INTERACTIVE AWARENESS OF UNAWARENESS

Burkhard C. Schipper
University of California, Davis

Motivation

- Unawareness refers to the lack of conception rather than the lack of information
- Development of the foundations of awareness: Fagin & Halpern (1988), …, Heifetz, Meier & Schipper (2006, 2008), … (see the chapter in the *Handbook of Epistemic Logic*, 2015)
- Increasing number of applications

Applications

Game Theory

Halpern & Rego (2014), Rego & Halpern (2012), Heifetz, Meier & Schipper (2013, 2021), Meier & Schipper (2011, 2014), Schipper (2018, 2021), Feinberg (2005, 2009, 2021), Sadzik (2021), Heinsalu (2014), Fukuda & Kamada (2022) …

Decision Theory

Karni & Viero (2013, 2015, 2017), Schipper (2013), Dominiak & Tserenjigmid (2018, 2021), Piermont (2017, 2021), Dietrich (2018), Schipper (2022) …

Finance

Lui (2017), Auster & Pavoni (2021), Gui, Huang & Zhao (2021), Schipper & Zhou (2021), Guerdjikova & Quiggin (2021), Viero (2021), Heifetz, Meier & Schipper (2013), Meier & Schipper (2014), Galanis (2018)

Business Strategy

Bryan, Ryall & Schipper (2021)

Electoral Campaigning

Schipper & Woo (2019)

Contract Theory

van Thadden & Zhao (2012), Auster (2013), Filiz-Ozbay (2012), Chung & Fortnow, 2016, Auster & Pavoni (2021), Lei & Zhao (2021), Francetich & Schipper (2021), Grant, Kline & Quiggin (2012), Herweg & Schmidt (2020), Lee (2008), Pram & Schipper (2022), …

General Equilibrium

Modica, Rustichini & Tallon (1998), Kawamura (2001), Teeple (2021)

Experiments

Ma & Schipper (2017), Araujo & Piermont (2020), Li & Schipper (2018)

Strategic Network Formation

Schipper (2016)

Motivation

One open foundational question relevant to many applications: How to model awareness of unawareness in a tractable way?

"I know that there exists something that I am unaware."

This is not really lack of conception (unawareness) or lack of information (ignorance) but more the lack of comprehension.

Is it even possible to model awareness of unawareness in event-based structures? Wouldn't properties of unawareness preclude awareness of unawareness?

$$K(U(E)) = \emptyset$$ (KU Introspection)

"You never know that you are unaware of the event E."

$$U(E) = U(U(E))$$ (AU Introspection)

"You are unaware of E if and only if you are unaware of that."

Motivation

- Prior work on knowledge of unawareness via first-order modal logic with awareness (Board & Chung, 2021, Sillari, 2008) or second-order awareness logic with quantification over formulae (Halpern & Rego, 2009, 2012): great inspiration but semantics is not syntax-free or requires "objects" rather than being purely event-based; challenge for applied researchers in economics
- Agotnes & Alechina (2007), Walker (2014), Karni & Viero (2017), … : Use of "catch all" event/proposition; not sufficiently expressive
- Halpern & Rego (2012): *"It is not clear how to capture knowledge of unawareness directly in the HMS approach."*
- This paper: Awareness of unawareness among multiple agents in a purely event-based approach with quantification despite leaving all properties of knowledge, belief, and unawareness intact.

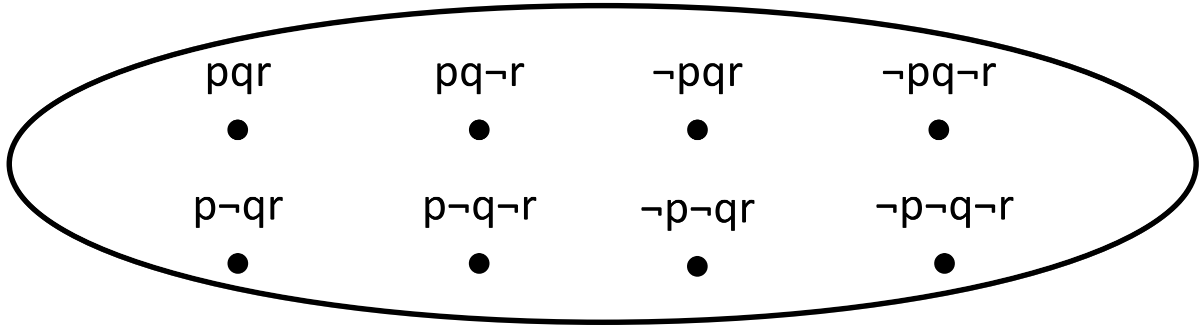
Quantified Unawareness Structures

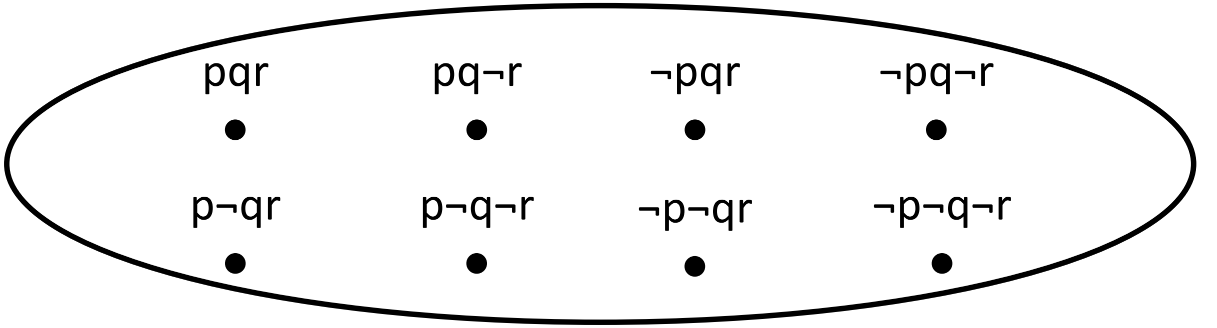
Starting point: Event structures of Heifetz, Meier, Schipper (2006, 2008)

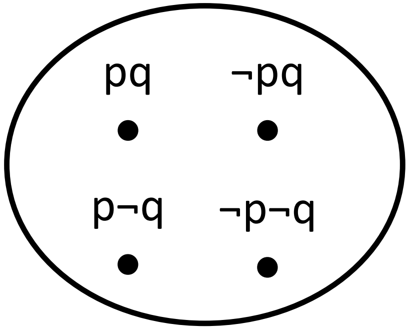
$\langle \mathcal{S}, \preceq \rangle$ nonempty complete lattice of nonempty disjoint spaces

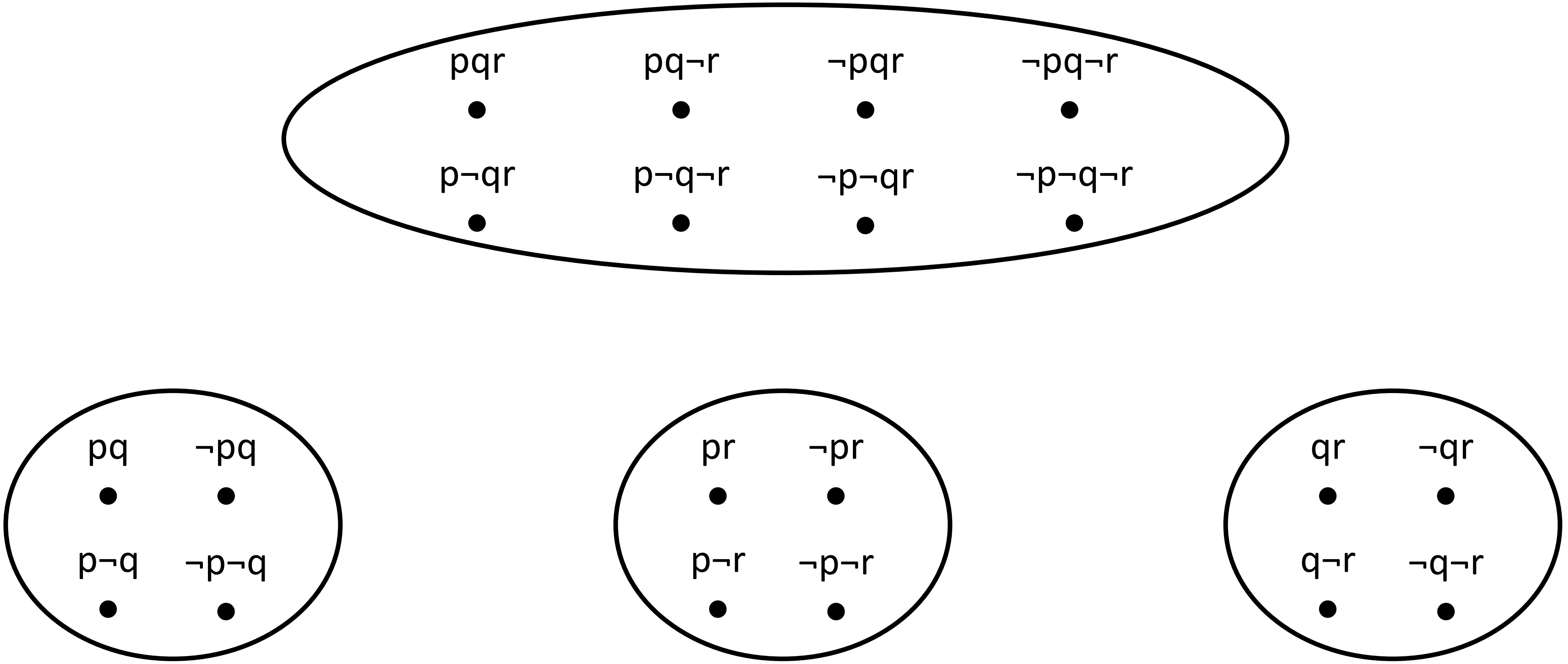
For $S, S' \in \mathcal{S}$, $S' \succeq S$ stands for "S' is more expressive than S"

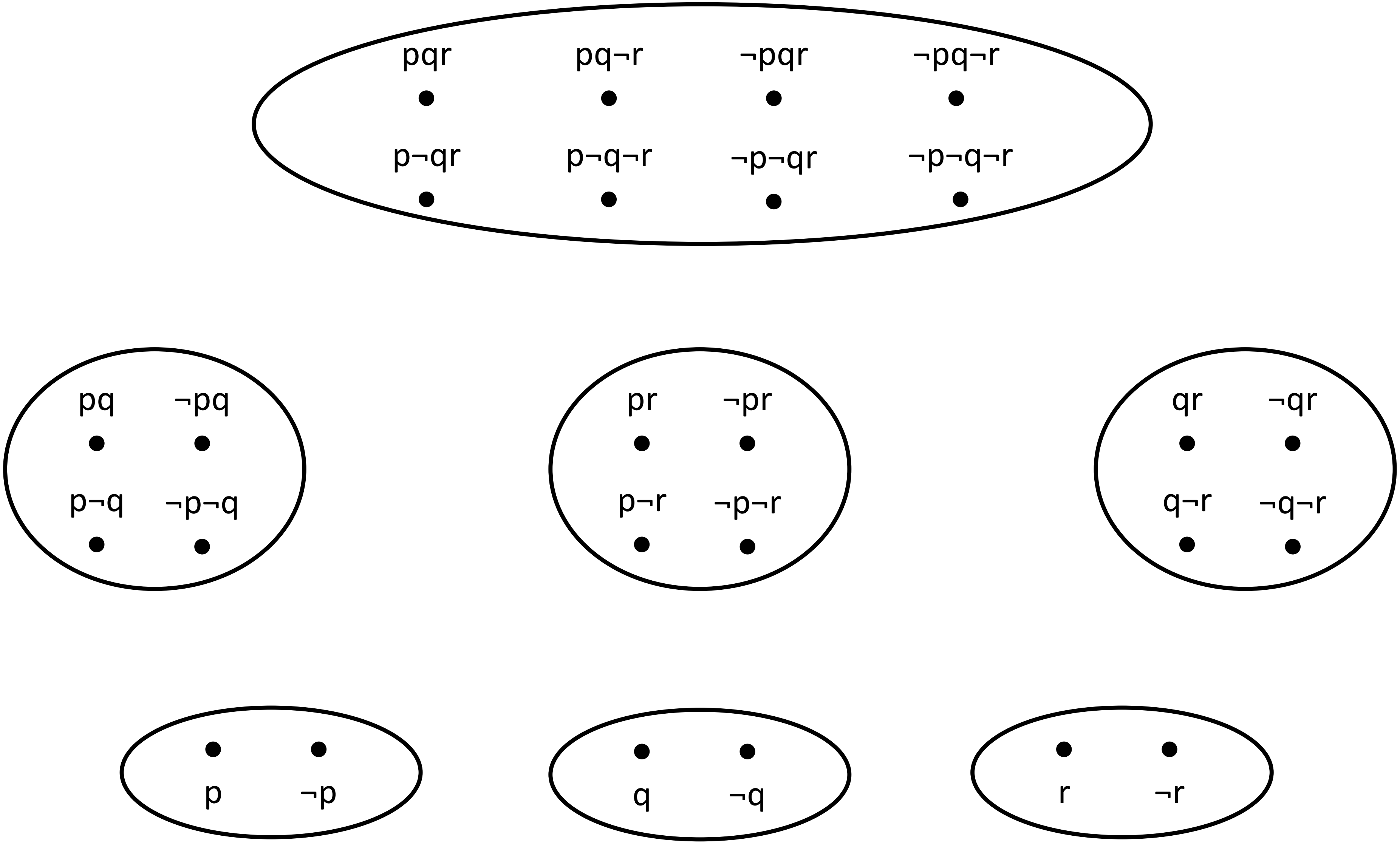
$\Omega := \bigcup_{S \in \mathcal{S}} S$

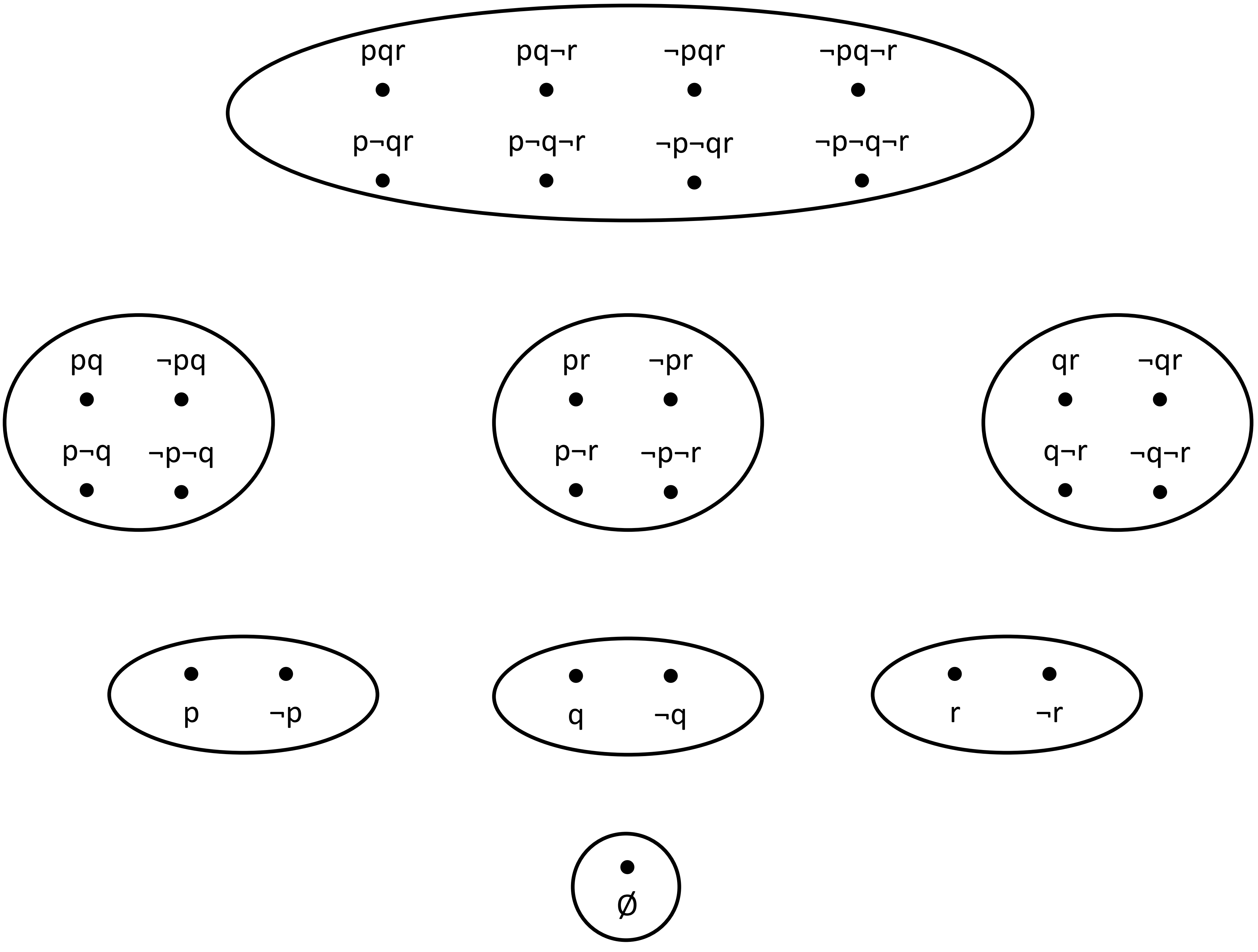

pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r


pq
¬pq
p¬q
¬p¬q


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
∅

Quantified Unawareness Structures

Starting point: Event structures of Heifetz, Meier, Schipper (2006, 2008)

$\langle \mathcal{S}, \preceq \rangle$ nonempty complete lattice of nonempty disjoint spaces

For $S, S' \in \mathcal{S}$, $S' \succeq S$ stands for "S' is more expressive than S"

$\Omega := \bigcup_{S \in \mathcal{S}} S$

Quantified Unawareness Structures

Starting point: Event structures of Heifetz, Meier, Schipper (2006, 2008)

$\langle \mathcal{S}, \preceq \rangle$ nonempty complete lattice of nonempty disjoint spaces

For $S, S' \in \mathcal{S}$, $S' \succeq S$ stands for "S' is more expressive than S"

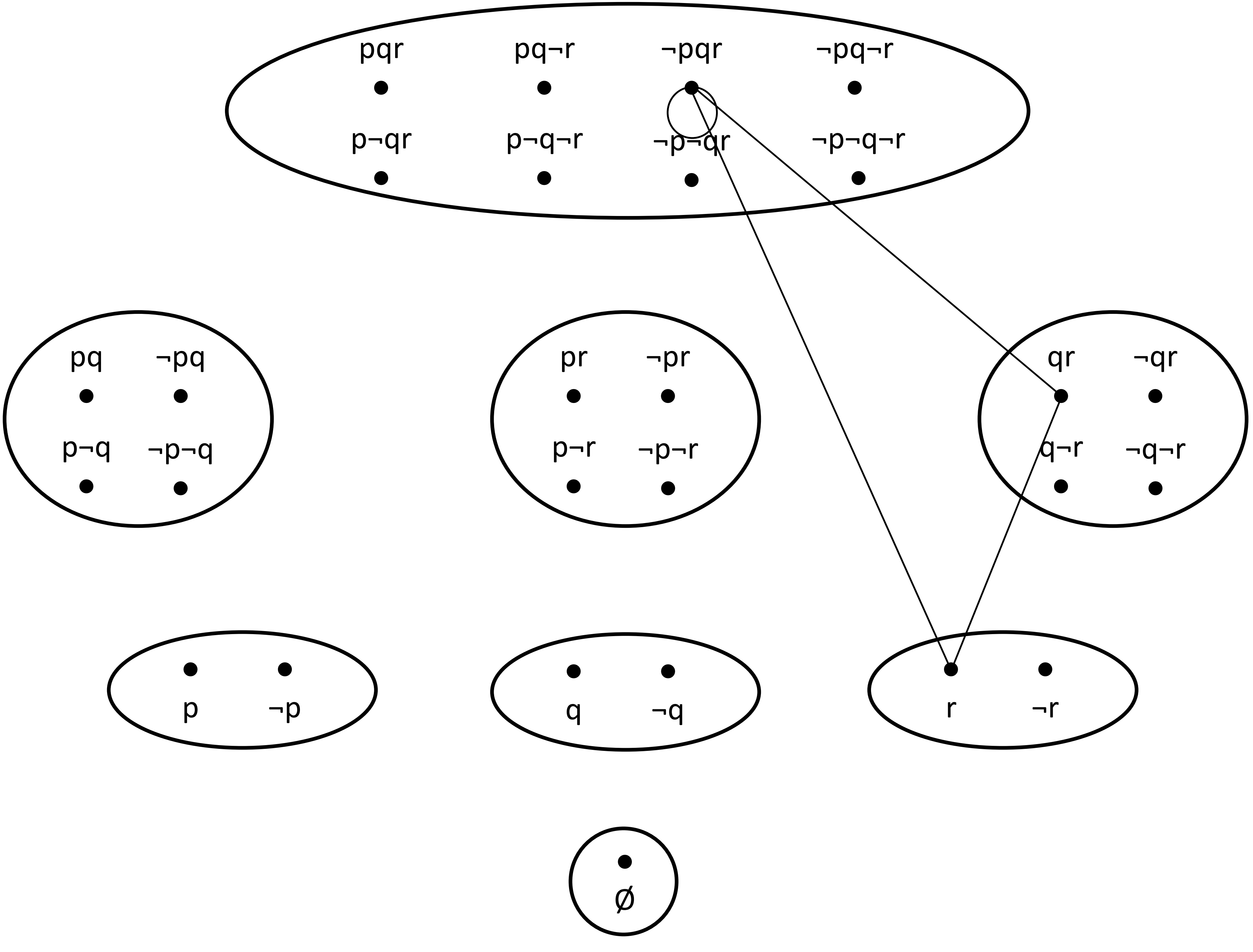
$\Omega := \bigcup_{S \in \mathcal{S}} S$

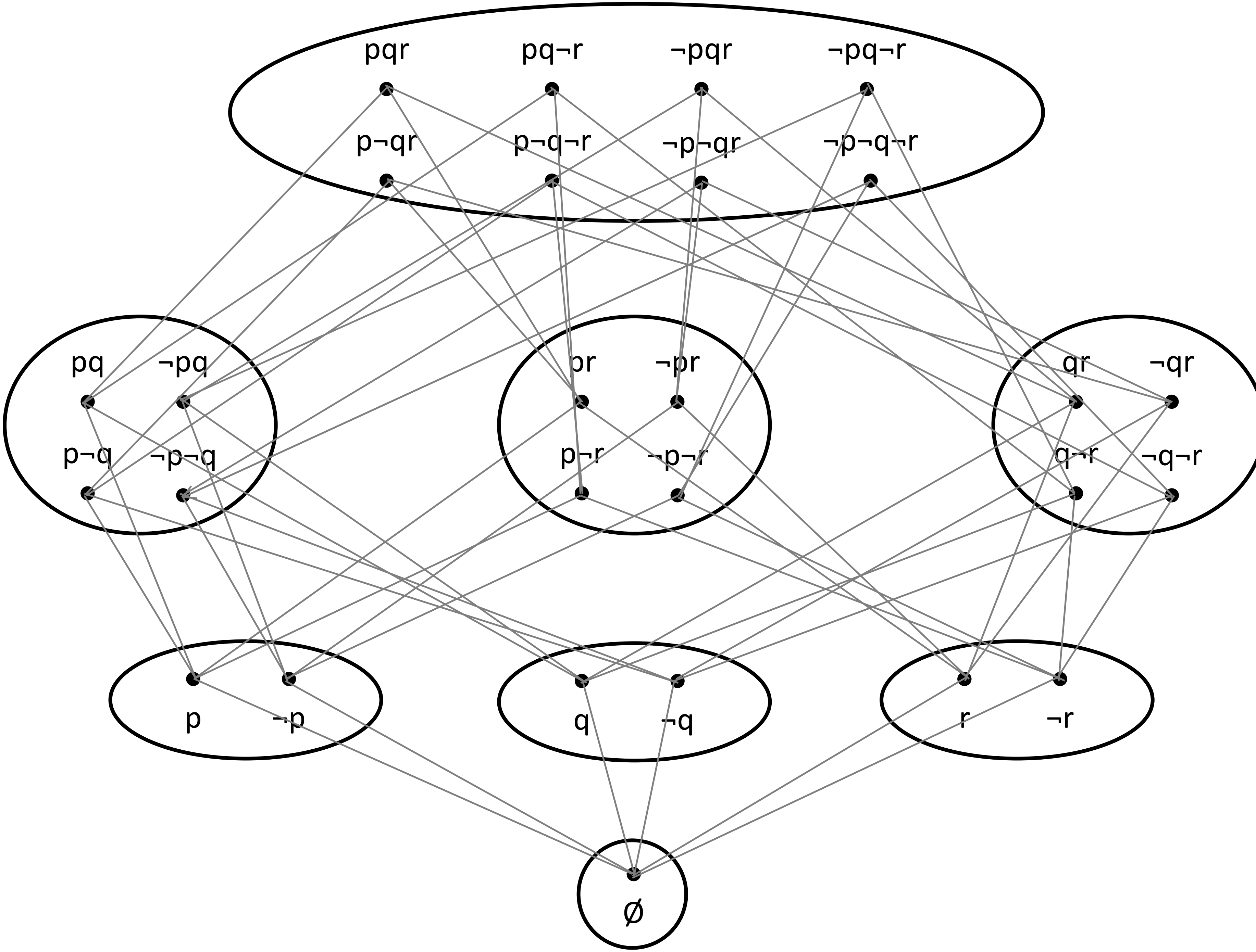
For $S, S' \in \mathcal{S}$ with $S' \succeq S$, $r_S^{S'} : S' \longrightarrow S$ surjective projection.

For any $S \in \mathcal{S}$, $r_S^S = id_S$.

For any $S, S', S'' \in \mathcal{S}$, $S'' \succeq S' \succeq S$, $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$.

For $\omega \in S'$, $\omega_S = r_S^{S'}(\omega)$. For $D \subset S'$, $D_S = \{\omega_S : \omega \in D\}$.


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
∅


pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
∅

Quantified Unawareness Structures

Starting point: Event structures of Heifetz, Meier, Schipper (2006, 2008)

$\langle \mathcal{S}, \preceq \rangle$ nonempty complete lattice of nonempty disjoint spaces

For $S, S' \in \mathcal{S}$, $S' \succeq S$ stands for "S' is more expressive than S"

$\Omega := \bigcup_{S \in \mathcal{S}} S$

For $S, S' \in \mathcal{S}$ with $S' \succeq S$, $r_S^{S'} : S' \longrightarrow S$ surjective projection.

For any $S \in \mathcal{S}$, $r_S^S = id_S$.

For any $S, S', S'' \in \mathcal{S}$, $S'' \succeq S' \succeq S$, $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$.

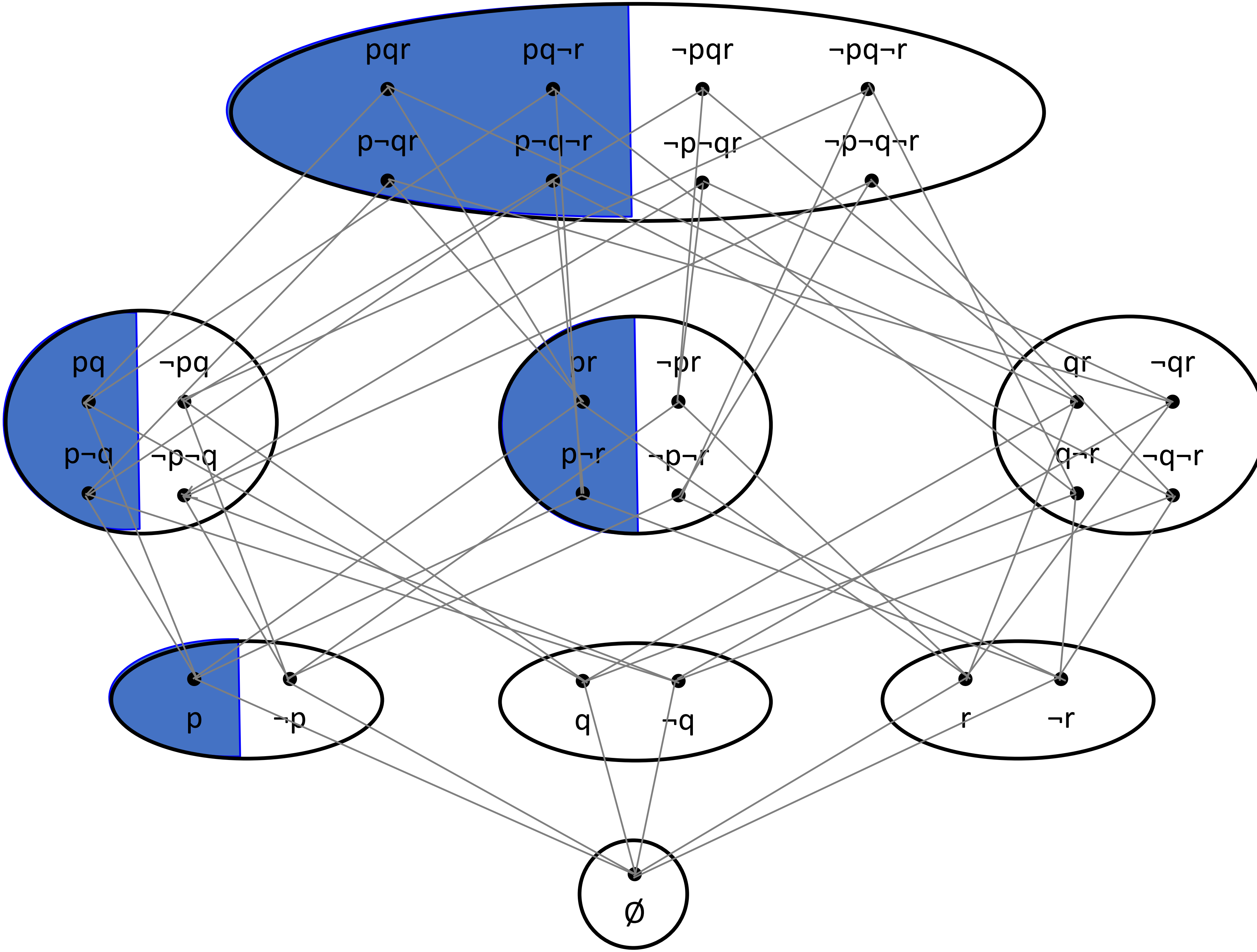
For $\omega \in S'$, $\omega_S = r_S^{S'}(\omega)$. For $D \subset S'$, $D_S = \{\omega_S : \omega \in D\}$.

For $D \subseteq S$, $D^{\uparrow} := \bigcup_{S' \in \mathcal{S} : S' \succeq S} \left(r_S^{S'} \right)^{-1} (D)$.

$E \subseteq \Omega$ is an *event* if $E = D^{\uparrow}$ for some *base* $D \subseteq S$ in some *base-space* $S \in \mathcal{S}$. (denoted by $S(E)$)

We write $\emptyset^S$ for the vacuous event with base-space S.

Not every subset of Ω is an event.



pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
∅

For $D \subseteq S$, $D^{\uparrow} := \bigcup_{S' \in \mathcal{S}: S' \succeq S} \left(r_S^{S'} \right)^{-1} (D)$.

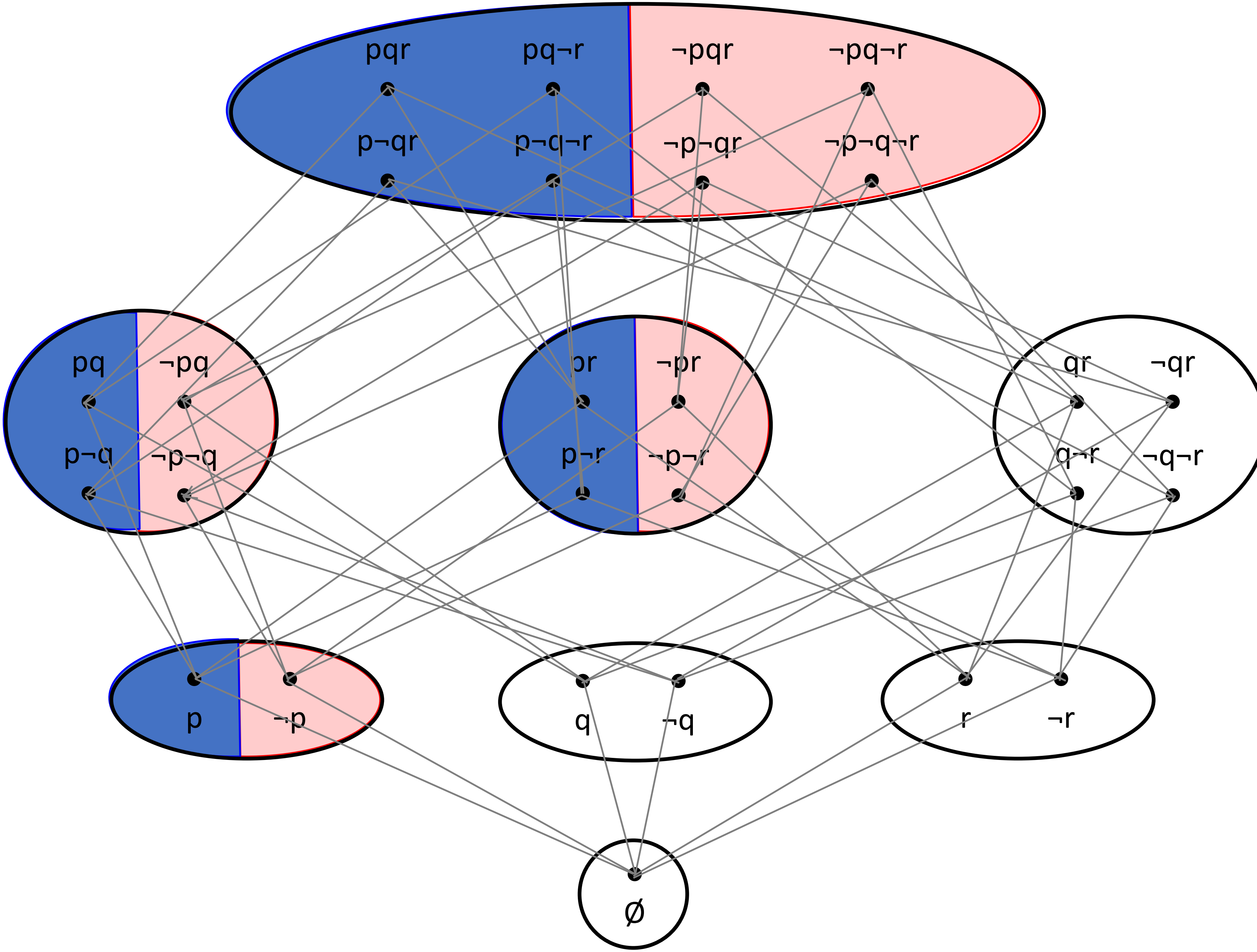
$E \subseteq \Omega$ is an *event* if $E = D^{\uparrow}$ for some *base* $D \subseteq S$ in some *base space* $S \in \mathcal{S}$. (denoted by $S(E)$)

We write $\emptyset^S$ for the vacuous event with base space S.

Not every subset of Ω is an event.

Negation: For $D^{\uparrow}$ an event,
$\neg D^{\uparrow} := \bigcup_{S' \in \mathcal{S}: S' \succeq S} \left(r_S^{S'} \right)^{-1} (S \setminus D)$.

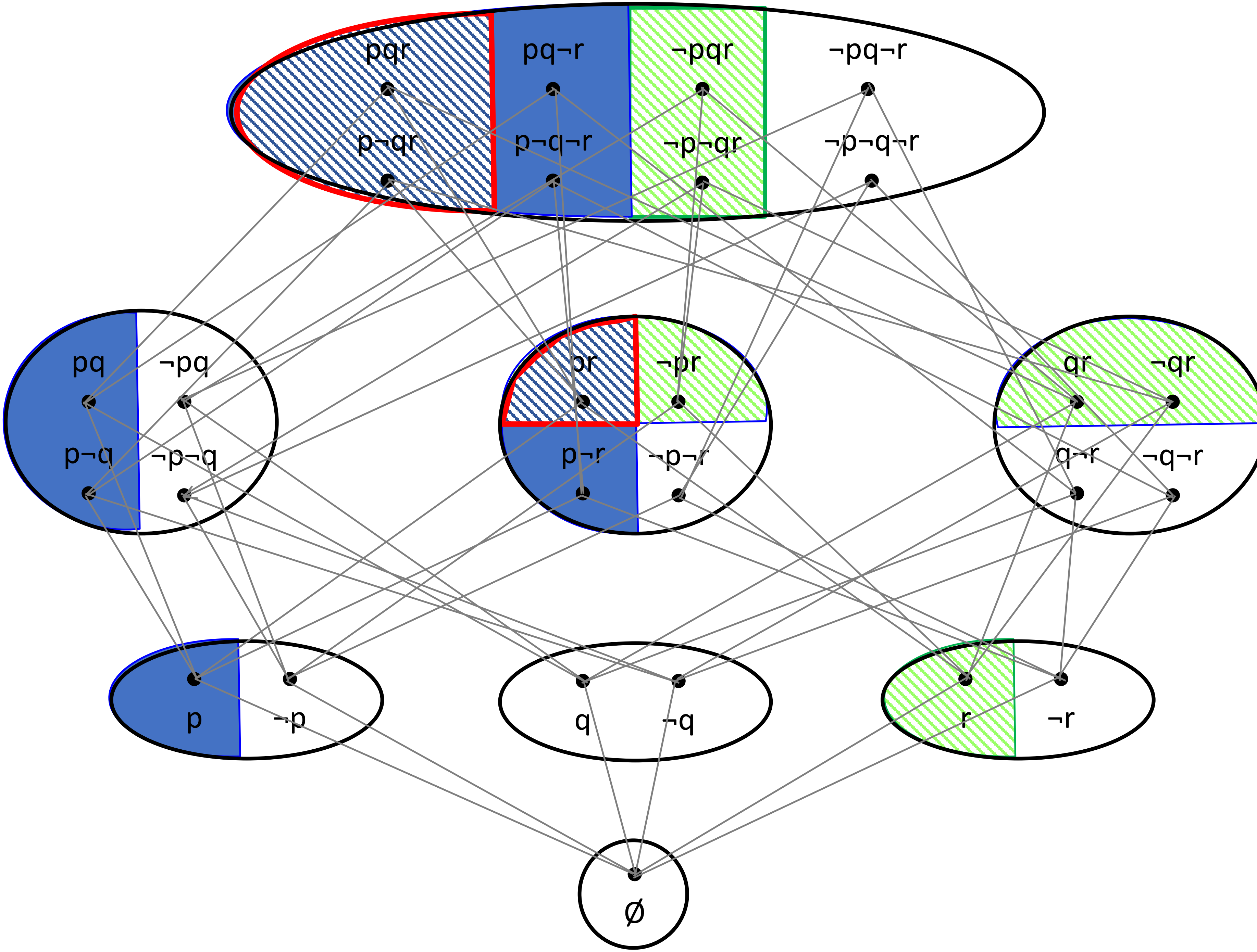
Typcially $\neg E \subsetneq \Omega \setminus E$.



pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
∅

$\{E_j\}$ collection of events

Conjunction: $\bigwedge_j E_j := \bigcap_j E_j$



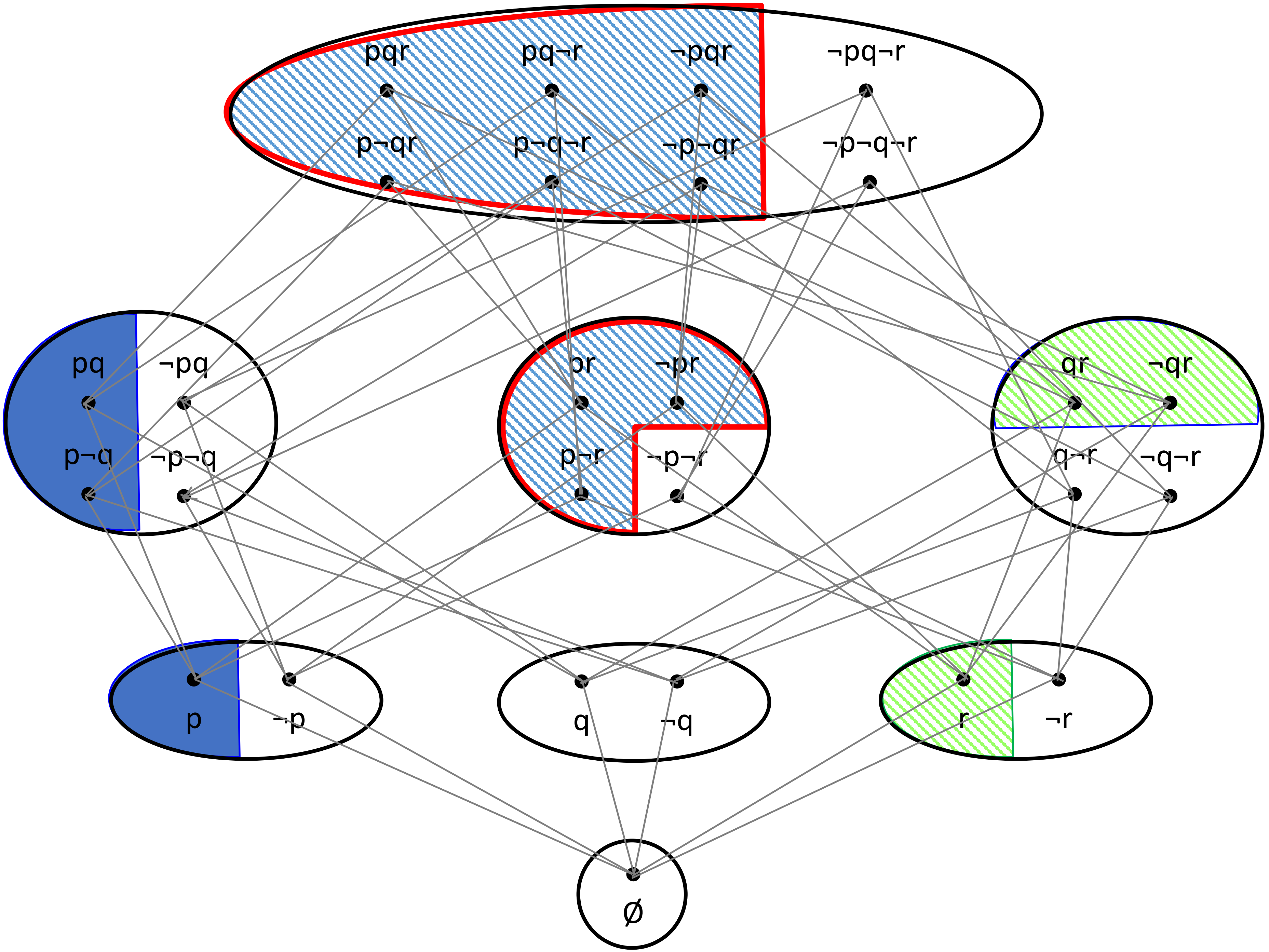
pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
∅

$\{E_j\}$ collection of events

Conjunction: $\bigwedge_j E_j := \bigcap_j E_j$

Disjunction: $\bigvee_j E_j := \neg \left(\bigwedge_j \neg E_j \right)$

Typically $\bigvee_j E_j \subsetneq \bigcup_j E_j$



pqr
pq¬r
¬pqr
¬pq¬r
p¬qr
p¬q¬r
¬p¬qr
¬p¬q¬r
pq
¬pq
p¬q
¬p¬q
pr
¬pr
p¬r
¬p¬r
qr
¬qr
q¬r
¬q¬r
p
¬p
q
¬q
r
¬r
∅

$\{E_j\}$ collection of events

Conjunction: $\bigwedge_j E_j := \bigcap_j E_j$

Disjunction: $\bigvee_j E_j := \neg\left(\bigwedge_j \neg E_j\right)$

Typically $\bigvee_j E_j \subsetneqq \bigcup_j E_j$

$E \subseteq F$ if and only if $E \subseteq F$ and $S(E) \succeq S(F)$.

For any space, $\Sigma(S)$ denotes the set of all events E with $S(E) \preceq S$. (Not necessarily an algebra because of many empty events.)

$S' \preceq S$ implies $\Sigma(S') \subseteq \Sigma(S)$

X set of variables

For $S \in \mathcal{S}$ and finite n, $E(x_1, ..., x_n)$ is S-based n-ary *event operator*.

If $x_1, ..., x_n$ are replaced by $F_1, ..., F_n \in \Sigma$, then $E(F_1, ..., F_n)$ is an event with base-space $S \vee \bigvee_{k=1}^{n} S(F_k)$.

Examples: $I(x) = x$ and $N(x) = \neg x$ are unary $\underline{S}$-based event operators.

The knowledge, mutual knowledge, common knowledge, awareness, unawareness, mutual awareness ... operators are unary $\underline{S}$-based event operators.

$F(x_1, x_2) = F \cap x_1 \cap x_2$ is a binary $S(F)$-based event operator.

$\textit{Domain correspondence } \mathcal{D} : \Omega \longrightarrow \{\Sigma(S)\}_{S \in \mathcal{S}}$ such that

(i) $\omega \in S$ implies $\mathcal{D}(\omega) \supseteq \Sigma(S)$,

(ii) $\mathcal{D}(\omega_S) \subseteq \mathcal{D}(\omega)$ for all $\omega \in \Omega$ and $S \preceq S_\omega$.

Can there be statements in a discourse that are not (yet) recognized as particular events?

“Today we got muruaneq.”

(That’s a particular sort of snow in an Inuit language.)

Domain correspondence $\mathcal{D} : \Omega \longrightarrow \{\Sigma(S)\}_{S \in \mathcal{S}}$ such that

(i) $\omega \in S$ implies $\mathcal{D}(\omega) \supseteq \Sigma(S)$,

(ii) $\mathcal{D}(\omega_S) \subseteq \mathcal{D}(\omega)$ for all $\omega \in \Omega$ and $S \preceq S_\omega$.

Domain correspondence $\mathcal{D} : \Omega \longrightarrow \{\Sigma(S)\}_{S \in \mathcal{S}}$ such that

(i) $\omega \in S$ implies $\mathcal{D}(\omega) \supseteq \Sigma(S)$,

(ii) $\mathcal{D}(\omega_S) \subseteq \mathcal{D}(\omega)$ for all $\omega \in \Omega$ and $S \preceq S_\omega$.

For each event operator $E(x)$ and space $S \in \mathcal{S}$,

$$\forall_S x E(x) := \left\{ \omega \in S^{\uparrow} : \ \omega \qquad \in \bigcap_{F \in \mathcal{D}(\omega)} E(F) \right\}$$

"Given S, for all x, we have $E(x)$."

Domain correspondence $\mathcal{D} : \Omega \longrightarrow \{\Sigma(S)\}_{S \in \mathcal{S}}$ such that

(i) $\omega \in S$ implies $\mathcal{D}(\omega) \supseteq \Sigma(S)$,

(ii) $\mathcal{D}(\omega_S) \subseteq \mathcal{D}(\omega)$ for all $\omega \in \Omega$ and $S \preceq S_\omega$.

For each event operator $E(x)$ and space $S \in \mathcal{S}$,

$$\forall_S x E(x) := \left\{ \omega \in S^{\uparrow} : \; \omega \qquad \in \bigcap_{F \in \mathcal{D}(\omega)} E(F) \right\}$$

"Given S, for all x, we have $E(x)$."

Domain correspondence $\mathcal{D} : \Omega \longrightarrow \{\Sigma(S)\}_{S \in \mathcal{S}}$ such that

(i) $\omega \in S$ implies $\mathcal{D}(\omega) \supseteq \Sigma(S)$,

(ii) $\mathcal{D}(\omega_S) \subseteq \mathcal{D}(\omega)$ for all $\omega \in \Omega$ and $S \preceq S_\omega$.

For each event operator $E(x)$ and space $S \in \mathcal{S}$,

$$\forall_S x E(x) := \left\{ \omega \in S^{\uparrow} : \{\omega\}^{\uparrow} \cap S \left(\bigcap_{F \in \mathcal{D}(\omega)} E(F) \right) \subseteq \bigcap_{F \in \mathcal{D}(\omega)} E(F) \right\}$$

"Given S, for all x, we have $E(x)$."

$$\exists_S x E(x) := \neg \forall_S x \neg E(x)$$

"Given S, there might exist x for which we have $E(x)$."

For standard state-space (i.e., singleton lattice), simply $\forall x E(x) = \bigcap_{F \in \Sigma} E(F)$ and $\exists x E(x) = \bigcup_{F \in \Sigma} E(F)$.

Lemma For the join of the lattice, $\overline{S}$,

$$\begin{aligned} \forall_{\overline{S}} x E(x) &= \bigcap_{F \in \Sigma} E(F) \\ \exists_{\overline{S}} x E(x) &= \bigcup_{F \in \Sigma} E(F) \end{aligned}$$

Lemma For any $S \in \mathcal{S}$ and any event operators $E(x)$ and $F(x)$,

$$\begin{aligned} \forall_S x E(x) \cap \forall_S x F(x) &= \forall_S x (E(x) \cap F(x)) \\ \exists_S x E(x) \cup \exists_S x F(x) &= \exists_S x (E(x) \cup F(x)) \end{aligned}$$

Lemma For any space $S \in \mathcal{S}$ and any event operator $E(x)$,

$$\begin{aligned} \forall_S x \neg E(x) &\subseteq \neg \forall_S x E(x) \\ \neg \exists_S x E(x) &\subseteq \exists_S x \neg E(x) \end{aligned}$$

Lemma The following properties hold:

(i) For any space $S \in \mathcal{S}$ and any event operators $E(x)$ and $F(x)$

$$\forall_S x(\neg E(x) \cap F(x)) \subseteq \neg\forall_S x E(x) \cap \forall_S x F(x).$$

(ii) For any S'-based event operator $E(x)$ and $S \succeq S'$

$$\forall_S x E(x) \subseteq E(F)$$

for any event F with base-space $S(F)$ such that $S \succeq S(F)$.

Lemma For any event operators, $E(x)$ and $F(x)$, if $E(G) \subseteq F(G)$ for all $G \in \Sigma$, then for any $S \in \mathcal{S}$,

$$\begin{aligned} \forall_S x E(x) &\subseteq \forall_S x F(x) \\ \exists_S x E(x) &\subseteq \exists_S x F(x) \end{aligned}$$

As in Heifetz, Meier, and Schipper (2006), for each individual $i \in I$ there is a possibility correspondence $\Pi_i : \Omega \longrightarrow 2^{\Omega}$ such that

(0) Confinement: If $\omega \in S$ then $\Pi_i(\omega) \subseteq S'$ for some $S' \preceq S$.

(i) Generalized Reflexivity: $\omega \in \Pi_i^{\uparrow}(\omega)$ for every $\omega \in \Omega$.

(ii) Stationarity: $\omega' \in \Pi_i(\omega)$ implies $\Pi_i(\omega') = \Pi_i(\omega)$.

(iii) Projections Preserve Ignorance: If $\omega \in S'$ and $S \preceq S'$ then $\Pi_i^{\uparrow}(\omega) \subseteq \Pi_i^{\uparrow}(\omega_S)$.

(iv) Projections Preserve Knowledge: If $S \preceq S' \preceq S''$, $\omega \in S''$ and $\Pi_i(\omega) \subseteq S'$, then $(\Pi_i(\omega))_S = \Pi_i(\omega_S)$.

For each $i \in I$, define the knowledge operator by

$$K_i(E) := \{\omega \in \Omega : \Pi_i(\omega) \subseteq E\},$$

if there is ω s.t. $\Pi_i(\omega) \subseteq E$, and by $K_i(E) := \emptyset^{S(E)}$ otherwise.

For any $E \in \Sigma$ and $i \in I$, $K_i(E)$ is an $S(E)$-based event (Heifetz, Meier, and Schipper, 2006, Proposition 1).

For each $i \in I$, define the awareness and unawareness operators on events, respectively by

$$\begin{aligned} A_i(E) &:= K(S(E)^{\uparrow}) \\ U_i(E) &:= \neg A_i(E) \end{aligned}$$

$K_i(x)$, $A_i(x)$, and $U_i(x)$ are all $\underline{S}$-based event operators.

Knowledge has "standard" properties.

Proposition (Heifetz, Meier & Schipper, 2006) For any agent $i \in I$ and events $E, F, E_1, E_2, \ldots$,

1. Monotonicity: $E \subseteq F$ implies $K_i(E) \subseteq K_i(F)$
2. Conjunction: $K_i\left(\bigcap_{n=1,2,\ldots} E_n\right) \subseteq \bigcap_{n=1,2,\ldots} K_i(E_n)$
3. Truth: $K_i(E) \subseteq E$
4. Positive Introspection: $K_i(E) \subseteq K_i(K_i(E))$
5. Weak Negative Introspection: $\neg K_i(E) \cap A_i(E) \subseteq K_i(\neg K_i(E))$
6. $\neg K_i(E) \cap \neg K_i(\neg K_i(E)) \subseteq \neg K_i(\neg K_i(\neg K_i(E)))$

The previous proposition (together previous lemma) implies:

Corollary For any $i \in I$, $E(x), F(x), E_1(x_1), E_2(x_2), ...$, and space S,

1. $\forall_S x E(x) \subseteq \forall_S x F(x)$ implies $\forall_S x K_i(E(x)) \subseteq \forall_S x K_i(F(x))$
2. $\forall_S x_1, x_2, ... K_i \left(\bigcap_{n=1,2,...} E_n(x_n) \right) \subseteq \bigcap_{n=1,2,...} \forall_S x_n K_i(E_n(x_n))$
3. $\forall_S x K_i(E(x)) \subseteq \forall_S x E(x)$
4. $\forall_S x K_i(E(x)) \subseteq \forall_S x K_i(K_i(E(x)))$
5. $\forall_S x(\neg K_i(E(x)) \cap A_i(E(x))) \subseteq \forall_S x K_i(\neg K_i(E(x)))$
6. $\forall_S x(\neg K_i(E(x)) \cap \neg K_i \neg K_i(E(x))) \subseteq \forall_S x \neg K_i \neg K_i \neg K_i(E(x))$

All properties above hold also when all $\forall_S$ are replaced by $\exists_S$.

"Standard" properties of unawareness hold:

Proposition (Heifetz, Meier & Schipper, 2006) For any agent $i \in I$ and events $E, E_1, E_2, ...,$

1. $U_i(E) = \neg K_i(E) \cap \neg K_i \neg K_i(E)$
2. $U_i(E) = \bigcap_{n=1}^{\infty} (\neg K_i)^n \neg K_i(E)$
3. $K_i(U_i(E)) = \emptyset^S$
4. $U_i(E) = U_i(U_i(E))$
5. $A_i(E) = A_i(\neg E)$
6. $A_i \left(\bigcap_{n=1,2,...} E_n \right) \subseteq \bigcap_{n=1,2,...} A_i(E_n)$
7. $A_i(E) = A_i K_i(E)$
8. $A_i(E) = A_i A_i(E)$
9. $A_i(E) = K_i A_i(E)$

Corollary For any $i \in I$, $E(x), E_1(x_1), E_2(x_2), \ldots$, and space S,

1. $\exists_S x U_i(E(x)) = \exists_S x(\neg K_i(E(x)) \cap \neg K_i \neg K_i(E(x))$
2. $\exists_S x U_i(E(x)) = \exists_S x \bigcap_{n=1}^{\infty} (\neg K_i)^n \neg K_i(E(x))$
3. $\exists_S x K_i U_i(E(x)) = \emptyset^S$
4. $\exists_S x U_i(E(x)) = \exists_S x U_i U_i(E(x))$
5. $\exists_S x A_i(E(x)) = \exists_S x A_i(\neg E(x))$
6. $\forall_S x_1, x_2, \ldots A_i \left(\bigcap_{n=1,2,\ldots} E_n(x_n) \right) \subseteq \bigcap_{n=1,2,\ldots} \forall_S x_n A_i(E_n(x_n))$
7. $\exists_S x A_i(E(x)) = \exists_S x A_i K_i(E(x))$
8. $\exists_S x A_i(E(x)) = \exists_S x A_i A_i(E(x))$
9. $\exists_S x A_i(E(x)) = \exists_S x K_i A_i(E(x))$
10. $\exists_S x A_i(E(x)) = K_i(S^{\uparrow})$

All properties above hold also when all $\exists_S$ are replaced by $\forall_S$.

How do the knowledge and awareness operator interact with quantifiers? How is awareness/knowledge within the quantifier related to awareness/knowledge "outside" the quantifier?

Proposition (Barcan for Awareness) For any $i \in I$, $S \in \mathcal{S}$, and $E(x)$,

$$\forall_S x A_i(E(x)) \subseteq A_i(\forall_S x E(x)).$$

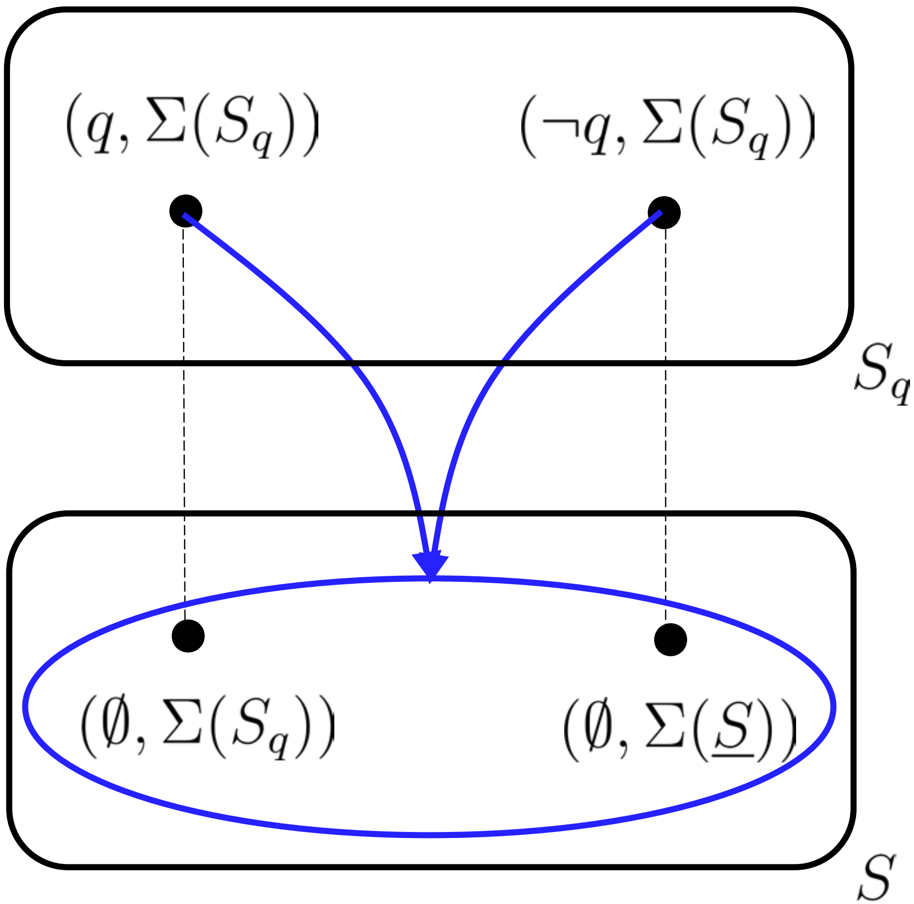
Proposition (Barcan for Knowledge) For any individual $i \in I$ if for any $\omega \in \Omega$, $\omega' \in \Pi_i(\omega)$ implies that $\mathcal{D}(\omega') \subseteq \mathcal{D}(\omega)$, then for any $S \in \mathcal{S}$ and $E(x)$,

$$\forall_S x K_i(E(x)) \subseteq K_i(\forall_S x E(x)).$$

Corollary (Barcan for Knowledge in $\overline{S}$) For any individual $i \in I$, $E(x)$,

$$\forall_{\overline{S}} x K_i(E(x)) \subseteq K_i(\forall_{\overline{S}} x E(x)).$$

Example: Possibility of being unaware of something



$$[q] = \{(q, \Sigma(S_q))\}$$
$$A_i([q]) = \emptyset^{S_q},\ U_i([q]) = S_q$$
$$A_i(\Omega) = \Omega$$

$$\forall_{\underline{S}} x A_i(x) = \{(\emptyset, \Sigma(\underline{S}))\}^{\uparrow}$$
$$\exists_{\underline{S}} x U_i(x) = \{(\emptyset, \Sigma(S_q))\}^{\uparrow}$$

$$K_i(\forall_{\underline{S}} x A_i(x)) = \emptyset^{\underline{S}}$$
$$K_i(\exists_{\underline{S}} x U_i(x)) = \emptyset^{\underline{S}}$$

$$\neg K_i(\forall_{\underline{S}} x A_i(x)) = \Omega$$
$$\neg K_i(\exists_{\underline{S}} x U_i(x)) = \Omega$$

$$\neg K_i(\forall_{\underline{S}} x A_i(x)) \cap \neg K_i(\exists_{\underline{S}} x U_i(x)) = \Omega$$

$$K_i(\neg K_i(\forall_{\underline{S}} x A_i(x)) \cap \neg K_i(\exists_{\underline{S}} x U_i(x))) = \Omega$$

Remark The converse Barcan property for awareness,

$$A_i(\forall_S x E(x)) \subseteq \forall_S x A_i(E(x))$$

does **not** hold.

I am so happy about it because this property would be equivalent to:

$$\exists_S x U_i(E(x)) \subseteq U_i(\exists_S x U_i(E(x)))$$

Sure, we don't want this!

Further Properties

Remark Property $\exists_S x A_i(E(x)) \subseteq A_i(\exists_S x E(x))$ does not hold.

Remark For any i, $S \in \mathcal{S}$, and $E(x)$, $A_i(\exists_S x E(x)) \subseteq \exists_S x A_i(E(x))$.

Proposition

1. For any space $S \in \mathcal{S}$, $\exists_S x A_i(x) = S^{\uparrow}$.
2. FA: For all spaces $S \in \mathcal{S}$, $\forall_S x U_i(x) \subseteq K_i(\forall_S x U_i(x))$.
3. $A_i \left(\exists_{\underline{S}} x U_i(x)\right) = \Omega$
4. For all $S \in \mathcal{S}$, $A_i \left(\exists_S x U_i(x)\right) = A_i(S^{\uparrow})$
5. $K_i \left(\forall_{\overline{S}} x A_i(x)\right) = K_i(\overline{S})$

Mutual knowledge

$$K(E) := \bigcap_{i \in I} K_i(E)$$

Common knowledge

$$CK := \bigcap_{n=1}^{\infty} K^n(E)$$

Mutual awareness

$$A(E) := \bigcap_{i \in I} A_i(E)$$

Common awareness

$$CA(E) := \bigcap_{n=1}^{\infty} A^n(E)$$

Corollary For any $i, j \in I$, $S \in \mathcal{S}$ and $E(x)$,

1. $\exists_S x A_i(x) = \exists_S x A_i A_j(x)$
2. $\exists_S x A_i(x) = \exists_S x A_i K_j(x)$
3. $\exists_S x K_i(x) \subseteq \exists_S x A_i K_j(x)$
4. $\exists_S x A(x) = \exists_S x K(S(x)^{\uparrow})$
5. $\exists_S x A(x) = \exists_S x CA(x)$
6. $\exists_S x K(x) \subseteq \exists_S x A(x)$
7. $\exists_S x CK(x) \subseteq \exists_S x CA(x)$
8. $\exists_S x CK(S(x)^{\uparrow}) \subseteq \exists_S x CA(x)$

These properties hold also when $\exists_S$ is replaced by $\forall_S$.

Proposition (Barcan for Mutual/Common Awareness)
For any $S \in \mathcal{S}$ and $E(x)$,

$$\begin{aligned} \forall_S x A(E(x)) &\subseteq A \forall_S x E(x) \\ \forall_S x CA(E(x)) &\subseteq CA \forall_S x E(x) \end{aligned}$$

Proposition (Barcan for Mutual/Common Knowledge)
Assume that for all $i \in I$, $\omega' \in \Pi_i(\omega)$ implies $\mathcal{D}(\omega') \subseteq \mathcal{D}(\omega)$.
Then for any space $S \in \mathcal{S}$ and event operator $E(x)$,

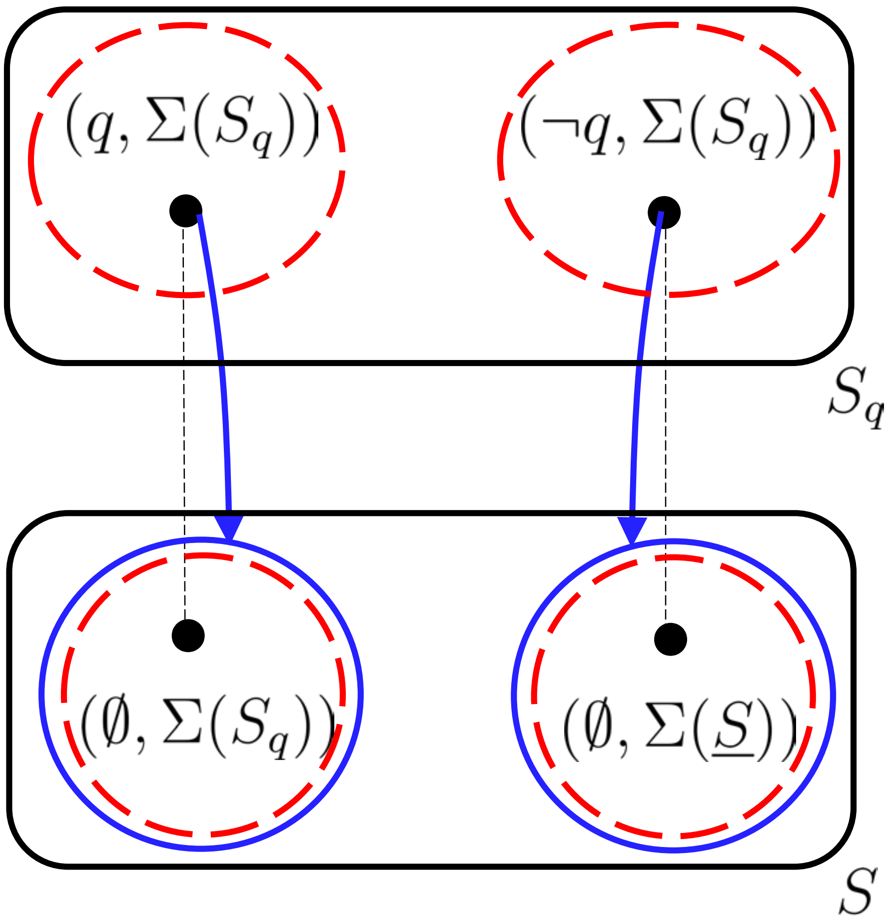
$$\begin{aligned} \forall_S x K(E(x)) &\subseteq K \forall_S x E(x) \\ \forall_S x CK(E(x)) &\subseteq CK \forall_S x E(x) \end{aligned}$$

Illustration: Consulting with Experts

We often consult with experts not just because they have better information but also because we consider it possible that they are aware of things that we are unaware of.

- Patients with doctors
- General medial practitioners refer to medial specialists
- Clients with lawyers
- Investors with financial advisors
- Governments with scientific advisory bodies
- Students with professors
- Music students with music instrument teachers
- Editors consult with referees
- …

Example: Person 1 Person 2



Person 1 wants to consult Person 2 only in the event $K_1 \exists_S x(U_1(x) \cap A_2(x))$

$$U_1([q]) = S_q$$
$$A_2([q]) = S_q$$

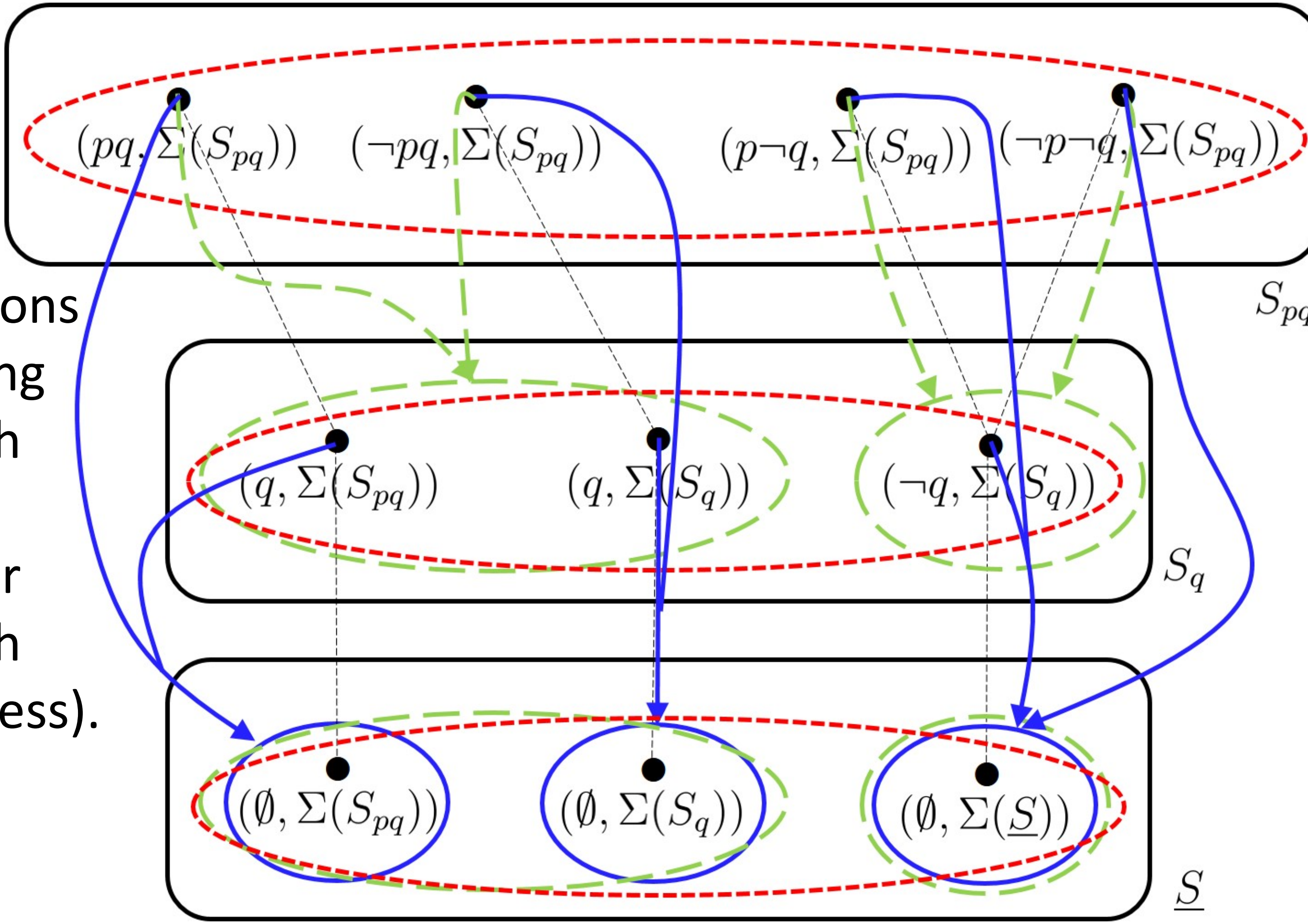
$$\{(\emptyset, \Sigma(S_q))\}^{\uparrow}$$
$$= \exists_{\underline{S}} x(U_1(x) \cap A_2(x))$$
$$= K_1(\exists_{\underline{S}} x(U_1(x) \cap A_2(x)))$$
$$= K_2 K_1(\exists_{\underline{S}} x(U_1(x) \cap A_2(x)))$$
$$= CK(\exists_{\underline{S}} x(U_1(x) \cap A_2(x)))$$

KU-Introsp.: $K_1 U_1([q]) = \emptyset^{S_q}$ Yet, $K_1(\exists_{\underline{S}} x U_1(x)) = \{(\emptyset, \Sigma(S_q))\}^{\uparrow}$

AU-Introsp.: $U_1([q]) = U_1 U_1([q])$

Yet, $U_1([q]) \cap U_1 U_1([q]) \cap A_1(\exists_{\underline{S}} x U_1(x)) = \{(q, \Sigma(S_q))\}$

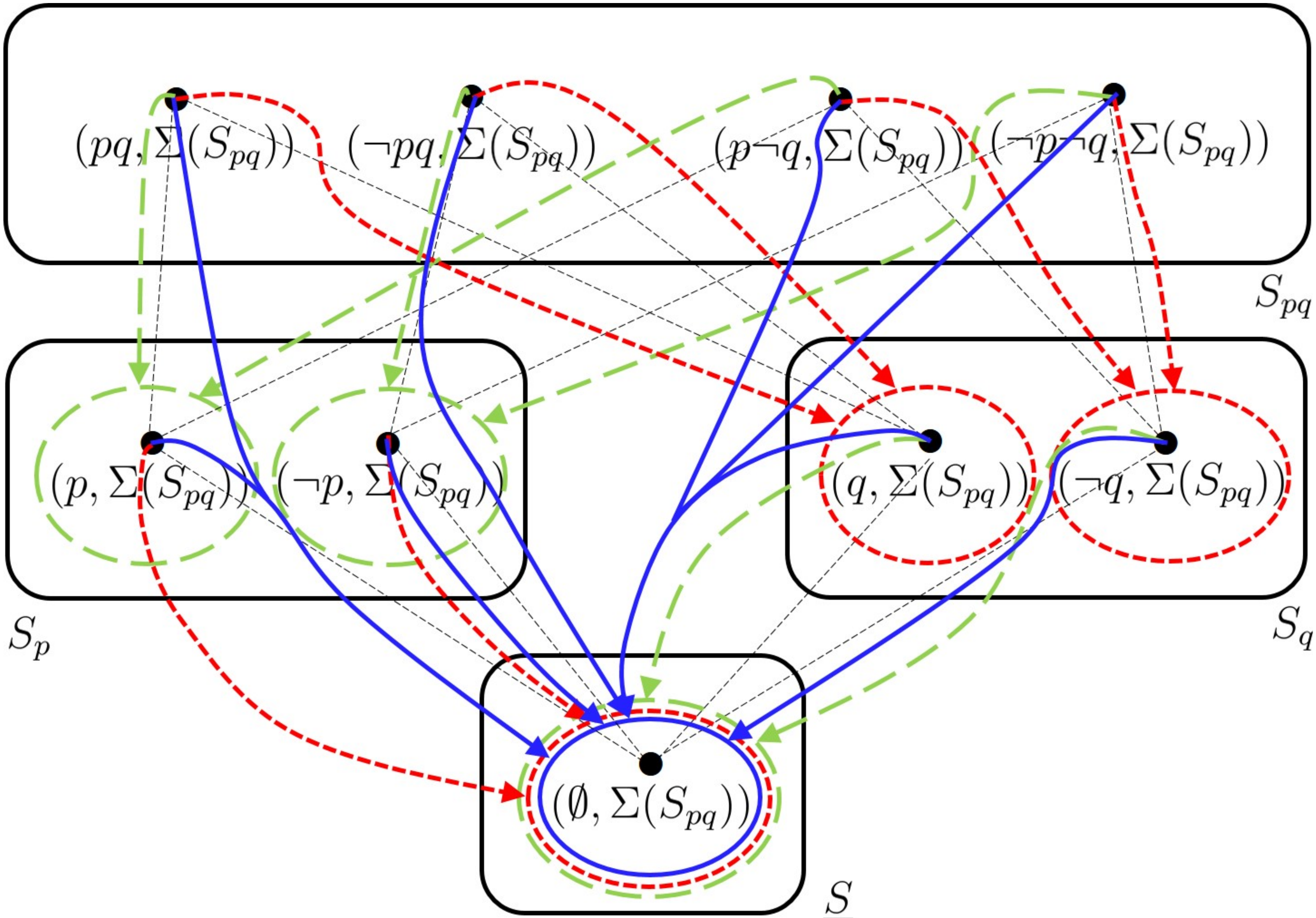
Example:



Person 1 reasons about engaging Person 2 (with better knowledge) or Person 3 (with more awareness).

$$\{(\emptyset, \Sigma(S_{pq})\}^{\uparrow}$$
$$= K_1(\exists_{\underline{S}} x(U_1(x) \cap A_2(x) \cap A_3(x)) \cap \exists_{\underline{S}} x(U_1(x) \cap U_2(x) \cap A_3(x)))$$
$$= K_1(\exists_{\underline{S}} x(U_1(x) \cap K_2(x) \cap \neg K_3(x))).$$

Example: Common knowledge of diverse awareness of experts



Discussion

- Implicitly it is common knowledge that whenever there is a statement that I am unaware of, this statement is an event at some awareness level.
- "God is special" (i.e., having a possibility set on the upmost space is special)
- Event-based structures sometimes make life easier compared to propositional quantification: E.g., semantics of $\forall x x$
- Given S, there might exist x for which we have E(x)

$$\exists_S x E(x) = \left\{ \omega \in S : \{\omega\}^{\uparrow} \cap S \left(\bigcup_{F \in \mathcal{D}(\omega)} E(F) \right) \cap \bigcup_{F \in \mathcal{D}(\omega)} E(F) \neq \emptyset \right\}^{\uparrow}$$

- Fukuda (2021) gives up AU-Introspection to capture awareness of unawareness if structures are *infinite.* We show that this is not necessary.

Next Steps

- Construct the structure from a second-order awareness logic via canonical model
- Detailed comparison to Halpern & Rego (2012), Board & Chung (2011), and Sillari (2008).
- Probabilistic type spaces with awareness of unawareness (extending Heifetz, Meier & Schipper, 2013)
- Bayesian games with awareness of unawareness (extending Meier and Schipper, 2014)
- Decisions in the face of awareness of unawareness
- Dynamic extensions of above
- Applications, applications, applications
- …?

Thank you!