstract belief operators, Monderer and Samet, 1989, Proposition 4). Since neither Geanakoplos (2021) nor Monderer and Samet (1989) present a proof for possibility correspondences under nondelusion, we include one in Appendix B for completeness.

Lemma 7. *Let each Π_i be nondeluded and let $E \in 2^\Omega$. Then $CK(E)$ is public and $CK(E) \subseteq E$. Moreover, $\omega \in CK(E)$ if and only if there is a public event G with $\omega \in G \subseteq E$.*

Nondelusion implies that every public event is covered by each agent's possibility sets:

$$G = \bigcup_{\omega \in G} \Pi_i(\omega) \quad \text{for every public event } G \text{ and every } i \in I. \tag{14}$$

In contrast to equation (4), the sets on the right-hand side need not be disjoint.

We maintain Assumption 1. Posterior ranks, the events $[\kappa_i(E)]$, belief with firmness at least f, tropical expectations, bets, and acceptance sets $\☺_i(X_i)$ are defined as before.

Proposition 2. *Let each Π_i be nondeluded, and let Assumption 1 hold. For any event $E \in 2^\Omega$,*

(i) If $CK\left(\bigcap_{i\in I}[\kappa_i(E)]\right) \neq \emptyset$, then $\kappa_1(E) = \cdots = \kappa_n(E)$.

(ii) For any $f \in \{1, 2, \ldots\} \cup \{\infty\}$ and $i, j \in I$, it cannot be common knowledge that agent i believes E with firmness at least f while agent j does not.

(iii) For every bet $(X_i)_{i\in I}$, $CK\left(\bigcap_{i\in I} \☺_i(X_i)\right) = \emptyset$.

Proof. The idea of the proof is as follows. If the relevant event is common knowledge, then it contains a nonempty public event G of finite rank. Each agent's possibility sets at states in G cover G, and on each of them the agent's posterior rank, belief, or acceptance has the same property. By conglomerability of conditional ranks (Lemma 2), of beliefs (Lemma 3), and of tropical expectations (Lemma 5 (iii)), this property carries over to unions of these sets, even though they may overlap, and hence to G itself. Since G is the same for all agents, this yields (i), (ii), and (iii).

Step 1: A public event of finite rank. In (i), let $V := \bigcap_{i\in I}[\kappa_i(E)]$. In (ii), let $V := B_i^f(E) \cap \neg B_j^f(E)$. In (iii), let $V := \bigcap_{i\in I} \☺_i(X_i)$. Suppose that V is common knowledge at some state, and let $G := CK(V)$. By Lemma 7, G is public and $G \subseteq V$, and $G \neq \emptyset$ since V

is common knowledge at some state. For $k \in I$, let $C_1^k, \ldots, C_{m_k}^k$ be the distinct sets $\Pi_k(\omega)$ with $\omega \in G$. There are finitely many since Ω is finite. By equation (14),

$$G = C_1^k \cup \cdots \cup C_{m_k}^k \quad \text{for every } k \in I. \tag{15}$$

Hence

$$\rho(G) = \min_{\ell=1,\ldots,m_k} \rho(C_\ell^k) < \infty.$$

The equality follows from equation (15) and property (3). The inequality holds since every possibility set has finite rank by Assumption 1. So $G \in \mathcal{D}_\rho$.

Step 2: From possibility sets to G. Let $k \in I$, and let $\mathcal{U} \subseteq \mathcal{D}_\rho$ be closed under unions, i.e., $P, Q \in \mathcal{U}$ implies $P \cup Q \in \mathcal{U}$. Then $\mathcal{U}$ is closed under finite unions. Hence, by equation (15),

$$C_\ell^k \in \mathcal{U} \text{ for } \ell = 1, \ldots, m_k \quad \text{implies} \quad G \in \mathcal{U}. \tag{16}$$

Step 3: The three claims. In each of the three parts, we apply (16) to collections of events in $\mathcal{D}_\rho$. Recall that $\mathcal{D}_\rho$ is closed under finite unions by property (3).

(i) Fix $i \in I$, and let $\mathcal{U} := \{S \in \mathcal{D}_\rho : \rho(E \mid S) = \kappa_i(E)\}$. Let $P, Q \in \mathcal{U}$. Then $P \cup Q \in \mathcal{D}_\rho$ and

$$\begin{aligned} \kappa_i(E) &= \min\{\rho(E \mid P), \rho(E \mid Q)\} \\ &\leq \rho(E \mid P \cup Q) \\ &\leq \max\{\rho(E \mid P), \rho(E \mid Q)\} \\ &= \kappa_i(E). \end{aligned}$$

The two equalities hold since $P, Q \in \mathcal{U}$. The two inequalities follow from Lemma 2, which does not require disjointness. Hence $P \cup Q \in \mathcal{U}$, so $\mathcal{U}$ is closed under unions. Each C_ℓ^i equals $\Pi_i(\omega)$ for some $\omega \in G \subseteq V \subseteq [\kappa_i(E)]$, so $C_\ell^i \in \mathcal{U}$. By (16), $G \in \mathcal{U}$, i.e., $\rho(E \mid G) = \kappa_i(E)$. The left-hand side does not depend on i. Hence $\kappa_i(E) = \rho(E \mid G) = \kappa_j(E)$ for all $i, j \in I$.

(ii) Let $\mathcal{U}_{\geq} := \{S \in \mathcal{D}_\rho : \rho(\neg E \mid S) \geq f\}$ and $\mathcal{U}_{<} := \{S \in \mathcal{D}_\rho : \rho(\neg E \mid S) < f\}$. By Lemma 3, both collections are closed under unions. Each C_ℓ^i equals $\Pi_i(\omega)$ for some $\omega \in G \subseteq V \subseteq B_i^f(E)$, so $C_\ell^i \in \mathcal{U}_{\geq}$. Each C_ℓ^j equals $\Pi_j(\omega)$ for some $\omega \in G \subseteq V \subseteq \neg B_j^f(E)$, so $C_\ell^j \in \mathcal{U}_{<}$. By (16), applied with $k = i$ and $\mathcal{U}_{\geq}$ and with $k = j$ and $\mathcal{U}_{<}$, we obtain $\rho(\neg E \mid G) \geq f$ and $\rho(\neg E \mid G) < f$, a contradiction.

(iii) Fix $i \in I$, and let $\mathcal{U} := \{S \in \mathcal{D}_\rho : \mathbb{E}_\rho[X_i \mid S] > 0\}$. For $P, Q \in \mathcal{U}$, Lemma 5 (iii) yields $P \cup Q \in \mathcal{U}$, so $\mathcal{U}$ is closed under unions. Each C_ℓ^i equals $\Pi_i(\omega)$ for some $\omega \in G \subseteq V \subseteq \text{☺}_i(X_i)$, so $C_\ell^i \in \mathcal{U}$. By (16), $G \in \mathcal{U}$. Since $i \in I$ was arbitrary,

$$\mathbb{E}_\rho[X_i \mid G] > 0 \quad \text{for all } i \in I. \tag{17}$$

Since $G \in \mathcal{D}_\rho$ by Step 1, Lemma 5 (ii), applied to $F := G$, yields $\omega_0 \in G$ with $\mathbb{E}_\rho[X \mid G] \leq X(\omega_0)$ for all random variables X. The state ω_0 depends only on ρ and G, not on i. Hence

$$\begin{aligned} \sum_{i \in I} X_i(\omega_0) &\geq \sum_{i \in I} \mathbb{E}_\rho[X_i \mid G] \\ &> 0. \end{aligned}$$

The first inequality follows from Lemma 5 (ii), applied to each X_i. The second follows from (17). This contradicts $\sum_{i\in I} X_i(\omega_0) = 0$, which must hold since $(X_i)_{i\in I}$ is a bet. □

Proposition 2 can be understood as an instance of a strengthening of Proposition 2 of Rubinstein and Wolinsky (1990). On p. 192 of their paper, they state without proof, for two agents, that their Proposition 2 holds under nondelusion (their condition P-1) alone, without positive introspection (their condition P-2), provided the agents' properties of events are preserved under unions. That is, if properties of events that cannot hold simultaneously are each preserved under unions, then it cannot be common knowledge that each agent's information has her property. The argument above establishes this for n agents. As we show, conditional ranks, firmness of belief, and acceptance of bets based on tropical expectations all have this property, by Lemmata 2, 3, and 5 (iii). Example 4 and Conclusion 4 of Rubinstein and Wolinsky (1990), on minima of functions, are the closest precursors to our result without probabilities.

The contrast with probabilistic beliefs is stark. Conditional probabilities are preserved under disjoint unions and under differences, but not under arbitrary unions. With a common prior probability, Aumann's theorem therefore requires more than nondelusion. It holds if agents also know what they know (Samet, 1990), and Geanakoplos (2021, Theorem 6) relates the absence of agreeable disagreement for all priors to a condition he calls balancedness.[6] The absence of commonly known speculation requires a stronger condition, positive balancedness (Geanakoplos, 2021, Corollary 5.1), which fails even for some structures in which agents know what they know (Rubinstein and Wolinsky, 1990, Section 4). Pacuit and Yang (2026b) show that, beyond these cases, common knowledge of posterior probabilities in nondeluded structures yields only approximate agreement. In contrast, for ranking functions, we show that nondelusion alone delivers exact agreement and rules out commonly known speculation. In the terminology of Geanakoplos (2021), decisions based on ranking functions satisfy his generalized sure-thing principle not only for positively balanced collections of events but for every collection that covers an event.

Example 6. *The following example shows that, under a common prior, commonly known speculation is possible in a nondeluded structure with probabilities but not with ranking functions. The structure is from Geanakoplos (2021, Example 4.1). Let* $I = \{1, 2\}$, $\Omega = \{a, b, c\}$, $\Pi_1(a) = \{a, b\}$, $\Pi_1(b) = \{b\}$, $\Pi_1(c) = \{b, c\}$, *and* $\Pi_2(\omega) = \Omega$ *for all* ω. *Agent 1 is nondeluded and knows what she knows, but* Π_1 *is not nested and not positively balanced. Since* $\Pi_2(\omega) = \Omega$, *the only nonempty public event is* Ω. *So by Lemma 7, an event is common knowledge at some state if and only if it equals* Ω.

Consider the bet $X_1 = (-1, 1, -1)$ *and* $X_2 = -X_1$. *With the common prior probability* $\pi(a) = \pi(c) = 2/7$ *and* $\pi(b) = 3/7$, *agent 1's conditional expected gain is* $1/5$ *at* a *and* c *and* 1 *at* b, *while agent 2's is* $1/7$. *Both agents accept the bet at every state, so this is common knowledge.*

In contrast, with any common prior ranking function, no bet is commonly known to be acceptable, by Proposition 2 (iii). For instance, let the common prior ρ *be such that*

[6] Pacuit and Yang (2026b) point out that balancedness must be slightly strengthened for this characterization.

$\rho(\{a\}) = \rho(\{c\}) = 1$ *and* $\rho(\{b\}) = 0$*, so that* b *is the least disbelieved state, just as it is the most likely state under* π*. Then*

$$\begin{aligned}
\mathbb{E}_\rho[X_1 \mid \{a,b\}] &= \min\{-1+1, 1+0\} = 0,\\
\mathbb{E}_\rho[X_1 \mid \{b\}] &= 1,\\
\mathbb{E}_\rho[X_1 \mid \{b,c\}] &= \min\{1+0, -1+1\} = 0,\\
\mathbb{E}_\rho[X_2 \mid \Omega] &= \min\{1+1, -1+0, 1+1\} = -1.
\end{aligned}$$

So agent 1 accepts X_1 *only at* b*, and agent 2 rejects* X_2 *at every state.*

More generally, let ρ *be any common prior ranking function and* (X_1, X_2) *any bet, not necessarily the one above. Suppose that acceptance of* (X_1, X_2) *is common knowledge at some state. Then both agents accept at every state. By property (1), there is a state* ω_0 *with* $\rho(\{\omega_0\}) = 0$*. Then* $\rho(\{\omega_0\} \mid \Omega) = 0$*, and* $\rho(\{\omega_0\} \mid \Pi_1(\omega_0)) = 0$ *since* $\rho(\Pi_1(\omega_0)) \leq \rho(\{\omega_0\}) = 0$*. By the definition of the tropical expectation,*

$$\begin{aligned}
X_1(\omega_0) &\geq \mathbb{E}_\rho[X_1 \mid \Pi_1(\omega_0)] > 0,\\
X_2(\omega_0) &\geq \mathbb{E}_\rho[X_2 \mid \Omega] > 0,
\end{aligned}$$

where the strict inequalities hold since both agents accept at ω_0*. Hence* $X_1(\omega_0) + X_2(\omega_0) > 0$*, which contradicts that* (X_1, X_2) *is a bet.*

Several of our results remain to be extended. The proofs of Theorem 2 (common belief with firmness f) and Theorem 4 (no betting without a common prior) use the partition structure. Without partitions, the fixed-point and iterative definitions of common p-belief may come apart (Pacuit and Yang, 2026b), and a state may belong to several possibility sets of the same agent with different posterior ranks, so the notion of a common unsurprising state needs to be reformulated. Moreover, Samet (1990, Theorem 8) shows that Aumann's theorem survives even without nondelusion if agents know what they know, no possibility set is empty, and the prior assigns probability zero to states at which some agent knows a false event. An analogous result for ranking functions, with rank ∞ in place of probability zero, seems within reach. We leave these extensions for future work.

11 Discussion

11.1 Priors and Tropical Convex Hulls

Samet (1998) considers type functions $t_i : \Omega \longrightarrow \Delta(\Omega)$, where $\Delta(\Omega)$ denotes the set of probability measures on Ω, such that $t_i(\omega)(\Pi_i(\omega)) = 1$ for all $\omega \in \Omega$ and $\omega' \in \Pi_i(\omega)$ implies $t_i(\omega') = t_i(\omega)$. He calls $p_i \in \Delta(\Omega)$ a prior for agent i if $p_i(\Pi_i(\omega)) > 0$ implies $p_i(\cdot \mid \Pi_i(\omega)) = t_i(\omega)$, and observes that the set of priors of agent i is the convex hull of her types. A tropical analog holds in ranking type structures. Call a ranking function ρ_i a *prior* for agent i if $\rho_i(\Pi_i(\omega)) < \infty$ implies $\rho_i(\cdot \mid \Pi_i(\omega)) = r_i(\omega)$.[7] Compare convex combinations

[7]Unlike in Definition 12, we do not require here that a prior assigns finite rank to every possibility set.

of types with tropical convex combinations of ranking-function types,

$$\sum_{\omega\in\Omega} \alpha_\omega\, t_i(\omega), \quad \alpha_\omega \geq 0,\ \sum_{\omega\in\Omega}\alpha_\omega = 1, \qquad\qquad \bigoplus_{\omega\in\Omega} \lambda_\omega \odot r_i(\omega), \quad \lambda_\omega \in \mathbb{N}\cup\{\infty\},\ \bigoplus_{\omega\in\Omega}\lambda_\omega = 0.$$

In min-plus notation, $\bigoplus_\omega \lambda_\omega \odot r_i(\omega)$ is the ranking function $E \mapsto \min_{\omega\in\Omega}\{\lambda_\omega + r_i(\omega)(E)\}$, where $\min_{\omega\in\Omega}\lambda_\omega = 0$. *The set of priors of agent i is the set of tropical convex combinations of her ranking-function types, i.e., their tropical convex hull.*[8] Indeed, every prior ρ_i satisfies

$$\rho_i(E) = \min_{\omega\in\Omega}\{\rho_i(\{\omega\}) + r_i(\omega)(E)\} \quad \text{for all } E\subseteq\Omega,$$

so it is the tropical convex combination of the agent's ranking-function types weighted by itself, just as a probabilistic prior satisfies $p_i(E) = \sum_{\omega\in\Omega} p_i(\{\omega\})\, t_i(\omega)(E)$. Conversely, let $\rho_i := \bigoplus_{\omega\in\Omega}\lambda_\omega \odot r_i(\omega)$ be a tropical convex combination of agent i's ranking-function types with weights λ_ω. Since $r_i(\omega') = r_i(\omega)$ for $\omega' \in \Pi_i(\omega)$, and $r_i(\omega')$ assigns rank ∞ to every state outside $\Pi_i(\omega')$, we have for every $\omega\in\Omega$

$$\rho_i(\Pi_i(\omega)) = \min_{\omega'\in\Pi_i(\omega)} \lambda_{\omega'} \quad \text{and} \quad \rho_i(\{\omega\}) = \min_{\omega'\in\Pi_i(\omega)} \lambda_{\omega'} + r_i(\omega)(\{\omega\}).$$

Hence the conditional ranking function given each possibility set of finite rank is the agent's ranking-function type there, so ρ_i is a prior. The weights of the states in a possibility set matter only through their minimum, which becomes the rank of the possibility set, and the rank of a state ω generally differs from its weight λ_ω. Convex combinations of probabilistic types behave analogously. The convex combination $p_i := \sum_{\omega\in\Omega}\alpha_\omega\, t_i(\omega)$ satisfies

$$p_i(\Pi_i(\omega)) = \sum_{\omega'\in\Pi_i(\omega)} \alpha_{\omega'} \quad \text{and} \quad p_i(\{\omega\}) = \sum_{\omega'\in\Pi_i(\omega)} \alpha_{\omega'}\cdot t_i(\omega)(\{\omega\}),$$

so the weights matter only through their sum over each possibility set, and the probability of ω generally differs from α_ω. Finally, the priors of Definition 12 are the tropical convex combinations that assign finite rank to every possibility set, i.e., in which every possibility set contains a state of finite weight.

11.2 Tropical vs. Classical Linearity: Money is Alien to the Tropics

Section 11.1 suggests that tropical convexity might take the place of convexity in Samet's (1998) characterization of common priors. We now explain why it does not. The probabilistic characterization of common priors by the absence of commonly known acceptable bets rests on linearity. Bets redistribute money, so the condition $\sum_{i\in I} X_i = 0$ is linear in the ordinary sense, and so are probabilistic expectations. This has three consequences. First, priors and

[8] Develin and Sturmfels (2004) study tropical convexity in $\mathbb{R}^n$ with $a\oplus b := \min\{a,b\}$ and $a\odot b := a+b$, and show that the tropical convex hull of a set consists of all tropical linear combinations of its elements (Develin and Sturmfels, 2004, Proposition 4). We allow ranks and weights to take the value ∞, and we normalize weights by $\bigoplus_{\omega\in\Omega}\lambda_\omega = 0$ instead of working in tropical projective space.

bets are dual to each other, so that a separation argument turns the absence of a common prior into a bet whose acceptance by all agents is common knowledge (Samet, 1998). Second, $E[-X] = -E[X]$, so that an agent's price for buying a random variable equals her price for selling it. For two agents, a commonly known disagreement about the expectation of a random variable is therefore equivalent to a commonly known acceptable bet, and the absence of such disagreement characterizes a common prior (Samet, 1998, Claim; Feinberg, 2000, Theorem 2). Third, $E[Y + X] - E[Y] = E[X]$, so that it is without loss of generality to evaluate bets from a riskless position.

Tropical expectations are linear in the tropical sense. Recall from Section 2 that $a \oplus b := \min\{a, b\}$ and $a \odot b := a + b$. Then

$$\mathbb{E}_\rho[X \oplus Z \mid F] = \mathbb{E}_\rho[X \mid F] \oplus \mathbb{E}_\rho[Z \mid F] \quad \text{and} \quad \mathbb{E}_\rho[c \odot X \mid F] = c \odot \mathbb{E}_\rho[X \mid F]$$

for all random variables X and Z and constants c, and, as we saw in Section 11.1, the priors of an agent form a tropically convex set. *However, money remains linear in the ordinary sense.* With money linear in the ordinary sense and evaluation linear in the tropical sense, all three consequences fail. First, there is no such duality between bets and tropically convex sets of priors, and the absence of speculative betting characterizes only the weaker condition of Theorem 4. Second, an agent's price for buying a random variable, $\mathbb{E}_\rho[X \mid F]$, is generally lower than her price for selling it, $-\mathbb{E}_\rho[-X \mid F]$, by Remark 3, so that a disagreement need not be tradable (Section 6). Third, a common endowment Y enters as tropical multiplication by a random variable, $Y + X = Y \odot X$, while tropical expectations are tropically linear only in constants. In general, $\mathbb{E}_\rho[Y \odot X \mid F] \neq \mathbb{E}_\rho[Y \mid F] \odot \mathbb{E}_\rho[X \mid F]$, just as $E[YX] \neq E[Y]E[X]$ in probability theory. This is why Theorem 7 requires the absence of commonly known acceptable bets for all common endowments.

Does it help to replace bets by a tropical analog? The literal translation of $\sum_{i \in I} X_i = 0$ fails, since the neutral element of tropical addition is ∞ and tropical addition has no inverse. What the condition $\sum_{i \in I} X_i = 0$ delivers in the proofs of Theorems 3 and 4 is only that not all agents gain at the same state. Its tropical expression is

$$\bigoplus_{i \in I} X_i = \min_{i \in I} X_i \leq 0.$$

Call a profile of random variables $(X_i)_{i \in I}$ with this property a *tropical bet*. Every bet is a tropical bet, but not conversely, since a tropical bet need not balance. Yet, changing from bets that are linear in the ordinary sense to tropical bets does not help to detect the absence of a common prior. In Example 1, there is no common prior since $\rho_1 \neq \rho_2$. Yet no tropical bet is acceptable to both agents, because ω_1 has rank 0 for both. Indeed, for each agent i and every random variable X_i,

$$\mathbb{E}_{\rho_i}[X_i \mid \Omega] = \min_{\omega \in \Omega}\{X_i(\omega) + \rho_i(\{\omega\})\} \leq X_i(\omega_1) + \rho_i(\{\omega_1\}) = X_i(\omega_1).$$

The first equality is the definition of the tropical expectation, using $\rho_i(\Omega) = 0$. The inequality holds since the minimum is weakly smaller than the term for ω_1. The last equality holds since $\rho_i(\{\omega_1\}) = 0$. If both agents accepted their sides of a tropical bet (X_1, X_2), then $X_1(\omega_1) > 0$ and $X_2(\omega_1) > 0$, so $\min\{X_1(\omega_1), X_2(\omega_1)\} > 0$, which contradicts that (X_1, X_2) is a tropical

bet. This is the argument via a common unsurprising state behind Theorem 4, and it never uses that a bet balances. What makes the absence of speculative betting weak is not the class of bets but the riskless position from which bets are evaluated.

11.3 Interpreting Common Prior Ranking Functions

Morris (1995) critically reviews the arguments in support of the common prior assumption for probabilities. We contrast his discussion with common prior ranking functions.

Morris (1995, Section 3) distinguishes frequentist, logical, and personalistic interpretations of probability. He argues that personalistic probabilities are derived from individual choices, so that there is no presumption that they are the same, and that it makes little sense to assume a common prior unless it exists logically prior to the decision problem (Morris, 1995, pp. 232–233). Ranking functions are personalistic as well. They represent an agent's degrees of disbelief (Spohn, 2012), and Schipper (2026a) derives them from her preferences. Thus, as an assumption on primitives, a common prior ranking function is as difficult to interpret as a common prior probability. This is why the converse of the no-trade theorem matters. It translates the common prior into a condition on behavior. For probabilities, a common prior exists if and only if there is no bet whose acceptance by all agents is common knowledge (Morris, 1994; Samet, 1998; Feinberg, 2000). For ranking functions with full support, a common prior exists if and only if there is no such bet from any common endowment (Theorem 7), which is more demanding compared to the probabilistic case.

This behavioral interpretation bears on an empirical argument for the common prior discussed by Morris (1995, Section 4.3). If differences in posterior beliefs were due to differences in prior beliefs, individuals would bet on every difference in their posterior beliefs, yet betting is rare. Morris replies that with asymmetric information, willingness to bet exhibits a bid–ask spread, so that heterogeneous priors need not lead to betting. With ranking functions, the rarity of betting is even weaker evidence for a common prior. The bid–ask spread is built into the evaluation itself (Remark 3), so that agents without a common prior may reject every bet even without private information (Example 1). By Theorem 4, the absence of commonly known acceptable bets only implies that every cell of the meet contains a common unsurprising state. Only the absence of commonly known acceptable bets from every common endowment reveals a common prior ranking function.

Finally, Morris (1995, Section 4.2) discusses the argument that learning from common experience removes differences in beliefs that are not due to differences in information. He notes that consistency results for Bayesian learning require that individuals agree on which events have positive probability, and that this requirement seems scarcely weaker than a common prior (Morris, 1995, p. 237). Ranking functions reveal requirements in between. Agreeing on which events have probability zero corresponds to agreeing on which events have rank ∞. A common prior ranking function requires more, namely agreement on how strongly each event is disbelieved. When we interpret ranks as orders of magnitude of infinitesimal probabilities, it still requires less than a common prior probability. Agents whose probabilistic priors differ but have the same orders of magnitude share a common prior ranking function.

Despite the subtleties with interpreting the common prior assumption, we expect it to be useful in the analysis of interaction of agents with ranking functions. By eliminating ex ante

differences in disbeliefs, the modeler can cleanly derive doxastic and behavioral implications of interim asymmetric information among agents.

11.4 What Speculation Reveals

While the absence of speculative betting is weak evidence for a common prior, the existence of speculative betting is more informative with ranking functions than with probabilities. With probabilities, a bet whose acceptance by all agents is common knowledge only reveals that there is no common prior. With ranking functions it reveals by Theorem 4 more: *The agents disagree about which states are not disbelieved.* That is, it reveals where the disagreement is, namely about which states are not disbelieved, instead of just that there is disagreement somewhere in the priors.[9] Disagreement about disbelieved states alone never leads to such a bet (see Example 1). From a common endowment Y, a commonly known acceptable bet reveals by Theorem 6 that the *agents disagree about the states that matter for valuing* Y, namely the states at which Y plus disbelief is lowest. Varying Y moves this window across the agents' ranking functions, which is why the absence of commonly known acceptable bets from every common endowment characterizes a common prior (see Theorem 7). These inferences presume that agents are not optimistic (see our discussion in Section 11.5 below) and that betting and its absence are due to the agents' beliefs alone, not to other motives.

Our results have implications for speculation in financial markets and business strategy. Acquiring an asset or strategic resource reveals not just some disagreement among buyers and sellers but disagreement about which states are not disbelieved. Two firms that value an asset or strategic resource equally in each scenario and differ only on how far-fetched a long shot is, such as a technological breakthrough that both regard as unlikely, cannot agree on a price at which one acquires the asset from the other, because a scenario that both firms take seriously settles their evaluations first. In this sense, any business idea, whether it turns out to be a success or a failure, starts with a fundamental disagreement about what is not disbelieved. In strategic management, Barney (1986, p. 1235) argues that differences in firms' expectations about the future value of strategic resources are the central source of above-normal returns from acquiring these resources. Our results suggest that, as long as firms are aware of the same scenarios, only differences about which scenarios are not disbelieved can give rise to such acquisitions, i.e., one firm must not disbelieve a scenario that the other firm disbelieves.[10] With probabilities, in contrast, any difference in the likelihood of a long shot supports a price between the two valuations and leads to trade. This difference yields a hypothesis that might be testable in financial or betting markets. Holding fixed disagreement about central scenarios, disagreement about upside long shots, such as a technological breakthrough, should not increase trading with ranking functions,

[9] If ranks are interpreted as orders of magnitude of infinitesimal probabilities, this comparison needs qualification. A commonly known acceptable bet then reveals the order of magnitude at which the agents disagree, namely the highest, but not where among the states of this order the disagreement lies, just as with probabilities it does not reveal where the priors differ.

[10] Under unawareness, a firm may not even form a disbelief about a scenario that another firm does not disbelieve, so that acquisitions can arise without the other firm realizing what it gives up (Bryan, Ryall, and Schipper, 2022). Combining ranking functions with unawareness is left for future work.

but should increase it with probabilities. Disagreement about downside long shots, such as disasters, can increase trading in both models, because a sufficiently bad outcome makes a disbelieved scenario relevant.

11.5 The Role of Risk Attitudes

Our agents evaluate a random variable X by the min-plus integral $\mathbb{E}_\rho[X \mid F] = \min_{\omega\in F}\{X(\omega) + \rho(\{\omega\} \mid F)\}$. That is, they value X by its worst outcome discounted by disbelief. Maximizing this pessimistic criterion is equivalent to minimizing $-\mathbb{E}_\rho[X \mid F] = \max_{\omega\in F}\{-X(\omega) - \rho(\{\omega\} \mid F)\}$, the max-plus integral of the loss $-X$. Giang and Shenoy (2000) use the pessimistic criterion for lotteries over prizes better than a neutral prize. The literature also uses an optimistic criterion, the max-plus integral $\max_{\omega\in F}\{X(\omega) - \rho(\{\omega\} \mid F)\}$, which values X by its best outcome discounted by disbelief. Maximizing this optimistic criterion is equivalent to minimizing $\min_{\omega\in F}\{-X(\omega) + \rho(\{\omega\} \mid F)\}$, the min-plus integral of the loss $-X$. This is what Spohn (2012, Chapter 10.4; 2025) does with disutilities, and what the order-of-magnitude expected utility does (see Fargier and Sabbadin, 2005, Definitions 19 and 20). Giang and Shenoy (2000) use the optimistic criterion for lotteries over prizes worse than a neutral prize. The max-plus integral is also the $(\max, +)$-expectation of idempotent probability theory (Del Moral and Doisy, 1999, Definition 4). In both criteria, a state counts less the more it is disbelieved: the pessimistic criterion adds the rank before taking the minimum, and the optimistic criterion subtracts it before taking the maximum. Schipper (2026a, Corollary 1) characterizes both by the same axioms, with tropical independence in min-plus form for pessimism and in max-plus form for optimism. The orientation, and not the utility function, decides whether an agent is risk averse or risk loving (Schipper, 2026b). We speak of attitudes toward risk because the pessimistic and optimistic criteria are the limits of expected utility with constant absolute risk aversion a and $-a$, respectively, as $a \to \infty$ and beliefs $p_a(\omega) \propto e^{-a\rho(\{\omega\}|F)}$ on F concentrate on the states of conditional rank 0 (Schipper, 2026a, Proposition 2).

The following example shows that our no-betting results require pessimism. Optimistic agents bet against each other, and this can be common knowledge even with a common prior and no private information.

Example 7. *Let $I = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2\}$, $\Pi_1(\omega) = \Pi_2(\omega) = \Omega$, and $\rho(\{\omega_1\}) = \rho(\{\omega_2\}) = 0$, so that there is a common prior, there is no private information, and neither agent disbelieves either state. Consider the bet with $(X_1(\omega_1), X_1(\omega_2)) = (1, -1)$ and $X_2 = -X_1$. That is, agent 1 wins a dollar from agent 2 in state ω_1 and pays a dollar to agent 2 in state ω_2. Under the pessimistic criterion, agent 1 values her side of the bet at*

$$\mathbb{E}_\rho[X_1 \mid \Omega] = \min\{X_1(\omega_1)+\rho(\{\omega_1\} \mid \Omega), X_1(\omega_2)+\rho(\{\omega_2\} \mid \Omega)\} = \min\{1+0, -1+0\} = -1 < 0,$$

and, likewise, agent 2 values her side at $\min\{-1 + 0, 1 + 0\} = -1 < 0$. Since neither agent disbelieves the state in which she loses, the loss is not discounted, and neither agent accepts. Under the optimistic criterion, agent 1 values her side of the bet at

$$\max\{X_1(\omega_1) - \rho(\{\omega_1\} \mid \Omega), X_1(\omega_2) - \rho(\{\omega_2\} \mid \Omega)\} = \max\{1 - 0, -1 - 0\} = 1 > 0,$$

and, likewise, agent 2 values her side at $\max\{-1-0, 1-0\} = 1 > 0$. *Each agent evaluates the bet by the state in which she wins, which she does not disbelieve. So both agents accept the bet at both states. Since the meet of the agents' partitions is* $\{\Omega\}$, *acceptance by both agents is common knowledge. Agents who evaluate random variables by their best outcome discounted by disbelief bet against each other even under a common prior and symmetric information, just as risk-loving expected utility maximizers do.*

The bet in Example 7 is not speculative. The agents have no private information to speculate on. They bet because each values her side of the bet by its best outcome discounted by disbelief, and without private information their acceptance is trivially common knowledge. Theorem 3 is a statement about speculation because pessimistic agents with a common prior do not bet without private information. A commonly known acceptable bet would therefore have to be due to private information, and Theorem 3 shows that there is none. Optimistic agents, like risk-loving expected utility maximizers, bet already without private information, so for them the absence of commonly known acceptable bets cannot be expected, whatever role private information plays. Pessimistic agents, in contrast, do not bet without private information, whatever their utility functions.[11]

A Proof of Theorem 6

Proof of Theorem 6. *If:* Suppose that every cell of the meet of the agents' partitions contains a common bleakest state for Y, and suppose to the contrary that there are a bet $(X_i)_{i\in I}$ and $\omega \in CK\left(\bigcap_{i\in I} \circledcirc_i^Y(X_i)\right)$. Let M be the cell of the meet of the agents' partitions containing ω. By Aumann's (1976) characterization of common knowledge, $M \subseteq \bigcap_{i\in I} \circledcirc_i^Y(X_i)$. Let $\omega^* \in M$ be a common bleakest state for Y. For every $i \in I$,

$$\begin{aligned} Y(\omega^*) + r_i(\omega^*)(\{\omega^*\}) &= \mathbb{E}_i[Y](\omega^*) \\ &< \mathbb{E}_i[Y + X_i](\omega^*) \\ &\leq Y(\omega^*) + X_i(\omega^*) + r_i(\omega^*)(\{\omega^*\}). \end{aligned} \tag{18}$$

The equality holds because ω^* is a common bleakest state for Y. The strict inequality holds because $\omega^* \in \circledcirc_i^Y(X_i)$. The weak inequality holds because $\mathbb{E}_i[Y + X_i](\omega^*)$ is a minimum over Ω, which contains ω^*. Since $r_i(\omega^*)(\{\omega^*\})$ is finite by the definition of a common bleakest state, (18) yields $X_i(\omega^*) > 0$ for all $i \in I$, contradicting $\sum_{i\in I} X_i(\omega^*) = 0$.

Only if: Suppose that some cell M of the meet of the agents' partitions contains no common bleakest state for Y. Note that $\mathbb{E}_i[Y](\omega)$ is finite for all $i \in I$ and $\omega \in \Omega$, since $r_i(\omega)(\Pi_i(\omega)) = 0$. For $\omega \in M$, let $b(\omega)$ be the number of agents i with $Y(\omega) + r_i(\omega)(\{\omega\}) > \mathbb{E}_i[Y](\omega)$. Then $b(\omega) \geq 1$, since ω is not a common bleakest state for Y. Let $\gamma \in (0, \infty]$ be the smallest value of $Y(\omega) + r_i(\omega)(\{\omega\}) - \mathbb{E}_i[Y](\omega)$ among those that are positive, over $i \in I$

[11] Let agent i accept X_i at ω if $\min_{\omega'\in\Omega}\{u_i(X_i(\omega')) + r_i(\omega)(\{\omega'\})\} > u_i(0)$ for a strictly increasing utility function $u_i : \mathbb{R} \longrightarrow \mathbb{R}$ that is continuous at 0. Then Theorems 3 and 4 continue to hold with essentially the same proofs, except that the bet in the 'only if' part of the proof of Theorem 4 is scaled down. We require continuity at 0.

and $\omega \in M$, and fix a real number δ with $0 < \delta < \gamma$. For $\omega \in M$ and $i \in I$, let

$$X_i(\omega) := \begin{cases} \delta\, b(\omega)/n & \text{if } Y(\omega) + r_i(\omega)(\{\omega\}) = \mathbb{E}_i[Y](\omega), \\ -\delta\,(1 - b(\omega)/n) & \text{otherwise,} \end{cases}$$

and let $X_i(\omega) := 0$ for $\omega \notin M$. At $\omega \in M$, exactly $n - b(\omega)$ agents receive the first payment and $b(\omega)$ agents the second, so

$$\sum_{i \in I} X_i(\omega) = (n - b(\omega))\, \frac{\delta\, b(\omega)}{n} - b(\omega)\, \delta \left(1 - \frac{b(\omega)}{n}\right) = 0.$$

Thus $(X_i)_{i \in I}$ is a bet. Now fix $i \in I$ and $\omega \in M$, and let $v := \mathbb{E}_i[Y](\omega)$. By condition (a) of Definition 11, only states $\omega' \in \Pi_i(\omega)$ have finite rank under $r_i(\omega)$, and $\Pi_i(\omega) \subseteq M$ by property (a) of the meet of the agents' partitions. For $\omega' \in \Pi_i(\omega)$, we have $r_i(\omega') = r_i(\omega)$ by condition (b) of Definition 11, so $\mathbb{E}_i[Y](\omega') = v$. If $Y(\omega') + r_i(\omega)(\{\omega'\}) = v$, then $X_i(\omega') = \delta\, b(\omega')/n > 0$. Otherwise, $Y(\omega') + r_i(\omega)(\{\omega'\}) \geq v + \gamma$ by the definition of γ, and $X_i(\omega') \geq -\delta > -\gamma$. In both cases,

$$Y(\omega') + X_i(\omega') + r_i(\omega)(\{\omega'\}) > v.$$

So $\mathbb{E}_i[Y + X_i](\omega)$ is the minimum of finitely many terms that exceed v and of terms equal to ∞, and hence $\mathbb{E}_i[Y + X_i](\omega) > \mathbb{E}_i[Y](\omega)$, i.e., $\omega \in \text{☺}_i^Y(X_i)$. Since $i \in I$ and $\omega \in M$ were arbitrary, $M \subseteq \bigcap_{i \in I} \text{☺}_i^Y(X_i)$. Since M is a cell of the meet of the agents' partitions, Aumann's (1976) characterization of common knowledge yields $M \subseteq CK\left(\bigcap_{i \in I} \text{☺}_i^Y(X_i)\right)$. □

B Proof of Lemma 7

Proof of Lemma 7. First, $K_i(B) \subseteq B$ for every event B and every $i \in I$. Indeed, if $\omega' \in K_i(B)$, then $\omega' \in \Pi_i(\omega') \subseteq B$ by nondelusion. Hence $K(B) = \bigcap_{i \in I} K_i(B) \subseteq B$. Applied to $B = K^\ell(E)$, this yields $K^{\ell+1}(E) \subseteq K^\ell(E)$ for all $\ell \geq 1$, so the sequence $K^\ell(E)$ is decreasing. Hence $CK(E) \subseteq K(E) \subseteq E$.

Second, $CK(E)$ is public. Let $\omega' \in CK(E)$ and $i \in I$. For every $\ell \geq 1$, $\omega' \in K^{\ell+1}(E) = \bigcap_{j \in I} K_j(K^\ell(E)) \subseteq K_i(K^\ell(E))$, i.e., $\Pi_i(\omega') \subseteq K^\ell(E)$. Since this holds for every $\ell \geq 1$, $\Pi_i(\omega') \subseteq \bigcap_{\ell \geq 1} K^\ell(E) = CK(E)$. Since ω' and i are arbitrary, $CK(E)$ is public. This proves the "only if" part. Indeed, if $\omega \in CK(E)$, then $G := CK(E)$ is a public event with $\omega \in G \subseteq E$.

Conversely, let G be public with $\omega \in G \subseteq E$. Since G is public, $\Pi_i(\omega') \subseteq G$ for every $\omega' \in G$, i.e., $G \subseteq K_i(G)$ for all i. So $G \subseteq K(G)$. Each K_i is monotone: if $B \subseteq B'$ and $\Pi_i(\omega') \subseteq B$, then $\Pi_i(\omega') \subseteq B'$. Hence K is monotone. We show by induction that $G \subseteq K^\ell(G)$ for all $\ell \geq 1$. For $\ell = 1$, this was just shown. If $G \subseteq K^\ell(G)$, then monotonicity yields $K(G) \subseteq K^{\ell+1}(G)$, and so $G \subseteq K(G) \subseteq K^{\ell+1}(G)$. Moreover, since $G \subseteq E$, applying monotonicity ℓ times yields $K^\ell(G) \subseteq K^\ell(E)$. Hence $G \subseteq K^\ell(E)$ for all $\ell \geq 1$, and so $\omega \in G \subseteq CK(E)$. □

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