Forward Guidance in a New Keynesian Model with Unawareness*

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Abstract

The forward guidance puzzle refers to the standard New Keynesian model's prediction that a central bank announcement of a *future* interest rate change has the same effect as the corresponding *current* interest rate change. We address this puzzle by allowing for unawareness of agents in an otherwise standard New Keynesian model. There are two types of consumers. Aware consumers are aware of both a TFP shock and a monetary policy shock. They also know that there is a share of unaware consumers who are only aware of the TFP shock. Consistent with their awareness, those unaware consumers believe that all consumers are (only) aware of the TFP shock. We show that in temporary equilibrium, forward guidance has the largest effect when all consumers are aware, a smaller effect under heterogeneous awareness, and the smallest effect when all consumers are unaware. While forward guidance under homogeneous awareness or homogeneous unawareness does not depend on the time horizon, its effect diminishes under heterogeneous awareness as the time horizon increases, resolving the forward guidance puzzle. If the share of aware consumers is sufficiently large, central bank announcements that raise awareness among consumers enhance the effectiveness of forward guidance. Finally, we study self-confirming equilibrium in which unaware consumers become aware of their unawareness. They discover that they miss a component of the model without knowing what it is. As remedy, they invent a dummy shock and continue to learn about it.

Keywords: Unawareness, New Keynesian models, monetary policy, central bank announcements.

JEL Classification Numbers: D83, D84, E12, E52.

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1 Introduction

A central bank's policy of suggesting a path of interest rates changes is known as forward guidance. Forward guidance can influence the agents' expectations and thus affect current output. This is particularly useful when the interest rate is near the zero lower bound since the nominal interest change can be limited in such a situation (Eggertsson and Woodford (2003)).

Del Negro et al. (2023) pointed out the puzzling effectiveness of forward guidance in an estimated DSGE model: The textbook New Keynesian model's prediction is too powerful to be intuitively appealing or to be supported by data.[1] This has been dubbed the "forward guidance puzzle" by Del Negro et al. (2023). The reason for this excessive response in the model is as follows: First, any future real interest rate change has the same effect on current consumption. Second, the reaction of the inflation becomes larger the further the future interest rate change is away from the announcement date of the future interest rate change because the inflation responds to the cumulative consumption changes. Finally, forward guidance under the zero lower bound is essentially an interest rate peg, which lets the solution of the system explode.[2]

To resolve this puzzle, we build in Section 2 a simple New Keynesian model in which agents have heterogeneous awareness of shocks. In our example, there are two kinds shocks, the TFP shock and the monetary policy shock. While all consumers are aware of the TFP shock, only a share of consumers is aware also of the monetary policy shock. Unawareness refers to the lack of conception rather than just the lack of information (Heifetz et al. (2006)). Being unaware of a shock means that the consumers is just like any rational consumer except that he considers the economy without the shock and anything associated with it. In our context, this may be motivated with awareness generated by the experiences of the consumers. While most consumers supply labor and thus may be generally aware of shocks to the production sector of the economy, only a share of consumers have an idea of the workings of the central bank and shocks to monetary policy.[3] This heterogeneity in consumers' awareness of shocks is a novel departure from the standard homogeneous beliefs (and homogeneous awareness) rational expectation equilibrium models since agents with different awareness perceive the structure of the economy and market clearing differently. Unaware consumers interpret aggregates, prices, and the forward guidance only within their model restricted to the TFP shock while aware consumers use their awareness of both shocks to interpret aggregates, prices, and forward guidance, and also take into account that just the share of aware consumers are able to do that. In our model, consumers do not just lack common belief but also common awareness. When we shut off idiosyncratic noise in the private signals (which we do for some comparative statics exercises mentioned below), they form common beliefs with respect to the shock that both types of consumers perceive, but they are unable form common beliefs with respect to the

[1] Del Negro et al. (2023) find that maintaining the federal fund rate at 25bp for 12 quarters increases quarterly real GDP by 9%, which is 30 times larger than the actual response in their data. Carlstrom et al. (2015) observed that the forward guidance can be seen as exogenous interest rate pegs, and the New Keynesian model's predictions are sensitive to the duration of the peg. As the duration increases, the reaction of the current output explodes.

[2] One of the eigenvalues of the solution matrix is outside of the unit circle.

[3] For instance, Claus and Nguyen (2020) and Carvalho and Nechio (2014) present evidence for heterogeneity (associated, for instance, with education) among consumers in response to monetary policy shocks and behavior consistent with monetary policy rules, suggesting heterogeneous understanding of monetary policy and shocks. While some of the heterogeneity may be due to different information or differences in (ir-)rational (in)attention paid by consumers to monetary policy, a survey by Ekins (2017) reports that 6% of Americans have never even heard about the Fed and 22% have heard about it but do not know what it does, suggesting not just lack of information but also a lack of awareness.

shock that unaware consumers are not aware of.

In Section 3, we first focus on the 'temporary equilibrium' similar to Woodford (2013), and Farhi and Werning (2019). In this equilibrium, both heterogeneous beliefs and heterogeneous awareness generate general equilibrium discounting. From Angeletos and Lian (2018) we know that idiosyncratic noise in private signals generates a discount factor in the higher-order expectation. As the horizon of the forward guidance increases, agents are required to form higher and higher orders of expectations, and the effectiveness of forward guidance diminishes because of the discount factor. By shutting off the idiosyncratic noise in private signals in our model, we can isolate a new discount factor. Heterogeneous awareness generates a discount factor of general equilibrium effects because aware consumers correctly anticipate that only the share of aware consumers perceive the existence of the monetary shock. As the time horizon of forward guidance increases, the discounting effect is analogous to the effect of decreasing the share of aware consumers and the effectiveness of forward guidance diminishes as well. This suggests heterogeneous awareness as a novel resolution of the forward guidance puzzle (see Section 3.2).

We also compare our model with two "standard" models, a model with homogeneous unawareness of the monetary policy shock as well as a model with homogeneous awareness of both types of shocks. Since we shut off idiosyncratic noise in private signals for the comparative static exercise, the homogeneous cases correspond to the consumers in rational expectation New Keynesian models with one shock and two shocks, respectively. We can rank these models by the effect of forward guidance on the output gap. The effect of forward guidance is largest in the homogeneous awareness model followed by the model with heterogeneous awareness and followed by the homogeneous unawareness model. Note that both the homogeneous awareness and the homogeneous unawareness model feature common belief (and common awareness) among consumers.

We then investigate a central bank's incentive to raise the awareness of the consumers assuming that it wants to boost the economy (see Section 4). The central bank may not just announce a future interest change but also verbally communicate to the public its rationale for such a future interest change by citing the particular shocks anticipated and thereby raising the public's awareness of those shocks.[4] We show that if the share of aware consumers is sufficiently large, then the central bank will raise awareness of the monetary policy shock together with forward guidance. This observation suggests that forward guidance should not only entail announcing a future interest rate change but raising awareness of the shocks that provide a rational for the future interest rate change. This may actually correspond to the practice of central banking. For instance, Yellen (2013) remarked that "(s)tarting in 2000, the FOMC issued information after every meeting about its economic outlook. It also provided an assessment of the balance of risks to the economy and whether it was leaning toward increasing or decreasing the federal funds rate in the future. Such information about intentions and expectations for the future, known as forward guidance, became crucial in 2003, when the Committee was faced with a stubbornly weak recovery from the 2001 recession." However, we

[4]Although we are not aware of an empirical study of the effect of raising awareness via central bank announcements, there is a sizable literature demonstrating the impact of central bank communication. This literature comprises of indirect approaches quantifying the impact of announcements on financial markets rather than analyzing verbal statements (Jarociński and Karadi (2020), Goodhead and Kolb (2025), Gürkaynak et al. (2005), Brand et al. (2010)), manual coding of central bank statements (Ehrmann and Fratzscher (2009), Rosa and Verga (2007)), and automated text analysis of central bank statements (Klejdysz and Lumsdaine (2023), Jegadeesh and Wu (2017), Picault and Renault (2017), Hansen et al. (2017), Born et al. (2014), Apel and Grimaldi (2012)). We believe that recent developments of Large Language Models (LLMs) offer a promising avenue of automated text analysis of central bank statements w.r.t. raising awareness of shocks.

also show that if only a small share of consumers is aware of the monetary policy shock, then the bank will not announce the future change of interest rates and so a marginal increase of awareness has also no effect.

Finally, in Section 5 we introduce a self-confirming equilibrium in which unaware consumers adjust their model to rationalize observed aggregates.[5] In temporary equilibrium with heterogeneous awareness, observed current market clearing prices may differ from current market clearing prices expected by unaware consumers. Thus, unaware consumers may become aware that they are unaware in the sense that they realize that their model of the economy misses an relevant factor even though they do not know what factor it is.[6] This is different from the standard New Keynesian model (e.g., see Gali (2015) and Woodford (2003)), in which rational expectation equilibrium requires that the path of expected endogenous variable aligns with actual realizations in every period. Our notion of self-confirming equilibrium requires a similar consistency of beliefs with observed data. Since unaware consumers should become aware of their unawareness and realize that they miss a shock, they invent a dummy shock and then continuously learn about. This is in contrast to aware consumers who upon announcement fully anticipate its effect including how unaware consumers will discover and learn about it. We argue that the unaware consumers change their belief about the natural interest rate of the economy in order to make sense of the aggregates. While self-confirming equilibrium differs from rational expectations equilibrium at the agent's level, aggregates correspond to the ones of rational expectations equilibrium, i.e., equilibrium under homogeneous full awareness.

All proofs are relegated to Appendix A.

1.1 Related Literature

A number of resolutions to the forward guidance puzzle have been proposed. One direction is introducing an idiosyncratic income shock and an incomplete financial market. Most notably, McKay et al. (2016) and McKay et al. (2017) showed that the two assumptions generate a discounting intertemporal Euler equation, a less forward-looking IS relation. The uninsurable income risk weakens the intertemporal substitution with a precautionary savings motive and the possibility of hitting the binding financial constraint in the future limits the agent's planning horizon. Werning (2015) on the other hand, expounded that the incomplete market itself may not change how the consumption reacts to the future interest rate. With a vanishing liquidity assumption, the intertemporal Euler equation does not discount future real interest rates even with an incomplete market. This 'neutral benchmark' result comes from the fact that the income risk and liquidity in his model are acyclical. Acharya and Dogra (2020) confirmed Werning (2015) with CARA utility and Normal distribution. They derived an Euler equation that discounts the future if the income risk is pro-cyclical. At the same time, they derive an explosive Euler equation with a counter-cyclical income risk. As Werning (2015) and Farhi

[5]The terminology is borrowed from game theory. Self-confirming equilibrium is an equilibrium in which players play best responses to their beliefs and beliefs are consistent with observations on the equilibrium path (Fudenberg and Levine (1993), Battigalli and Guaitoli (1997)). In games with unawareness, self-confirming equilibrium requires additionally that awareness is consistent with observations on the equilibrium path. That is, the game model perceived by unaware players must be consistent with observations in equilibrium (see Schipper (2021)).

[6]Models of unawareness (e.g., Heifetz et al. (2006) and Heifetz et al. (2013)) require that if an agent is unaware of a shock then she is unaware that she is unaware of the shock. However, there are extensions of models of unawareness (Schipper (2024)) that allow for awareness of unawareness of "something" without an understanding yet of what it is.

and Werning (2019) pointed out, the cyclicalities of the shock and liquidity are endogenous. Hagedorn et al. (2019) found that the forward guidance puzzle could either disappear or worsen depending on the primitives of the model; the distribution of income, profits, and tax policies. If the redistribution is from high MPC households to low MPC households, then forward guidance is less effective in incomplete market models. If the distribution works in the other direction, then the incomplete market would exacerbate the puzzle.

Another strand of the literature on the resolution of the forward guidance puzzle is relaxing strong assumptions embodied in the standard equilibrium concept of the New Keynesian model, namely 'full information rational expectation' (FIRE). Angeletos and Lian (2018) gave up the first half of FIRE, the (F)ull (I)nformation assumption. Specifically, by removing common knowledge of the news (i.e., the announcement of the central bank), they introduced a higher-order uncertainty in the aggregate action. This information friction attenuates general equilibrium effects in the Euler equation and causes the agents to react to the news as if they were myopic. It is clear that our approach also relaxes FIRE, in particular (F)ull (I)information. However, whereas Angeletos and Lian (2018) relax full information by assuming that agents observe central bank announcements shrouded by some idiosyncratic noise (i.e., we may interpret it as agents trying to make sense of Alan Greenspan's mumbling), we model the heterogeneous lack of awareness of entire shocks to the economy. The idea is that economies are complex. Agents may have a better understanding of shocks in their own sectors of the economy but may not perceive shocks in other sectors. In reality, we believe both reasons for lack of full information are present. That's why our model nests both, noisy interpretation of monetary policy signals and lack of awareness of shocks. Wiederholt (2015) presented an earlier approach closely related to Angeletos and Lian (2018) in which heterogeneity in beliefs is generated by allowing only a fraction of households to update beliefs each period. This is interpreted as inattention due to information acquisition costs. Note that unawareness is different from inattention. Inattentive agents still form beliefs about the shock using their common correct prior. Moreover, they realize that there is a fraction of agents that is attentive. In contrast, unaware agents perceive an economy without the shock and therefore do not even form a belief about the shock. Moreover, they do not realize that some agents are aware of the shock.

A different approach is taken by Gabaix (2020) who relaxes the second part of FIRE, the (R)rational (e)xpectation, not the first half. He assumes that agents, when considering a shock in the future, shrink their expectations towards the steady-state of the economy, which is dubbed "cognitive discounting". This leads directly to discounting in the Euler equation. (See also Pfäuti and Seyrich (2024) for resolving the forward guidance puzzle in a HANK version of the cognitive discounting model). Both García-Schmidt and Woodford (2019) and Farhi and Werning (2019) also relax (R)ational (E)xpectations by using the idea of finite level reasoning from game theory. García-Schmidt and Woodford (2019) consider a temporary equilibrium with reflective expectations. The reaction to the change in nominal interest rate is muted if the levels of reflection are low, especially at the beginning of the reaction, but as the process of reflections goes on in the limit it converges to rational expectations equilibrium. Similarly, Farhi and Werning (2019) also adopt level-k reasoning together with the assumption of incomplete markets. They showed that each of the assumptions, level-k reasoning and incomplete market, is not enough separately. The interaction of the two assumptions, however, generates a desired much-muted reaction of the current output. Our notion of self-confirming equilibrium is the analogous to the level ∞ equilibrium in Farhi and Werning (2019) and the level ∞ reflection equilibrium in Woodford (2013).[7]

[7]Other approaches to resolving the forward guidance include Gaballo (2016) (heterogeneous beliefs in an

A closely related literature studies the signaling effect of monetary policy and optimal transparency. Campbell et al. (2012) empirically investigated whether the reaction to the forward guidance is aligned with the central bank's intention. They distinguish two types of forward guidance: Delphic forward guidance refers to the central bank's forecast of the future economic activity and an expected monetary policy reaction. In contrast, Odyssean forward guidance is about the bank's commitment to a nominal interest rate path. In the data, the FOMC forward guidance was successful in delivering the intention of the central bank, indicating the forward guidance was Odyssean. This justifies the usage of the forward guidance as a policy tool in the zero lower bound. On the other hand, Andrade et al. (2019) find evidence for both types of forward guidence, Delphic and Odyssean. They further showed that the powerful reaction of forward guidance may be counteracted by the pessimistic (Delphic) agents. Broadly, our paper touches on the reaction to the future news, a question explored by Coibion and Gorodnichenko (2015), Bordalo et al. (2020), Maćkowiak and Wiederholt (2009), and Kohlhas and Walther (2021). Baeriswyl and Cornand (2010) pointed out that monetary instruments take on a dual stabilizing role. Focusing on the role of central bank announcements as a public signal, they investigated the welfare implications of the transparency of the monetary policy. Their conclusion echoes Morris and Shin (2002) and Angeletos and Pavan (2007). Public information generally does not necessarily improve firms' coordination; rather, its effect depends on how it interacts with the policy action. Cornand and Heinemann (2008) focus on optimal transparency, or provision of the public signal, in a very similar setting as Morris and Shin (2002). By distinguishing the accuracy of a signal and the provision of it, they conclude that a central bank should limit the degree of publicity rather than the precision of information.

Our paper has a connection to the studies of misspecified models. Woodford (2010b) introduced a concept, 'near rational' equilibrium. An agent may have a different assessment of the distribution of the state. The difference between the agent's assessment and the actual distribution is measured with relative entropy (Kullback-Leibler divergence), and a robust policy is defined as a policy that minimizes the maximum of the loss function given an entropy constraint. Woodford (2013) reviewed different equilibrium concepts departing from rational equilibrium with a New Keynesian model. In a similar vein, Esponda and Pouzo (2016) establishes the 'Berk-Nash' equilibrium. Each player has a subjective model, a set of probability distributions over the consequences of the action. The subjective model may be misspecified, meaning that the set may not include the objective distribution. Then the Berk-Nash equilibrium is defined as a strategy profile that is optimal and minimizes the (K-L) distance. Fudenberg et al. (2021) investigated the learning dynamics of the Berk-Nash equilibrium showing that only uniform Berk-Nash can be a long-run outcome. Molavi (2019) built a general equilibrium model with the possibility of model misspecification and proposes constrained rational expectations equilibrium, which is Berk-Nash equilibrium in a dynamic model. Our unaware consumers live also an a misspecified model, in which the misspecification is due to lack of awareness of a shock.

Finally, since this is the first paper on unawareness in a macroeconomic model, we contribute to the recent growing literature on exploring the implications of unawareness in economics. Other applications of unawareness pertain to disclosure, moral hazard, contract theory, screening, efficient mechanism design, auctions, procurement, delegation, speculation, financial market microstructure, default in general equilibrium, electoral campaigning, business strategy,

OLG model), Michelacci and Paciello (2020) (ambiguity about the commitment of the monetary authority and dispersed wealth), Michaillat and Saez (2021) (wealth in the utility function), and Bilbiie (2024) (in a "T(ractable)HANK" model). Empirical studies of forward guidance include D'Acunto et al. (2022), Miescu (2022), Campbell et al. (2019), and Roth et al. (2021).

and conflict resolution; for a bibliography, see Schipper (2025). We believe that our paper's main idea of heterogeneous awareness of shocks and its implementation in an otherwise standard New Keynesian model could be of interest to other application beyond forward guidance.

2 Model

We consider a New Keynesian dynamic economy with consumers, producers, and a central bank (e.g., see Gali (2015) and Woodford (2003)). Time is discrete and indexed by $t = 0, 1, ...$ There is a continuum of consumers I whose measure is standardized to 1. The consumption of consumer $i \in I$ in period t is denoted by $c_{i,t} \in \mathbb{R}_+$. Her labor supply at period t is denoted by $n_{i,t} \in [0, 1]$. Consumer i's utility at period t is $U(c_{i,t}, n_{i,t}) := \frac{(c_{i,t})^{1-\frac{1}{\gamma}}}{1-\frac{1}{\gamma}} - \frac{1}{1+\psi} n_{i,t}^{1+\psi}$ with $\gamma > 0$ being interpreted as the elasticity of intertemporal substitution and $\psi > 0$ being the inverse of Frisch elasticity as in Woodford (2010a).

There are two types of producers. One is an intermediate good producer, and the other is a final good producer. There is a continuum of intermediate good producers denoted by J, and they are normalized to measure 1. Each intermediate good producer is a monopolist. It hires labor and produces its specialized product using a CRS technology given by $y_{j,t} = \exp(z_t) \cdot n_{j,t}$, where $y_{j,t}$ is the amount of the intermediate good that is produced by firm $j \in J$ at period t, $\exp(z_t) \in \mathbb{R}_+$ is the (common) productivity, and $n_{j,t} \in [0, 1]$ is the labor input hired by firm j. There is a representative final good producer who buys intermediate goods and combines them as a final good. Y_t represents the final good at period t, and the technology is the Dixit-Stiglitz aggregator $Y_t = \left(\int_0^1 (y_{j,t})^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}$ where $\varepsilon < 1$ is the elasticity of the substitution.

Finally, there is a central bank that sets the nominal interest rate $R_t \in \mathbb{R}_+$ for every period t via a Taylor rule that will be specified later.

2.1 Shocks and Awareness

There are two shocks in the economy. One is a TFP shock (z_t) in the intermediate good production function, and the other is a monetary policy shock (v_t) in the central bank's monetary policy rule. The two shocks are independently drawn from normal distributions, $N(0, \sigma_z^2)$ and $N(0, \sigma_v^2)$, respectively, for every period. We assume that there's no persistence in the shock process, and the variances of the distribution are finite (i.e., $\sigma_z^2 < \infty$ and $\sigma_v^2 < \infty$).

Central to our model is that we allow consumers to have heterogeneous awareness of shocks. There are two types of consumers that differ in their awareness. Consumers of type a are fully aware of both shocks. We call them *aware consumers*. The set of aware consumers is denote by I_a. Consumers of type u are only aware of the TFP shock. That is, this second type of consumer misses the monetary policy shock. We call them *unaware consumers* and denote their set by I_u. We have $I = I_a \cup I_u$. Let the measures of the two types of consumers be μ and $1 - \mu$, respectively. We use $\ell \in \{a, u\}$ as index for the awareness level/type of consumers.

Aware consumers realize that there is a measure of aware consumers and a measure of unaware consumers. In contrast, unaware consumers do not think about the monetary policy shocks (v_t) and thus also do not consider that others think about the monetary policy shock. That is, they consider all consumers to be the same type as themselves, namely unaware consumers considering only the TPF shocks (v_t). We illustrate the simple unawareness type

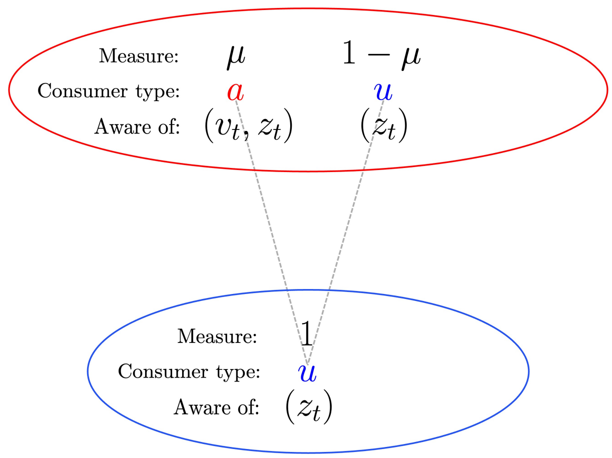

Figure 1: Unawareness Type Space

space in Figure 1. There are two spaces. In the lower space (in blue), all consumers are unaware and consider only one shock, (v_t). All consumers believe that all consumers are of the same type u. The upper space (in red) contains both aware and unaware types, a and u, respectively. The belief of aware consumers about the types of consumers coincides with the prior $(\mu, 1-\mu)$. They realize though that both types map to type u in the lower space and thus know that all unaware types consider all consumers to be aware of only (v_t). This simple type space is a very special case of Heifetz et al. (2013), who present a general extension of beliefs type spaces to unawareness. For simplicity, we focus here only on the awareness of consumers. That is, we will assume that producers and the central bank are fully aware of both shocks.

The central bank, in addition to observing past and present shocks, gets an 'early realization' of the future fundamentals/shocks (or perfect signals about them). The realization of the monetary policy shock can be interpreted for instance as internal information about upcoming changes in the management of the central bank. The TPF shock realization can be interpreted as internal research about future aggregate productivity. The fact that it is about aggregate productivity as compared to individual productivity also motivates our simplifying assumption that producers do not receive such a signal about future TPF shocks. The central bank can signal such future fundamentals via forward guidance in the form of the announcement of a future nominal interest rate, $\tilde{R}_{T|t^*}$, where T denotes the time of implementation of the interest rate and t^* is the time of announcement.[8] At the moment, we do not consider the case of the central bank directly communicating about future fundamentals/shocks to consumers. This will be considered later in Section 4. Let $z_{T|t^*}$ and $v_{T|t^*}$ be the early realization of the fundamentals. That is, $z_{T|t^*}$ is the early realization in t^* about the TPF shock in T (and likewise for $v_{T|t^*}$). Then the central bank announces a corresponding nominal interest rate following a Taylor rule that in our log-linearized model takes the general linear form

$$\tilde{R}_{T|t^*} := \xi_z z_{T|t^*} + \xi_v v_{T|t^*},$$

with parameters $\xi_z, \xi_v \in \mathbb{R}$.

We write the Taylor rule as a function of the two shocks rather than as a function of endogenous aggregate variables (such as the output gap or inflation). This is interpreted as the

[8]It is convenient to have $\tilde{R}_{T|t^*} := R_{T|t^*} + \ln \beta$ instead of the nominal interest rate itself. The advantage will become clear when we characterize the equilibrium in Proposition 1.

composition of endogenous variables as functions of the two shocks and the "usual" Taylor rule. Our formulation is more convenient in our setting because consumers with the private signal eventually want to infer the future shocks from the reaction of the central bank.

Given the central bank's announcement in period t^*, consumers try to infer shocks realized in T. We assume that the announcement works as a private signal at t^*, shrouded by idiosyncratic noise. This can be interpreted as consumer-specific attention to the central bank policy. To differentiate the signal from the announcement, we denote by

$$\omega_{i,T|t^*} := \tilde{R}_{T|t^*} + \eta_{i,t^*}$$

consumer i's private signal about the nominal interest rate in T, where the idiosyncratic noise η_{i,t^*} is drawn from $N(0, \sigma_\eta^2)$. Since consumers in I_u are unaware of the monetary policy shock, they cannot infer anything about the monetary policy shock. Thus, they will interpret nominal interest rates differently from consumers who are aware of the monetary policy shock. An aware consumer $i \in I_a$ forms at t^* conditional beliefs (inference) about the realized values of the future fundamentals in T given the announcement according to

$$\mathbb{E}_{a,t^*}[(z_{T|t^*}, v_{T|t^*}) \mid \omega_{i,T|t^*}] = \left(\lambda_z \frac{\omega_{i,T|t^*}}{\xi_z}, \lambda_v \frac{\omega_{i,T|t^*}}{\xi_v}\right) \tag{1}$$

with $\lambda_z = \frac{\xi_z^2\sigma_z^2}{\xi_z^2\sigma_z^2+\xi_v^2\sigma_v^2+\sigma_\eta^2}$, $\lambda_v = \frac{\xi_v^2\sigma_v^2}{\xi_z^2\sigma_z^2+\xi_v^2\sigma_v^2+\sigma_\eta^2}$. The conditional belief contains the relative variances of the two shocks and the noise. To see this, note that under a correct common prior $(N(0, \xi_z^2\sigma_z^2+\xi_v^2\sigma_v^2))$, the posterior mean of $\tilde{R}_{T|t^*}$ given the signal $\omega_{i,T|t^*}$ is $\mathbb{E}_{a,t^*}\left[\tilde{R}_{T|t^*} \mid \omega_{i,T|t^*}\right] = \left(\frac{\xi_z^2\sigma_z^2+\xi_v^2\sigma_v^2}{\xi_z^2\sigma_z^2+\xi_v^2\sigma_v^2+\sigma_\eta^2}\right)\omega_{i,T|t^*}$. Further, note that the conditional expectation on $\xi_z z_{T|t^*}$ given $\tilde{R}_{T|t^*}$ is $\left(\frac{\xi_z^2\sigma_z^2}{\xi_z^2\sigma_z^2+\xi_v^2\sigma_v^2}\right)\tilde{R}_{T|t^*}$ since $\tilde{R}_{T|t^*}$ is a sum of two normally distributed random variables. Combining these, the conditional expectation on $\xi_z z_{T|t^*}$ is $\left(\frac{\xi_z^2\sigma_z^2}{\xi_z^2\sigma_z^2+\xi_v^2\sigma_v^2+\sigma_\eta^2}\right)\omega_{i,T|t} := \lambda_z\omega_{i,T|t^*}$. This explains the above inference rules for each shock.

An unaware consumer $i \in I_u$, on the other hand, can only infer from the announcement something about the TFP shock. Her inference rule is given by

$$\mathbb{E}_{u,t^*}[z_{T|t^*} \mid \omega_{i,T|t^*}] = \lambda_z^u \frac{\omega_{i,T|t^*}}{\xi_z} \tag{2}$$

where $\lambda_z^u = \frac{\xi_z^2\sigma_z^2}{\xi_z^2\sigma_z^2+\sigma_\eta^2}$. Recall from Figure 1 that any unaware consumer $i \in I_u$ believes that all consumers (including the aware consumers) are unaware. Thus, we must also define expectations given by equation (2) for all $i \in I$, i.e., including consumers in I_a who are actually aware.

When we consider periods before the announcement in t^*, consumers form unconditional expectations. In the formal development, we avoid stating always two versions of the formulas with conditional and unconditional expectations, respectively. Instead, we only state the version with conditional expectations in order to save space.

As mentioned above, we focus on the unawareness of consumers and that's why we assume that both the central bank and producers are aware of both shocks. In contrast to consumers, they observe the central bank's announcement of the future nominal interest $R_{T|t^*}$ without noise. For instance, firms may have departments specialized in market research who can perfectly observe the central bank's announcements while consumers may lack such professional support.

2.2 Consumers

No matter whether the consumer is aware of the monetary policy shock or not, she solves a standard consumer maximization problem. The two types of consumers only differ in in what shocks they think about when forming their expectations. Recall that we use index $\ell \in \{a, u\}$ to denote their awareness level/type of the consumer. A consumer with awareness level $\ell \in \{a, u\}$ maximizes the expected discounted sum of utilities given budget constraints conditional on her signal at t^*,

$$\max_{\{c^\ell_{i,t}, s^\ell_{i,t}, n^\ell_{i,t}\}_{t=t^*}^\infty} \sum_{t=t^*}^{\infty} \beta^{t-t^*} \mathbb{E}_{\ell,t^*}\left[U(c^\ell_{i,t}, n^\ell_{i,t}) \middle| \omega_{i,T|t^*}\right] \text{ s.t.}$$
$$P_t c^\ell_{i,t} + \frac{1}{1+R_t} s^\ell_{i,t} = s^\ell_{i,t-1} + W_t n^\ell_{i,t} + D_t, \quad \forall t \in \{t^*, t^*+1, ...\}, \quad i \in [0,1]$$

where P_t is price level of the consumption good, R_t is the nominal interest rate, $s^\ell_{i,t}$ is savings, W_t is the nominal wage, and D_t is dividend, all at period t. At first glance, the superscript $\ell \in \{a, u\}$ seems redundant as either $i \in I_a$ or $i \in I_u$. However, for consumers in $i \in I_a$, we also need solutions $(c^u_{i,t})$ because unaware consumers think that every consumer, including consumers in I_a, are unaware when deciphering information from prices.

We assume that $s^\ell_{i,t}$ is in an open interval for which no Ponzi schemes can arise. It should be clear that no matter the awareness of agents, such an interval should exist.[9] For instance, take $s^\ell_{i,t} > 0$. However, restricting to strict positive savings will not be necessary. We form the Lagrangian,

$$\mathbb{E}_{\ell,t^*}\left[\sum_{t=t^*}^{\infty} \beta^{t-t^*} U(c^\ell_{i,t}, n^\ell_{i,t}) + \sum_{t=t^*}^{\infty} \zeta_{i,t}\left(s^\ell_{i,t-1} + W_t n^\ell_{i,t} + D_t - P_t c^\ell_{i,t} - \frac{1}{1+R_t} s^\ell_{i,t}\right) \middle| \omega_{i,T|t^*}\right]$$

and derive first-order conditions for $t = t^*, t^*+1, ...$ w.r.t. consumption $c_{i,t}$, savings $s_{i,t}$, and labor supply $n_{i,t}$, respectively,

$$\beta^{t-t^*} \frac{\partial U(c^\ell_{i,t}, n^\ell_{i,t})}{\partial c^\ell_{i,t}} - \zeta_{i,t} P_t = 0 \tag{3}$$
$$-\zeta_{i,t} \frac{1}{1+R_t} + \mathbb{E}_{\ell,t}\left[\zeta_{i,t+1} \middle| \omega_{i,T|t^*}\right] = 0 \tag{4}$$
$$\beta^{t-t^*} \frac{\partial U(c^\ell_{i,t}, n^\ell_{i,t})}{\partial n^\ell_{i,t}} - \zeta_{i,t} W_t = 0. \tag{5}$$

Solve equation (3) for $\zeta_{i,t}$ and substitute it into equations (4), and use the partial derivative of the expected utility function to obtain the intertemporal substitution condition

$$1 = \beta \mathbb{E}_{\ell,t}\left[\frac{P_t}{P_{t+1}} \left(\frac{c^\ell_{i,t+1}}{c^\ell_{i,t}}\right)^{-\frac{1}{\gamma}} (1+R_t) \middle| \omega_{i,T|t^*}\right]. \tag{6}$$

[9] For the case of heterogeneous expectations, this point must have been obvious to Angeletos and Liam (2018) as they do not explicitly state any conditions on savings.

Move the second term in equations (3) and (5) to the r.h.s. and divide (3) by (5). Use the partial derivatives of the utility function to obtain the labor supply condition

$$\left(n_{i,t}^{\ell}\right)^{\psi} = \frac{W_t}{P_t}\left(c_{i,t}^{\ell}\right)^{-\frac{1}{\gamma}}. \tag{7}$$

Define a steady state of the consumer's problem as a path with no shock and stable endogenous variables (for example, $c_{i,t}^{\ell} = c_{i,t+1}^{\ell}$). Use subscript ss for variables in the steady state. Since there are no shocks in the steady state, the process is deterministic. Awareness does not matter in the steady state. From the intertemporal substitution condition and the labor supply condition, we get $R_{ss} = -\ln\beta$ and $\psi \ln n_{i,ss}^{\ell} = \ln w_{ss} + \frac{1}{\gamma}\ln c_{i,ss}^{\ell}$, where w is the real wage defined by $w := \frac{W}{P}$.

We assume that the central bank announcement of the future nominal interest rate is not far away from the steady state. Hence, given normally distributed shocks and idiosyncratic noise in interpreting the central bank announcement, with a large probability consumers are not far away from their steady state no matter their awareness. Thus, we use the first-order Taylor approximation of equation (6) around its steady state to get a usual log-linearized representation of the consumption block of the New Keynesian model. That is, rewrite equation (6) for

$$1 = \mathbb{E}_{\ell,t}\left[\exp\left(\ln\beta - \pi_{t+1} - \frac{1}{\gamma}(\ln c_{i,t+1}^{\ell} - \ln c_{i,t}^{\ell}) + R_t\right)\middle|\omega_{i,T|t^*}\right]$$

and use the first-order Taylor approximation around its steady state

$$1 \approx 1 + \ln\beta + \mathbb{E}_{\ell,t}\left[-\pi_{t+1} - \frac{1}{\gamma}(\ln c_{i,t+1}^{\ell} - \ln c_{i,t}^{\ell})\middle|\omega_{i,T|t^*}\right] + R_t \tag{8}$$

where $\pi_{t+1} := \ln\frac{P_{t+1}}{P_t}$.

For equation (6), simply take log on both sides,

$$\psi \ln n_{i,t}^{\ell} = \ln w_t - \frac{1}{\gamma}\ln c_{i,t}^{\ell} \tag{9}$$

Define

$$\begin{aligned}
C_t^a &:= \int_{i\in I_a} c_{i,t}^a di + \int_{i\in I_u} c_{i,t}^u di \\
C_t^u &:= \int_{i\in I} c_{i,t}^u di \\
N_t^a &:= \int_{i\in I_a} n_{i,t}^a di + \int_{i\in I_u} n_{i,t}^u di \\
N_t^u &:= \int_{i\in I} n_{i,t}^u di.
\end{aligned}$$

Variable C_t^a is the aggregate consumption perceived by the aware consumer. It is also the actual aggregate consumption. In contrast, variable C_t^u is the aggregate consumption perceived by the unaware consumer. Analogous for aggregate labor supply N_t^a and N_t^u.

2.3 Firms and the Central Bank

The representative final good producer's profit maximization problem in period t given its packaging technology is,

$$\max_{(y_{j,t})} P_t Y_t - \int_0^1 P_{j,t} y_{j,t} dj \quad \text{s.t.}$$

$$Y_t = \left(\int_0^1 (y_{j,t})^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

and the solution to the problem gives the following factor (intermediate goods) demand functions,

$$y_{j,t} = \left(\frac{P_{j,t}}{P_t} \right)^{-\varepsilon} Y_t. \tag{10}$$

Substituting the factor demands into the technology constraint of the maximization problem allows us to derive the aggregate price index,

$$P_t = \left(\int_0^1 (P_{j,t})^{1-\varepsilon} dj \right)^{\frac{1}{1-\varepsilon}}. \tag{11}$$

In the intermediate good production, we assume Calvo price stickiness: With probability $1-\theta$, each intermediate goods firm can reset the price of her product, and with a probability θ, the firm maintains the price that is set in the previous period. The price setting opportunities are i.i.d. across firms. The firm chooses the current price $P_{j,t}$ considering that price re-setting opportunities arrive randomly in the future. Using the aggregate price index given in equation (11), denote by P_t^* the aggregate price resulting from intermediate goods prices optimized at t by intermediate goods firms. We now have

$$\begin{aligned} P_t &= \left(\int_{j \in S(t)} (P_{j,t-1})^{1-\varepsilon} dj + (1-\theta)(P_t^*)^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}} \\ &= \left(\theta (P_{t-1})^{1-\varepsilon} + (1-\theta)(P_t^*)^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}} \end{aligned}$$

where $S(t)$ is the realized group of firms that are allowed to adjust their price at period t. Then, by dividing both sides by P_{t-1}, taking the log, and considering the first-order Taylor approximation around its steady state (i.e., zero inflation), we obtain

$$\pi_t = (1-\theta)(P_t^* - P_{t-1}). \tag{12}$$

To get the expression for price P_t^*, consider the optimization problem of intermediate good producer j at any period $t \geq t^*$:

$$\max_{P_{j,t}} \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} \beta^\tau \theta^\tau \left(P_{j,t} \left(\frac{P_{t+\tau}}{P_{j,t}} \right)^{\varepsilon} Y_{t+\tau} - W_{t+\tau} n_{j,t+\tau} \right) \middle| \tilde{R}_{T|t^*} \right] \quad \text{s.t.}$$

$$\left(\frac{P_{j,t}}{P_t} \right)^{-\varepsilon} Y_t = \exp(z_t)\ n_{j,t}.$$

The left-hand side of the constraint is the factor demand of the aggregate final goods producer, and the right-hand side is the production technology of the intermediate goods producer. Substituting the constraint into the objective function, we get

$$\max_{P_{j,t}} \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} \beta^\tau \theta^\tau \left(P_{j,t} \left(\frac{P_{t+\tau}}{P_{j,t}} \right)^\varepsilon Y_{t+\tau} - W_{t+\tau} \left(\frac{P_{t+\tau}}{P_{j,t}} \right)^\varepsilon \frac{Y_{t+\tau}}{\exp(z_{t+\tau})} \right) \middle| \tilde{R}_{T|t^*} \right]$$

from which we can derive the first-order condition

$$0 = \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} (\beta\theta)^\tau Y_{j,t+\tau} \left(1 - \varepsilon + \varepsilon \frac{W_{t+\tau}/\exp(z_{t+\tau})}{P_{j,t}} \right) \middle| \tilde{R}_{T|t^*} \right]. \tag{13}$$

Since $\mathbb{E}_t[P_{j,t}] = P_{j,t}$, we get the following optimal price of the intermediate good j.

$$P^*_{j,t} = \frac{\varepsilon}{\varepsilon - 1} \frac{\mathbb{E}_t \left[\sum_{\tau=0}^{\infty} (\beta\theta)^\tau Y_{j,t+\tau} W_{t+\tau}/\exp(z_{t+\tau}) \middle| \tilde{R}_{T|t^*} \right]}{\mathbb{E}_t \left[\sum_{\tau=0}^{\infty} (\beta\theta)^\tau Y_{j,t+\tau} \mid \tilde{R}_{T|t^*} \right]} \tag{14}$$

and the optimum price is identical to every firm that reoptimizes at period t.

We replace $P^*_{j,t}$ with P^*_t. By dividing both sides P_{t-1} and taking the first order Taylor approximation around the steady state, $P^*_{j,t} = P_{t-1}$, to obtain

$$\begin{aligned}
\ln\left(\frac{P^*_t}{P_{t-1}}\right) &= \ln\left(\frac{\varepsilon}{\varepsilon-1}\right) + (1-\beta\theta)\sum_{\tau=0}^{\infty}(\beta\theta)^\tau \mathbb{E}_t \left[\ln w_{t+\tau} - z_{t+\tau} + \ln\left(\frac{P_{t+\tau}}{P_{t-1}}\right) \middle| \tilde{R}_{T|t^*} \right] \\
&= (1-\beta\theta)\sum_{\tau=0}^{\infty}(\beta\theta)^\tau \mathbb{E}_t \left[\ln w_{t+\tau} - z_{t+\tau} + \ln\left(\frac{\varepsilon}{\varepsilon-1}\right) + \ln\left(\frac{P_{t+\tau}}{P_{t-1}}\right) \middle| \tilde{R}_{T|t^*} \right]
\end{aligned}$$

where $mc^n := \ln\left(\frac{\varepsilon}{\varepsilon-1}\right)$. Take the difference between the two equations

$$\ln P^*_t - \ln P_{t-1} = (1-\beta\theta)\sum_{\tau=0}^{\infty}(\beta\theta)^\tau \mathbb{E}_t \left[\ln w_{t+\tau} - z_{t+\tau} + \ln\left(\frac{\varepsilon}{\varepsilon-1}\right) + \ln\frac{P_{t+\tau}}{P_{t-1}} \middle| \tilde{R}_{T|t^*} \right]$$

$$\begin{aligned}
&\beta\theta\left(\ln P^*_{t+1} - \ln P_t\right) \\
&= (1-\beta\theta)\sum_{\tau=1}^{\infty}(\beta\theta)^\tau \mathbb{E}_{t+1} \left[\ln w_{t+\tau+1} - z_{t+\tau+1} + \ln\left(\frac{\varepsilon}{\varepsilon-1}\right) + \ln\frac{P_{t+\tau+1}}{P_{t-1}} \middle| \tilde{R}_{T|t^*} \right]
\end{aligned}$$

to get the following difference equation.

$$\begin{aligned}
&\ln P^*_t - \ln P_{t-1} \\
&= \beta\theta\mathbb{E}_t\left[\ln P^*_{t+1} - \ln P_t \mid \tilde{R}_{T|t^*}\right] + (1-\beta\theta)\left(\ln w_t - z_t + \ln\left(\frac{\varepsilon}{\varepsilon-1}\right)\right) + \pi_t
\end{aligned}$$

Combine this with the inflation-intermediate good price relation (equation (12))

$$\frac{\pi_t}{1-\theta} = \beta\theta\frac{\mathbb{E}_t\left[\pi_{t+1} \mid \tilde{R}_{T|t^*}\right]}{1-\theta} + (1-\beta\theta)\left(\ln w_t - z_t + \ln\left(\frac{\varepsilon}{\varepsilon-1}\right)\right) + \pi_t$$

we derive the inflation dynamics as follows:

$$\pi_t = \beta \mathbb{E}_t \left[\pi_{t+1} \mid \tilde{R}_{T|t^*} \right] + \frac{(1-\beta\theta)(1-\theta)}{\theta} \left(\ln w_t - z_t + \ln \left(\frac{\varepsilon}{\varepsilon - 1} \right) \right) \tag{15}$$

Since the intermediate goods producers can have positive profits, they pay a dividend. The nominal dividend from firm j in period t is

$$D_{j,t} = \left(P_{j,t} - W_t \frac{1}{\exp(z_t)} \right) y_{j,t}.$$

Aggregating over all intermediate goods producers

$$D_t = \int_{j \in J} D_{j,t} dj$$

which, as we have seen in the consumers' problem, is captured by the consumers.

Finally, we close the model by specifying the central bank's reaction function (Taylor rule),

$$R_t = -\ln \beta + \phi_y \hat{X}_t + \phi_\pi \pi_t + v_t$$

or equivalently,

$$\tilde{R}_t = \phi_y \hat{X}_t + \phi_\pi \pi_t + v_t \tag{16}$$

where $\hat{X}_t$ is the log deviation of the output gap from its steady state, ϕ_y and ϕ_π are the exogenous coefficients, and v_t is the monetary policy shock. An output gap is the difference between aggregate output and 'natural' output level, $\hat{X}_t := \hat{Y}_t - \hat{Y}_t^n$. The natural output level is an output level under the full price flexibility assumption, which we will derive in the next section.

In the following, we use the hat symbol ˆ on variables to denote both its log deviation from the steady state or its relative deviation from the steady state, which are approximately equal to each other. E.g., $\hat{X} = \ln X_t - \ln X_{ss} \approx \frac{X_t - X_{ss}}{X_{ss}}$.

3 Forward Guidance

3.1 Temporary Equilibrium

Consider a situation in which the economy is in equilibrium but no monetary policy shocks occurred yet. Agents maximize their objective functions, the central bank follows her Taylor rule, and markets clear. Since no monetary policy shocks have occurred yet, consumers behave the same no matter their awareness. Note that aware consumers anticipate that there might be some monetary policy shock in the future but unless it is announced by the central bank, the expected nominal interest rate change is zero.

Definition 1 (Temporary equilibrium) *Aggregate consumption $\{C_t^\ell\}$, aggregate output $\{Y_t\}$, labor supply $\{N^\ell\}$ and labor demand $\{N^d\}$, a nominal interest rate $\{R_t\}$, aggregate dividends $\{D_t\}$, wages $\{W_t\}$, and inflation $\{\pi_t\}$ constitute a temporary equilibrium if, for every t,* *(i) Each consumer $i \in I$ optimizes leading to the intertemporal substitution condition given by equation (6) and the labor supply condition of equation (7) for $\ell = u$. Each consumer $i \in I_a$ also optimizes leading to the intertemporal substitution condition given by equation (6) and the labor supply condition of equation (7) for $\ell = a$.*

(ii) The representative final goods producer optimizes leading to factor demands given by equation (10). The intermediate goods producers set optimal factor prices given by equation (14).

(iii) The nominal interest rate is set by the central bank according to the Taylor rule given by equation (16).

(iv) Unaware consumers perceive market clearing prices to solve

$$Y_t^u = C_t^u = \int_{i \in I} c_{i,t}^u di$$

in the final goods market and

$$\int_{j \in J} n_{j,t} dj =: \quad N^d = N^u \quad = \int_{i \in I} n_{i,t}^u di$$

in the labor market.

Aware consumers, producers, and the central bank perceive market clearing prices to solve

$$Y_t^a = C_t^a = \int_{i \in I_u} c_{i,t}^u di + \int_{i \in I_a} c_{i,t}^a di$$

in the final goods market and

$$\int_{j \in J} n_{j,t} dj =: \quad N_t^d = N_t^a \quad = \int_{i \in I_u} n_{i,t}^u di + \int_{i \in I_a} n_{i,t}^a di$$

in the labor market.

Note that in terms of notation, C_t^u is not aggregate consumption of unaware consumers only but aggregate consumption of all consumers as perceived by unaware consumers (and analogous for C_t^a).

Before the announcement of the central bank, the behavior in the temporary equilibrium corresponds to the behavior in the standard rational expectations equilibrium for NK models. In particular, conditions (i), (ii), and (iii) are the building blocks of the 3-equation NK model, and (iv) is the usual market clearing condition. To see the latter, recall that as discussed above there is no difference in consumption of aware and unaware consumers before the central bank's announcement. Even though aware consumers anticipate that there will be a future central bank announcement of a monetary policy shock, the shock is mean zero ex-ante. Thus, it does not affect their behavior. There is also no difference in the labor supply of aware and unaware consumers. Thus, condition (i) is standard before the central bank's announcement.

Now consider the announcement by the central bank at period t^*. Agents continue to optimize like in the baseline equilibrium but are now taking into account the announcement. Due to differences in awareness among consumers, their optimal consumption and labor supply may differ. Moreover, the perceived market clearing of unaware consumers may differ from the perceived market clearing of aware consumers. Aware consumers fully perceive actual market clearing. Unaware consumers, however, perceive market clearing as follows: Unaware consumers form beliefs about the future aggregates based on their model lacking conception of the monetary policy shocks. All information contained in the central bank announcement is attributed by unaware consumers to TFP shocks. They believe markets clear, i.e.,

$$\mathbb{E}_{u,t^*}[Y_t^u \mid \omega_{i,T|t^*}] = \mathbb{E}_{u,t^*}[C_t^u \mid \omega_{i,T|t^*}]$$
$$\mathbb{E}_{u,t^*}[N_t^d \mid \omega_{i,T|t^*}] = \mathbb{E}_{u,t^*}[N_t^u \mid \omega_{i,T|t^*}]$$

given perceived price vectors, $\mathbb{E}_{u,t^*}[(R_t, w_t, \pi_t) \mid \omega_{i,T|t^*}]$, for all $t > t^*$.

We now characterize the temporary equilibrium. We start by deriving the unawareness augmented IS curve:

Proposition 1 *The aggregate reaction of consumers forms the following unawareness augmented IS relation for each type space. For the lower space, the IS curve is*

$$\hat{Y}_t^u = -\gamma \sum_{\tau=0}^{\infty} \beta^{\tau+1} \overline{\mathbb{E}}_{I,t}[\tilde{R}_{t+\tau}] + (1-\beta) \sum_{s=0}^{\infty} \beta^{\tau} \overline{\mathbb{E}}_{I,t} \left[\hat{Y}_{t+\tau}^u \right]$$

and for the upper space, the IS curve is

$$\begin{aligned} \hat{Y}_t = & -\gamma \sum_{\tau=0}^{\infty} \beta^{\tau+1} \left(\mu \overline{\mathbb{E}}_{I_a,t}[\tilde{R}_{t+\tau} - \pi_{t+\tau+1}] + (1-\mu) \overline{\mathbb{E}}_{I_u,t}[\tilde{R}_{t+\tau} - \pi_{t+\tau+1}] \right) \\ & + (1-\beta) \sum_{s=0}^{\infty} \beta^{\tau} \left(\mu \overline{\mathbb{E}}_{I_a,t} \left[\hat{Y}_{t+\tau}^a \right] + (1-\mu) \overline{\mathbb{E}}_{I_u,t} \left[\hat{Y}_{t+\tau}^u \right] \right) \end{aligned} \tag{17}$$

where $\overline{\mathbb{E}}_{a,t}[\cdot] := \frac{1}{\mu} \int_{i \in I_A} \mathbb{E}_{a,t}[\cdot \mid \omega_{i,T|t^*}]\, di$ is the average expectation among the aware consumers $i \in I_a$ (and likewise for the unaware consumers in I_u).

Next, we want to link both the aggregate demand of the consumption block of Proposition 1 and the inflation dynamics of the production block (equation (15)) with the monetary policy given by the Taylor rule (equation 16). To this end, we derive the natural rates of output and the output gaps. Start with the production side. The natural output level is defined as an output level under complete price flexibility. Recall the first-order condition of the intermediate good producer (equation (13)),

$$0 = \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} (\beta\theta)^{\tau} Y_{j,t+\tau} \left(1 - \varepsilon + \varepsilon \frac{W_{t+\tau} / \exp(z_{t+\tau})}{P_{j,t}} \right) \middle| \tilde{R}_{T|t^*} \right].$$

Since there is a continuum of intermediate goods producers, each of them is small. Thus, they take wages as given. Moreover, they can adjust prices each period under complete price flexibility assumed when considering the natural rate of output. Hence, their dynamic optimization problem is a sequence of one-period problems. Therefore, the above first-order condition can be written as

$$0 = 1 - \varepsilon + \varepsilon \frac{W_t^n / \exp(z_t)}{P_t}$$

where $P_{j,t}$ is replaced with P_t^n since every firm will choose the same price, and n in the superscript implies the natural level. Moving $\frac{W_t^n}{\exp(z_t)P_t^n}$ to the left side and taking the natural log gives,

$$\ln w_t^n - z_t = -\ln\left(\frac{\varepsilon}{\varepsilon - 1}\right).$$

Then, we can derive

$$\begin{aligned}-\ln\left(\frac{\varepsilon}{\varepsilon-1}\right) &= \psi \ln N_t^n + \frac{1}{\gamma}\ln Y_t^n - z_t \\ &= \psi(\ln Y_t^n - z_t) + \frac{1}{\gamma}\ln Y_t^n - z_t\end{aligned}$$

where the first equation comes from aggregate labor supply, i.e., $\psi \ln N_t^n = \ln w_t - \frac{1}{\gamma}\ln Y_t^n$, and the second equation comes from $\ln N_t^n = \ln Y_t^n - z_t$ which can be obtained from the intermediate good production technology and aggregation of the labor demand.[10] We can rearrange the above equation as

$$\ln Y_t^n = \frac{1+\psi}{\psi + \frac{1}{\gamma}} z_t - \frac{\ln\left(\frac{\varepsilon}{\varepsilon-1}\right)}{\psi + \frac{1}{\gamma}}$$

by collecting $\ln Y_t^n$. Finally, define the output gap $\hat{X}_t$ as the difference between the current output and the natural level of output. Then,

$$\begin{aligned}\hat{X}_t :=& \ln Y_t - \ln Y_t^n \\ =& \frac{1}{\psi+\frac{1}{\gamma}}\ln w_t + \frac{\psi}{\psi+\frac{1}{\gamma}} z_t - \frac{1+\psi}{\psi+\frac{1}{\gamma}} z_t + \frac{\ln\left(\frac{\varepsilon}{\varepsilon-1}\right)}{\psi+\frac{1}{\gamma}} \\ =& \frac{1}{\psi+\frac{1}{\gamma}}\ln w_t - \frac{1}{\psi+\frac{1}{\gamma}} z_t + \frac{\ln\left(\frac{\varepsilon}{\varepsilon-1}\right)}{\psi+\frac{1}{\gamma}}\end{aligned}$$

where the second equation is immediate if we combine $\ln Y_t = z_t + \ln N_t$ and $\psi \ln N_t = \ln w_t - \frac{1}{\gamma}\ln Y_t$ as $\ln Y_t = \frac{1}{\psi+\frac{1}{\gamma}}\ln w_t + \frac{\psi}{\psi+\frac{1}{\gamma}} z_t$. Multiplying $\psi + \frac{1}{\gamma}$ on both sides, we get

$$\left(\psi + \frac{1}{\gamma}\right)\hat{X}_t = \ln w_t - z_t + \ln\left(\frac{\varepsilon}{\varepsilon-1}\right)$$

and plugging this into equation (15) gives the following New Keynesian Phillips curve:

$$\pi_t = \beta \mathbb{E}_t[\pi_{t+1} \mid \tilde{R}_{T|t^*}] + \kappa\left(\psi + \frac{1}{\gamma}\right)\hat{X}_t \tag{18}$$

[10] We have

$$\begin{aligned}\ln N_t^n :=& \ln \int_{j\in J} n_{j,t}^n dj = \ln \frac{1}{\exp(z_t)} \int_{j\in J}\left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} dj Y_t \\ =& \ln Y_t^n - z_t + \ln \int_{j\in J}\left(\frac{P_{j,t}}{\int_{j\in J}\left(P_{j,t}^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}} dj}\right)^{-\varepsilon} dj \\ \approx& \ln Y_t^n - z_t.\end{aligned}$$

3.2 Effect of Forward Guidance in Temporary Equilibrium

We would like to analyze how heterogeneous awareness of consumers affects forward guidance in temporary equilibrium. The forward guidance puzzle is that a future interest rate change has the same effect on the IS curve as a corresponding change of the current interest rate (McKay et al. (2017), Angeletos and Lian (2018), Farhi and Werning (2019)). We show that under unawareness the effect of forward guidance is weaker than the effect of a current interest rate change.

We assume that the economy is initially in a steady state. Thus, there are no shocks. In this case, temporary equilibrium coincides with rational expectations equilibrium. Since there are no shocks, differences in awareness of shocks do not matter and the reaction of consumers is the same across the two types. Then we introduce the early realization of the future shocks, $(z_{T|t^*}, v_{T|t^*})$, and the central bank's announcement, $\tilde{R}_{T|t^*}$, at period t^*. We assume that the announcement is about the nominal interest rate only at period $T > t^*$ and no other period. That is, we fix the nominal interest rate of any other periods at the steady state level and assume that all other agents in the model do not change their beliefs about it. In principle, if there is an expected change in the nominal interest rate at T, it will change the agents' action at $T-1$. Responding to this, the monetary policy should adjust R_{T-1} according to the Taylor rule. As this backward recursion goes on, all nominal interest rates at the in-between periods should adjust. We exclude this consideration by assuming that the economy is in a steady state until period t^* and will deviate from the steady state for all periods after t^*+1. The nominal interest rate, however, is fixed at the steady state level until period $T-1$. There are three reasons for this assumption: First, we do not know how to solve the model analytically without this assumption. The related literature like Angeletos and Lian (2018) or Farhi and Werning (2019) uses the same assumption. Second, as suggested by Angeletos and Lian (2018), the unmodeled zero nominal interest rate lower bound may be binding for all periods before T, constraining how the central bank could react in periods before T. Third, like Farhi and Werning (2019) we are interested in the comparative statics between two announcements of the change of the nominal interest rate at different horizons keeping nominal interest rates for any other periods constant. In some sense, we isolate an upper bound on the potential effect of forward guidance.

As we have seen in the previous section, aware and unaware consumers have different evaluations of the fundamentals/shocks, respectively. Hence the shift of the aggregate IS curve differs from the benchmark of rational expectations equilibrium under full awareness. In the following proposition, we show how the current output gap changes when there is a central bank announcement at t^* about the nominal interest rate at period $T > t^*$ (i.e., forward guidance).

Proposition 2 *The temporary equilibrium reaction of the current output gap, $\hat{X}_{t^*}$, on the announcement of the future nominal interest rate, $\tilde{R}_{T|t^*}$, is*

$$\hat{X}_{t^*}(\tilde{R}_{T|t^*}) = \mu\left(\Phi^a_{t^*}\overline{\mathbb{E}}_{I_a,t^*}\left[\tilde{R}_{T|t^*}\right] - \Phi^{u*}_{t^*}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right]\right) + \Phi^u_{t^*}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right]$$

where Φ^a_{t*} and Φ^{u*}_{t*} are the average output gap reactions to $\tilde{R}_{T|t}$ of aware and unaware consumers, I_a and I_u, respectively. Φ^u_{t*} is the perceived average reaction of unaware consumers in the lower space. Φ^a_{t*}, Φ^{u*}_{t*}, Φ^u_{t*} are defined as follows:

$$\Phi^a_{t^*}\overline{\mathbb{E}}_{I_a,t^*}\left[\tilde{R}_{T|t^*}\right] - \Phi^{u*}_{t^*}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right] = \begin{pmatrix}1 & -1 & 0 & 0\end{pmatrix}\cdot(M_a)^{T-t^*-1}\, b_a\tilde{R}_{T|t^*}$$

$$\Phi_t^u = \begin{pmatrix}1 & 0\end{pmatrix} (M_u)^{T-t^*-1} b_u \tilde{R}_{T|t}$$

$$M_a := \begin{pmatrix} \beta + ((1-\beta)\mu + \gamma\Xi\mu)(\lambda_z + \lambda_v) & 0 & \gamma\beta & 0 \\ 0 & \beta + ((1-\beta)\mu + \gamma\Xi\mu)\lambda_z^u & 0 & \gamma\beta \\ \Xi\mu & 0 & \beta & 0 \\ 0 & \Xi\mu & 0 & \beta \end{pmatrix}$$

$$M_u := \begin{pmatrix} \beta + (1-\beta+\gamma\Xi)\lambda_z^u & \gamma \\ \beta\Xi & \beta \end{pmatrix}$$

$$b_a := \begin{pmatrix} \left(1 + \frac{(1-\beta)\mu+\gamma\Xi\mu}{\beta}\right)\left(\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z + \frac{\Lambda_{12}}{\Lambda_{22}}\lambda_v\right) \\ \left(1 + \frac{(1-\beta)\mu+\gamma\Xi\mu}{\beta}\right)\left(\frac{\Lambda_{11}}{\Lambda_{21}}\frac{\beta\lambda_z^u+(1-\beta)\mu\lambda_z}{\beta+(1-\beta)\mu} + \frac{\Lambda_{12}}{\Lambda_{22}}\frac{(1-\beta)\mu\lambda_v}{\beta+(1-\beta)\mu}\right) \\ \frac{\Xi\mu}{\beta}\left(\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z + \frac{\Lambda_{12}}{\Lambda_{22}}\lambda_v\right) \\ \frac{\Xi\mu}{\beta}\left(\frac{\Lambda_{11}}{\Lambda_{21}}\frac{\beta\lambda_z^u+(1-\beta)\mu\lambda_z}{\beta+(1-\beta)\mu} + \frac{\Lambda_{12}}{\Lambda_{22}}\frac{(1-\beta)\mu\lambda_v}{\beta+(1-\beta)\mu}\right) \end{pmatrix} \tilde{R}_{T|t^*}$$

$$b_u := \begin{pmatrix} (1+\gamma\Xi)\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z^u \\ \beta\Xi\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z^u \end{pmatrix}$$

where λ_z, λ_v, and λ_z^u are the relative variances defined previously, and $\Lambda_{11} := \frac{-\gamma}{1+\gamma\phi_y}\frac{\gamma+\gamma\psi}{1+\gamma\psi}$, $\Lambda_{12} := \frac{-\gamma\beta+\frac{1-\beta}{\phi_y}(1-\mu)}{\beta+\gamma\beta\phi_y}\frac{1}{\mu}$, $\Lambda_{21} := \frac{-\gamma\phi_y}{1+\gamma\phi_y}\frac{\gamma+\gamma\psi}{1+\gamma\psi}$, $\Lambda_{22} := \frac{1-(1-\beta)\mu}{\beta+\gamma\beta\phi_y}\frac{1}{\mu}$, *and* $\Xi := \kappa\left(\psi + \frac{1}{\gamma}\right)$.

To understand the proposition, first note that the aggregate output gap reaction, $\hat{X}_{t^*}$, is a weighted average of aware and unaware consumers' average reactions. The weights are the measures of the type of consumers $(\mu, 1-\mu)$. Then, notice that each type of the consumer's average reaction at period t^* is calculated by a backward recursion. The transition matrices from $t+1$ to t are M_a for the aware type and M_u for the unaware type. Finally, b_a and b_u are the reactions of aggregates (i.e., output gap and inflation) at period $T-1$ which is the "beginning" point of the recursion. Therefore, $M_a b_a$ is the reaction of $T-2$, $(M_a)^2 b_a$ is the reaction of $T-3$, etc.

The proof consists of six steps. First, we build a contemporaneous reaction of the output gap to the nominal interest rate change announcement in the lower space. Second, we derive the reaction of the output gap to the announcement for a general period in the lower space using backward induction. Third, given the lower space results, we move to the upper space where the market clearing prices may differ from the unaware consumers' perceived ones in the lower space. We derive the perceived-actual reaction relations for the unaware consumers. Fourth, we derive the contemporaneous reaction of the output gap among the aware consumers. Fifth, we invoke backward induction and get the result for a general period. Lastly, in the sixth step, we derive the aggregate output gap reaction by taking the weighted average between the reactions for the aware and unaware consumers.

To get a better idea of the Proposition 2, focus on the movement of the IS curve. Because we assumed heterogeneous unawareness among the consumers, it is enough to investigate the consumption block to get intuitions. To this end, assume that the probability of resetting the price is 0 (i.e., the fraction of firms that do not change their price is $\theta = 1$), hence the New Keynesian Philips Curve is fixed at the steady state level. Further, we also simplify the exposition by assuming that the signal is perfect (i.e., $\sigma_\eta^2 = 0$). Then, the proposition can be simplified as follows:

Corollary 1 *The reaction of the IS curve on the announcement $R_{T|t^*}$ with a perfect signal (i.e., $\sigma_\eta^2 = 0$) is as follows:*

$$\hat{X}_{t^*}^{IS}|_{\sigma_\eta^2=0} = \frac{\Lambda_{11}}{\Lambda_{21}}\tilde{R}_{T|t^*} + \mu\left((\beta+(1-\beta)\mu)\right)^{T-t^*-1}(1-\lambda)\left(\frac{\Lambda_{12}}{\Lambda_{22}} - \frac{\Lambda_{11}}{\Lambda_{21}}\right)\tilde{R}_{T|t^*} \tag{19}$$

where $\hat{X}_{t^*}^{IS}$ is the IS curve movement at period t^* after the announcement, and $\lambda = \frac{\sigma_v^2}{\sigma_z^2+\sigma_v^2}$.

For an easier interpretation of the result, it is instructive to separate equation (19) into three parts: the information content of the forward guidance (i.e., the announcement) for each type of consumers, the general equilibrium discounting, and the model misspecification correction:

$$\begin{aligned}\frac{\hat{X}_{t^*}^{IS}|_{\sigma_\eta^2=0}}{\tilde{R}_{T|t^*}} = & \underbrace{(1-\mu)\frac{\Lambda_{11}}{\Lambda_{21}}}_{\substack{\text{information content}\\ \text{for unaware consumers}}} + \underbrace{\mu\left(\lambda\frac{\Lambda_{11}}{\Lambda_{21}} + (1-\lambda)\frac{\Lambda_{12}}{\Lambda_{22}}\right)}_{\substack{\text{information content}\\ \text{for aware consumers}}}\left(\underbrace{(\beta+(1-\beta)\mu)}_{\text{GE discounting}}\right)^{T-t^*-1} \\ & + \underbrace{\mu\left(\frac{\Lambda_{11}}{\Lambda_{21}} - \frac{\Lambda_{11}}{\Lambda_{21}}(\beta+(1-\beta)\mu)^{T-t^*-1}\right)}_{\text{model misspecification correction}}\end{aligned} \tag{20}$$

The 'Information content of Forward Guidance' comes from the fact that the consumers cannot observe the fundamentals $(z_{T|t}, v_{T|t})$ directly and have to infer them from the announcement. The unaware consumers' interpretation of the forward guidance is the central bank's reaction to future productivity, and it is unambiguously positive $(\frac{\Lambda_{11}}{\Lambda_{21}})$. The aware consumers' interpretation depends on the distribution of the shocks and their relative variance $(\lambda,\ 1-\lambda)$.

If the aware consumers believe that the forward guidance is mostly the reaction to the monetary policy shock as in Angeletos and Lian (2018), which corresponds in our model to when λ is close to zero, then the information content part is close to $\frac{\Lambda_{12}}{\Lambda_{21}}$. In the language of Campbell et al. (2012), this case may be interpreted as corresponding to 'Odyssean' forward guidance when consumers think that the announcement is a binding commitment by the central bank. On the contrary, if the aware consumers think that the guidance mostly indicates the central bank's internal knowledge of the future productivity (i.e., λ is close to one), then the coefficient is close to $\frac{\Lambda_{11}}{\Lambda_{21}}$ and the forward guidance might be interpreted as 'Delphic' in the language of Campbell et al. (2012).

In the remainder of the text, we assume that the information content part of the aware consumers is negative so as to emphasize the difference between aware and unaware consumers. In other words, while the unaware consumers account for the announcement only on the TFP shock, the aware consumers account for the monetary policy shock more heavily than the TPF shock. The following assumption on primitives guarantees that the information content part of aware consumers is negative.

Assumption 1

$$\frac{1}{\phi_y} > 0 > \frac{1}{\phi_y}\lambda + \left(\frac{-\gamma\beta + \frac{(1-\beta)(1-\mu)}{\phi}}{1-(1-\beta)\mu}\right)(1-\lambda)$$

The general equilibrium discounting originally comes from two sources: One is the idiosyncratic noise of the signal and higher-order uncertainty as in Angeletos and Lian (2018). The parameters λ_z and λ_v in Proposition 2 are the higher-order expectation related discount factors. In equation (20), idiosyncratic noise of the signal is shut off because of the perfect signal assumption. Instead, we obtain general equilibrium discounting from heterogeneous awareness, i.e. $\mu \leq 1$. The measure of aware consumers μ functions as an unawareness-driven discounting factor. *The idea is that an aware consumer $i \in I_a$ can correctly anticipate that only a fraction of aware consumers perceive the existence of the monetary policy shock, hence the general equilibrium effect in the future is diminished. This is our novel resolution of the 'forward guidance puzzle'.* Recall that the puzzle stems from the IS reaction being independent of the time horizon (under complete information). When awareness is homogeneous, the reaction in the corollary is $\frac{\Lambda_{11}}{\Lambda_{21}}$ or $\frac{\Lambda_{11}}{\Lambda_{21}}\lambda + \frac{\Lambda_{12}}{\Lambda_{22}}(1-\lambda)$ for all being unaware or all being aware, respectively. In these cases, the reaction does not change when the horizon of the guidance $T - t^*$ differs, demonstrating that homogeneous unawareness itself does not resolve the forward guidance puzzle. With heterogeneous awareness, on the other hand, the reaction diminishes as the horizon increases. In the extreme, when the horizon is very long, the effect of forward guidance on the output gap disappears. That is, the heterogeneity of awareness is crucial to resolve the forward guidance puzzle; homogeneous awareness or homogeneous unawareness do not resolve it.

Finally, the model misspecification correction, the last part of equation (20) comes from the aware consumers' actual market clearing. Note that unaware consumers disregard GE discounting if the signal is perfect. Aware consumers, on the other hand, understand that the effect of the future event is discounted with $(\beta + (1-\beta)\mu)^{T-t^*-1}$. The model misspecification correction in equation (20) is the difference between what the unaware consumers actually do, i.e., $\frac{\Lambda_{11}}{\Lambda_{21}}$, and what they should do, namely $\frac{\Lambda_{11}}{\Lambda_{21}}(\beta + (1-\beta)\mu)^{T-t^*-1}$.

Proposition 3 (Comparative Statics) *The output gap reaction to the interest rate (cut) announcement in the heterogeneous awareness model is larger than in the homogeneous* unawareness *model, and smaller than in the homogeneous* awareness *model. Increasing the horizon of the forward guidance, $T - t^*$, decreases the reaction of the output gap. Reducing the share of aware consumers, μ, also decreases the reaction of the output gap.*

To see this, consider first the case when all consumers are unaware (i.e., $\mu = 0$) vis-a-vis heterogeneous awareness (i.e., $\mu \in (0,1)$). Like for the corollary, we continue to assume that information is perfect, $\sigma_\eta^2 = 0$. From the corollary we obtain

$$-\frac{\hat{X}_{t^*}^{IS} - \hat{X}_{t^*}^{IS}|_{\mu=0}}{\tilde{R}_{T|t^*}} = -\,\mu(1-\lambda)\left(\frac{\Lambda_{12}}{\Lambda_{22}} - \frac{\Lambda_{11}}{\Lambda_{21}}\right)((\beta + (1-\beta)\mu))^{T-t^*-1}$$

For any given horizon of the forward guidance $(T - t^*)$, the above difference is always positive for any μ with the Assumption 1. That is, compared to the homogeneous unawareness case, heterogeneous awareness lowers the reaction of the output gap.

Now consider the second case when all consumers are aware (i.e., $\mu = 1$) vis-a-vis heterogeneous awareness (i.e., $\mu \in (0,1)$). From the corollary, we obtain:

$$
\begin{aligned}
&-\frac{\hat{X}_{t^*}^{IS} - \hat{X}_{t^*}^{IS}|_{\mu=1}}{\tilde{R}_{T|t^*}} \\
&= -\mu\left((\beta + (1-\beta)\mu)\right)^{T-t-1}(1-\lambda)\left(\frac{\Lambda_{12}}{\Lambda_{22}} - \frac{\Lambda_{11}}{\Lambda_{21}}\right) - (1-\lambda)\left(\frac{\Lambda_{12}}{\Lambda_{22}} - \left.\frac{\Lambda_{11}}{\Lambda_{21}}\right|_{\mu\to 1}\right) \\
&= -\left(\mu\left((\beta + (1-\beta)\mu)\right)^{T-t-1} - 1\right)(1-\lambda)\left(\frac{\Lambda_{12}}{\Lambda_{22}} - \frac{\Lambda_{11}}{\Lambda_{21}}\right) - (1-\lambda)\left(\frac{\Lambda_{11}}{\Lambda_{21}} - \left.\frac{\Lambda_{11}}{\Lambda_{21}}\right|_{\mu\to 1}\right)
\end{aligned}
$$

where $\left.\frac{\Lambda_{12}}{\Lambda_{22}}\right|_{\mu\to 1}$ is the solution of the New Keyesian Model when all consumers are aware of the monetary policy shock. The sign of the difference is also strictly negative for any $\mu < 1$. The reaction in the heterogeneous awareness case is weaker than in the homogeneous awareness case. That is, compared to the homogeneous awareness case, heterogeneous awareness decreases the reaction of the output gap. To sum up, the reaction under heterogeneous awareness is between the reaction under homogeneous awareness and homogeneous unawareness.

Finally, we check how the two comparative statics change when we increase the horizon of the forward guidance. Intuitively, increasing the horizon of forward guidance diminishes the general equilibrium effect. Similarly, when more consumers are unaware, less consumers take into account the full general equilibrium effect. That's why both increasing the horizon or increasing the fraction of unaware consumers decreases the reaction of the output gap to shock(s). More formally, from equation (19), we can easily confirm that the current output reaction decreases as the horizon $T - t^*$ increases because of the general equilibrium discounting factor. It means that the difference to the homogeneous unawareness case becomes smaller, and the difference to the homogeneous awareness case tends to be larger. We borrow intuition from Angeletos and Lian (2018) for this observation. Increasing the horizon of forward guidance is similar to increasing the order of average expectation because as the horizon gets longer, we get more backward recursions. More backward recursion implies multiple iterations of expectation on an aggregate action, and the aware type consumer $i \in I_a$ expects that fewer consumers can understand the monetary policy shock. The result, therefore, is similar to decreasing the share of aware consumers. We illustrate these facts in Figure 2.

4 Raising Awareness

So far, we treated the measure of unaware consumers, $1 - \mu$, as being exogenously given. However, when the central bank communicates with the consumers, it could raise awareness of the monetary policy shock and thus change the effect of its monetary policy. Note that changing awareness is just one-directional. The central bank can raise awareness but cannot make them unaware of things that they are already aware of. While raising awareness maybe in interesting to study in a variety of macroeconomic models, let us consider it in our model.

Remember that the Taylor rule, $\tilde{R}_t = \phi_y \hat{X}_t + v_t$, is the central bank's reaction function. Once a shock realizes (i.e., z_t or v_t), then the bank sets the nominal interest rate accordingly. The early realization of the shock $z_{T|t^*}$ or $v_{T|t^*}$ also changes the future nominal interest rate $\tilde{R}_T$. Recall that forward guidance is about the central bank's announcement on the planned change of the nominal interest rate. Why does the bank want to *announce* the plan rather

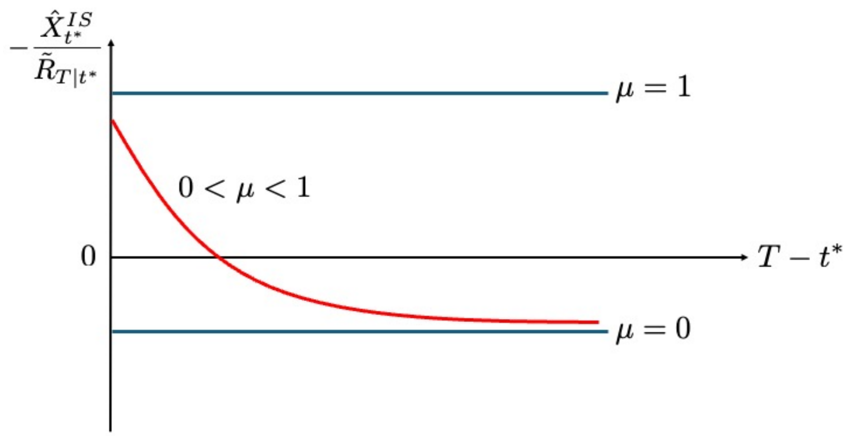

Figure 2: Output gap reaction to the announcement

than just implement it in the future? In an economy that is close to the zero lower bound, the monetary policy has limited room for further action even if the output gap is negative. Because of this, the central bank may want to announce the future policy so that it can boost the current economy. In what follows, we consider such a case. To focus on the effect of awareness and simplify the transition matrix, we assume as in Corollary 1 that signals are perfect and inflation is fixed at the steady state, thus eliminating asymmetric information.

When the central bank announces its future nominal interest rate cut, $\Delta \tilde{R}_{T|t^*} < 0$, it intends to boost the economy with an expansionary monetary policy at the current period, t^*. For that to be possible, the 'Information Content of Forward Guidance' in the equation (19) should be negative,[11] which is implied for aware consumers only by Assumption 1. In this sense, the assumption implies that the aware consumers' expected contemporaneous reaction is aligned with the central bank's intention. However, because of the presence of unaware consumers, the reaction in temporary equilibrium is biased toward the TFP shock, z_t, and the overall effect on the current economy may not be aligned with the central bank's intention. For example, if unawareness is widespread (i.e., $\mu \to 0$), the reaction in the temporary equilibrium is unambiguously negative:

$$\Delta \hat{X}^{IS}_{t^*} = \frac{\Lambda_{11}}{\Lambda_{21}} \Delta \tilde{R}_{T|t^*} < 0$$

In such a case, it is better not to make any announcement about the future nominal interest rate change. When there is no announcement, marginally raising awareness by marginally increasing μ has no effect on the output gap. The central bank has the incentive to make an announcement only if the measure of aware consumers $\mu > \tilde{\mu}$ is above a threshold $\tilde{\mu}$ that satisfies $\frac{\Delta \hat{X}^{IS}_{t^*}}{\Delta \tilde{R}_{T|t^*}} = 0$. It is in this case that raising awareness can amplify forward guidance. Raising awareness allows more people to have information content that is consistent with the central bank's intention. At the same time, as the central bank increases awareness, the general equilibrium discounting (see equilibrium (20) becomes smaller hence the positive reaction on the current output becomes larger. We summarize above observations as follows:

Proposition 4 *If Assumption 1 is satisfied, then the aware consumers' contemporaneous re-*

[11] I.e., $\frac{\Lambda_{11}}{\Lambda_{21}} \lambda + \frac{\Lambda_{12}}{\Lambda_{22}} (1 - \lambda) < 0$

action to forward guidance is in line with the central bank's intention. Raising awareness can assist the effectiveness of the forward guidance if $\mu > \tilde{\mu}$, where $\tilde{\mu}$ satisfies $\frac{\Delta \hat{X}^{IS}_{t^*}}{\Delta \tilde{R}_{T|t^*}} = 0$.

Alternatively, consider the case where Assumption 1 is violated and the information content of aware consumers is also positive. In this case, the current output gap reaction in the temporary equilibrium is always negative regardless of μ. This is because consumers, who don't observe the shocks directly, interpret the rate cut as a central bank's response to a negative $z_{T|t^*}$. Therefore, the contemporaneous reaction is at odds with the central bank's intention. In this case, there's no room for forward guidance, hence the bank will not announce the future nominal interest rate change, and will keep the 'early realization' as internal knowledge.

We just observed that when Assumption 1 is violated, the central bank does not want to announce. Suppose now they are required (e.g. by law) to announce nevertheless. In such a case, can raising awareness mitigate the negative effect of the announcement? Increasing awareness about the monetary policy shock (v_t) will balance the interpretation of the forward guidance between $z_{T|t^*}$ and $v_{T|t^*}$ because the aware consumers account for both shocks. Further, as more people become aware of the monetary policy shock, the Taylor rule relies more on the aware type output. At the same time, raising awareness weakens the general equilibrium discounting hence it increases aware consumers' reaction. To see this, differentiate equation (19) with respect to μ:

$$\begin{aligned}\frac{\partial \hat{X}^{IS}_{t^*}}{\partial \mu} =&(1-\lambda)\left(\frac{\Lambda_{12}}{\Lambda_{22}} - \frac{\Lambda_{11}}{\Lambda_{21}}\right)(\beta + (1-\beta)\mu)^{T-t^*-1}\tilde{R}_{T|t^*} \\ &- \mu(1-\lambda)(1-\beta)\frac{\beta(1+\gamma\phi_y)}{\phi_y(1-(1-\beta)\mu)^2}(\beta + (1-\beta)\mu)^{T-t^*-1}\tilde{R}_{T|t^*} \\ &+ \mu(1-\lambda)(1-\beta)(T-t^*-1)\left(\frac{\Lambda_{12}}{\Lambda_{22}} - \frac{\Lambda_{11}}{\Lambda_{21}}\right)(\beta + (1-\beta)\mu)^{T-t^*-2}\tilde{R}_{T|t^*} < 0\end{aligned}$$

The first line of the r.h.s. of the equation represents balancing the information content of forward guidance. As the central bank increases μ, more people interpret the announcement as a combination of two shocks and fewer people interpret it as driven only by the TFP shock. The second term of the r.h.s. of the equation shows the change of the solution in the contemporaneous case. Recall that $\frac{\Lambda_{12}}{\Lambda_{22}}$ is a function of μ. This is because the monetary policy is a function of the total output gap $\hat{X}_t$ which is again a weighted average of the output gap for aware types and the output gap for unaware types. As the bank increases the awareness, the Taylor rule itself relies more on the output gap of aware types, hence the (contemporaneous) solution of the model changes. Finally, the third term of the r.h.s. of the equation comes from a weakening of the general equilibrium discounting. The direction of the derivative is negative regardless of parameters, which implies that the central bank can mitigate the negative effect of announcement by increasing the awareness of the monetary policy shock.

5 Self-Confirming Equilibrium

The agents' reaction functions are purely forward-looking, hence beliefs about the future variables in addition to observe current prices fix the current equilibrium allocation. However, one important question remains: Why do consumers who are unaware of the second shock do not realize that their model is in some sense misspecified? Upon announcement by the central bank about future interest rates, agents form expectations about future and *current* market clearing prices. However, observed current market clearing prices may differ from expected current market clearing prices. At this point, shouldn't unaware consumers realize that their model is missing something? In this section, we define an equilibrium concept that on top of temporary equilibrium requires that behavior is consistent with beliefs and beliefs are consistent with observations. In particular, we allow unaware consumers to change their model in order to make it consistent with observed current aggregates when the aggregates reveal that they are unaware of some shock.

Definition 2 (Self-Confirming Equilibrium) *The sequence of aggregates* $(C_t, Y_t, N_t^d, N_t^\ell)$ *and price vectors* (d_t, R_t, w_t, π_t) *constitutes a self-confirming equilibrium if*

(i.) it is a temporary equilibrium, and

(ii.) it is common belief that any unaware consumer $i \in I_u$ *chooses an inference rule* $\mathbb{E}^{sc}_{u,t}\left[z_{T|t^*} \mid \omega_{i,T|t^*}\right]$ *that in every period* t *is consistent with observed current aggregates, i.e.,*

$$C_t^u = Y_t \tag{21}$$

Equation (21) implies that market clearing perceived by unaware consumers also clears the actual market (i.e., $\hat{Y}_t^u = \hat{Y}_t = \hat{Y}_t^a$) so that the unaware consumers can rationalize their observations with their self-confirming inference rule *given* their awareness level.

In the following proposition, we show that there exists a self-confirming equilibrium with a self-confirming inference rule that in some sense is a minimal departure from Bayesian inference because it is a linear transformation of it. Moreover, it allows unaware consumers to perceive a Taylor rule that adds an additional factor similar to the monetary policy shock even though they remain unaware of the monetary policy shock. It is as if they are aware that they are unaware of some shock even though they do not know what it is.

Proposition 5 *There exists a self-confirming inference rule of the unaware consumers* $i \in I_u$,

$$\mathbb{E}^{sc}_{u,t}\left[z_{T|t^*} \mid \omega_{i,T|t^*}\right] = \lambda_z^u \frac{\omega_{i,T|t^*}}{\xi_z} + \delta_{i,t}$$

and an associated perceived Taylor rule,

$$R_t = \phi_x \hat{X}_t + e_t$$

such that the economy is in self-confirming equilibrium. The individual modification, $\delta_{i,t}$*, and the aggregate constant on the Taylor rule,* $\overline{\mathbb{E}}_{I,t}\left[e_t\right]$ *are given by*

$$\begin{aligned}\delta_{i,t} =& \frac{1}{\phi_y}\left(\frac{\frac{1}{\gamma}+\psi}{1+\psi}\right)\left(\lambda_z^u - \frac{\begin{pmatrix}1 & 0\end{pmatrix}(M_a)^{T-t-1}\begin{pmatrix}1+\gamma\Xi\mu \\ \beta\Xi\end{pmatrix}\left(\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z + \left.\frac{\Lambda_{12}}{\Lambda_{22}}\right|_{\mu\to 1}\lambda_v\right)}{\begin{pmatrix}1 & 0\end{pmatrix}(M_u)^{T-t-1}\begin{pmatrix}(1+\gamma\Xi) \\ \beta\Xi\end{pmatrix}\frac{\Lambda_{11}}{\Lambda_{21}}}\right)\tilde{R}_{T|t^*} \\ &+ \frac{\lambda_z^u}{\xi_z}\left(\tilde{R}_{T|t^*} - \omega_{i,T|t^*}\right)\end{aligned}$$

$$\overline{\mathbb{E}}_{I,t}\left[e_T\right] = \left(\lambda_z^u - \frac{\begin{pmatrix}1 & 0\end{pmatrix}(M_a)^{T-t-1}\begin{pmatrix}1+\gamma\Xi\mu \\ \beta\Xi\end{pmatrix}\left(\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z + \left.\frac{\Lambda_{12}}{\Lambda_{22}}\right|_{\mu\to 1}\lambda_v\right)}{\begin{pmatrix}1 & 0\end{pmatrix}(M_u)^{T-t-1}\begin{pmatrix}(1+\gamma\Xi) \\ \beta\Xi\end{pmatrix}\frac{\Lambda_{11}}{\Lambda_{21}}} \right)\tilde{R}_{T|t^*}$$

where λ_z, λ_v, Λ_{11}, Λ_{12}, Λ_{21}, and Λ_{22} are defined in Proposition 2, and

$$\tilde{M}_a := \begin{pmatrix}\beta + (1-\beta+\gamma\Xi)(\lambda_z+\lambda_v) & \gamma\beta \\ \Xi & \beta\end{pmatrix}.$$

Two comments are in order about Proposition 5. First, the unaware consumers' modified inference rule and perceived Taylor rule depend on the time of the announcement. That is, $\overline{\mathbb{E}}^{sc}_{I,t^*}\left[e_T\right]$ and δ_{i,t^*} change when T and t^* change. For example, let $R_{T|t^*}$ be the announcement and $(\delta_{i,t^*}, \overline{\mathbb{E}}_{I,t^*}\left[e_T\right])$ are the associated modifications. The unaware consumer has to adjust her belief about the Taylor rule again in the next period in order to be consistent with market clearing ($\hat{C}^u_{t^*+1} = \hat{Y}_{t^*+1}$). More generally, in the self-confirming equilibrium, the unaware consumers update their beliefs continuously between periods t^* and T such that their modified parameters $(\delta_{i,t}, \overline{\mathbb{E}}_{I,t}\left[e_T\right])$ satisfy the condition in the proposition for every $t \in \{t^*, ..., T\}$. That is, unaware consumers continue to learn about the parameters between announcement of an interest change and the period of interest rate change. It is as if unaware consumers discover that they are unaware of something in period t^*, change their inference rule such as to add a place holder or dummy for what they could be unaware of, and then continuously learn about the factor they are unaware of. This is in contrast to aware consumers who upon announcement fully anticipate its effect including how unaware consumers will learn about it. The aware consumers' response at period t^* foresees future market clearing. Aware consumers understand the unaware consumers' problem. The aware consumer can put herself in the unaware consumer's shoes and anticipates the lower space adjustments $(\delta_{i,t}, \overline{\mathbb{E}}_{I,t}\left[e_T\right])$ for the current and every future period $t \in \{t^*, ..., T\}$. Then, when she derives her best response in the current period, she considers all future market clearing based on the $(\delta_{i,t}, \overline{\mathbb{E}}_{I,t}\left[e_T\right])_{t\in\{t^*,...,T\}}$.

Second, the dummy (δ_{i,t^*}) in the modified inference rule has a tight connection to the perceived Taylor rule. The average expectation $\overline{\mathbb{E}}^{sc}_{I,t^*}\left[e_T\right]$ is an aggregate of linear transformations of δ_{i,t^*}. There is a one-to-one relationship between δ_{i,t^*} and the individual e_T. Further, $\overline{\mathbb{E}}^{sc}_{I,t^*}\left[e_T\right]$ is a deterministic constant in the Taylor rule, which we can interpret as the 'natural interest rate'. That is, the unaware consumers change their belief about the natural interest rate of the economy in order to make sense of the aggregate unless they become aware of the other shock.

Above discussion makes clear that at the agents' level, self-confirming equilibrium is different from rational expectations equilibrium. Yet, at the aggregate level self-confirming equilibrium recovers the aggregate outcome of rational expectations equilibrium, i.e., equilibrium under homogeneous full awareness. When we fix inflation, previous proposition implies the following observation for the current aggregate reaction of the output gap:

Corollary 2 *In the self-confirming equilibrium, the aggregate IS reaction to the announcement recovers aggregate IS reaction under homogeneous full awareness:*

$$\frac{\hat{X}^{IS}_{t^*}}{\tilde{R}_{T|t^*}} = \left(\lambda_z \frac{\Lambda_{11}}{\Lambda_{21}} + \lambda_v \left.\frac{\Lambda_{12}}{\Lambda_{22}}\right|_{\mu\to 1}\right)\left(\beta + (1-\beta)(\lambda_z+\lambda_v)\right)^{T-t^*-1}$$

In our self-confirming equilibrium, the output gap corresponds to rational expectation equilibrium, rendering the effect of forward guidance identical to the predictions of the standard New Keynesian model under complete information or those of Angeletos and Lian (2018)'s under incomplete information.

6 Conclusion

What is the impact of forward guidance across different models? In a standard New Keynesian model, the IS movement upon the announcement of future nominal interest rate changes (i.e., forward guidance) remains independent of the timing of the rate change. Angeletos and Lian (2018) showed that the effect of forward guidance decreases w.r.t. the timing of the interest rate change if the signal is noisy. With heterogeneous awareness, we show that the effect diminishes as the announcement is about a distant future. In contrast to Angeletos and Lian (2018), however, the central bank can influence awareness of the shock, providing an additional policy lever. Under certain conditions, the central bank can enhance the effectiveness of forward guidance by increasing public awareness of the shock. In practice, raising awareness of shocks via central bank announcements may also make these announcements more complex and difficult to decipher. Greater difficulty in deciphering central bank announcements increases heterogeneity of beliefs and diminishes the effect of forward guidance. We have therefore two competing predictions for central bank announcements accompanying forward guidance. Raising awareness of shocks should enhance forward guidance while the associated increased complexity of the announcements should hamper forward guidance. Ultimately, it is an empirical question which effect is larger. We believe recent advances in automatic textual analysis with large language models (LLMs) should provide a feasible avenue for carrying out such study. LLMs could list topics and events mentioned in central bank announcements, press releases and press conferences, which can then we tracked over time.

The main idea of the paper of agents having heterogeneous awareness of shocks in an otherwise standard New Keynesian model should be useful for other applications beyond forward guidance. That agents have awareness of shocks in their own sector of the economy but lack of awareness of shocks in other sectors of the economy is intuitively compelling and empirically plausible. The current model is the simplest setting with just two shocks and one-side unawareness of one of the shock. Using ideas from Heifetz et al. (2013), it would be possible to consider more than two shocks and two-sided incomparable awareness. For instance, one set of agents is aware of shock 1 but not of shock 2, the others are aware of shock 2 but not of shock 1, and both are aware of shock 3. See for instance Schipper and Zhou (2024) for a rational expectations model with a continuum of informed traders and heterogeneous awareness of random variables making up the fundamental of an asset.

A Proofs

Proof of Proposition 1

Consider the consumer problem. Similar to Angeletos and Lian (2018), the budget constraint can also be log linearized as follows: The budget constraint of consumer i at period t is

$$\frac{1}{1+R_t} s^{\ell}_{i,t} = s^{\ell}_{i,t-1} + W_t n^{\ell}_{i,t} + D_t - P_t c^{\ell}_{i,t}$$

In the next period, the budget constraint is

$$\frac{1}{1+R_{t+1}} s^{\ell}_{i,t+1} = s^{\ell}_{i,t} + W_{t+1} n^{\ell}_{i,t+1} + D_{t+1} - P_{t+1} c^{\ell}_{i,t+1}.$$

Multiplying both sides of the previous equation by $\frac{1}{1+R_t}$ and taking the difference with the period t budget constraint cancels out $\frac{1}{1+R_t} s^{\ell}_{i,t}$ and we obtain:

$$\begin{aligned}
&\frac{1}{(1+R_{t+1})(1+R_t)} s^{\ell}_{i,t+1} \\
&\quad = \quad s^{\ell}_{i,t-1} + W_t n^{\ell}_{i,t} + D_t - P_t c^{\ell}_{i,t} + \frac{1}{1+R_t}\left(W_{t+1} n^{\ell}_{i,t+1} + D_{t+1} - P_{t+1} c^{\ell}_{i,t+1}\right).
\end{aligned}$$

Iterating this process generates

$$\prod_{\tau=0}^{\infty} \frac{1}{1+R_{t+\tau}} s^{\ell}_{i,\infty} = s^{\ell}_{i,t-1} + \sum_{\tau=0}^{\infty}\left(\prod_{k=1}^{\tau} \frac{1}{1+R_{t+k}}\right)\left(W_{t+\tau} n^{\ell}_{i,t+\tau} + D_{t+\tau} - P_{t+\tau} c^{\ell}_{i,t+\tau}\right).$$

Since $s^{\ell}_{i,\infty}$ is bounded, the l.h.s. converges to 0. Hence the budget constraint can be stated as

$$\sum_{\tau=0}^{\infty}\left(\prod_{k=1}^{\tau} \frac{1}{1+R_{t+k}}\right) P_{t+\tau} c^{\ell}_{i,t+\tau} = s^{\ell}_{i,t-1} + \sum_{\tau=0}^{\infty}\left(\prod_{k=1}^{\tau} \frac{1}{1+R_{t+k}}\right)\left(W_{t+\tau} n^{\ell}_{i,t+\tau} + D_{t+\tau}\right).$$

Now, we approximate the above budget constraint around the steady state. First, take the total derivative of the above equation evaluated at the steady state. The left-hand side of it becomes

$$\sum_{\tau=0}^{\infty}\left(\frac{(P_{t+\tau} - P_{ss}) c_{ss}}{(1+R_{ss})^{\tau}} + \frac{P_{ss}\left(c^{\ell}_{t+\tau} - c_{ss}\right)}{(1+R_{ss})^{\tau}} + \sum_{k=1}^{\tau}\left(\frac{P_{ss} c_{ss}}{R_{t+k} - R_{ss}}\right)\right),$$

whereas the right-hand side is

$$s^{\ell}_{i,t-1} + \sum_{\tau=0}^{\infty}\left(\frac{(W_{t+\tau} - W_{ss}) n_{i,ss}}{(1+R_{ss})^{\tau}} + \frac{W_{ss}(n^{\ell}_{i,t+\tau} - n_{i,ss})}{(1+R_{ss})^{\tau}} + \frac{D_{t+\tau} - D_{ss}}{(1+R_{ss})^{\tau}} + \sum_{k=1}^{\tau}\left(\frac{W_{ss} n_{ss} + D_{ss}}{R_{t+k} - R_{ss}}\right)\right).$$

Setting both sides equal, dividing both sides by $P_{ss} c^{\ell}_{ss}$, and using $\frac{1}{1+R_{ss}} = \beta$ yields

$$\sum_{\tau=0}^{\infty} \beta^{\tau} \hat{c}^{\ell}_{i,t+\tau} = \frac{s^{\ell}_{i,t-1}}{P_{ss} c^{\ell}_{ss}} + \sum_{\tau=0}^{\infty} \beta^{\tau}\left(\underbrace{\frac{W_{ss} n_{i,ss}}{P_{ss} c_{ss}}}_{:=L_{ss}} (\hat{w}_{t+\tau} + \hat{n}^{\ell}_{i,t+\tau}) + \underbrace{\frac{D_{ss}}{P_{ss} c_{ss}}}_{=1-L_{ss}} \hat{d}_{i,t+\tau}\right) \tag{22}$$

where L denotes the labor share of income in the steady state, $w_t := \frac{W_t}{P_t}$, $d_t := \frac{D_t}{P_t}$, and the hat variables are the log deviations from their steady state as before.

Recall the two optimality conditions for the consumer (i.e., equations (8) and (9)),

$$1 \approx 1 + \ln\beta + \mathbb{E}_{\ell,t}\left[-\pi_{t+1} - \frac{1}{\gamma}(\ln c^{\ell}_{i,t+1} - \ln c^{\ell}_{i,t}) \middle| \omega_{i,T|t^*}\right] + R_t$$

$$\psi \ln n^{\ell}_{i,t} = \ln w_t - \frac{1}{\gamma} \ln c^{\ell}_{i,t}$$

Moving $\ln c^\ell_{i,t}$ to the left-hand side, taking the difference from its steady state, and defining $\tilde{R}_t := R_t + \ln\beta$ gives the following equations for each $\ell \in \{a, u\}$:

$$\hat{c}^\ell_{i,t} = -\gamma\left(\tilde{R}_t - \mathbb{E}_{\ell,t}[\pi_{t+1} \mid \omega_{i,T|t^*}]\right) + \mathbb{E}_{\ell,t}[\hat{c}^\ell_{i,t+1} \mid \omega_{i,T|t^*}] \tag{23}$$

$$\hat{c}^\ell_{i,t} = \gamma(\hat{w}_t - \psi\hat{n}^\ell_{i,t}) \tag{24}$$

Using equations (23) and (24) as well as the above log linearized budget constraint given by equation (22), we can rewrite the consumer block as a dynamic beauty contest,

$$\begin{aligned}\hat{c}^\ell_{i,t} = &\left(\frac{(1-\beta)\psi\gamma}{\psi\gamma + L_{ss}}\right)\frac{s^\ell_{i,t-1}}{P_{ss}c^\ell_{ss}} - \gamma\sum_{\tau=0}^{\infty}\beta^{\tau+1}\mathbb{E}_{\ell,t}[\tilde{R}_{t+\tau} - \pi_{t+1+\tau} \mid \omega_{i,T|t^*}] \\ &+ (1-\beta)\sum_{\tau=0}^{\infty}\beta^\tau\mathbb{E}_{\ell,t}\left[\underbrace{(1+\psi)L_{ss}\frac{\gamma}{\psi\gamma+L_{ss}}\hat{w}_{t+\tau} + \psi(1-L_{ss})\frac{\gamma}{\psi\gamma+L_{ss}}\hat{d}_{t+\tau}}_{:=\hat{m}_{i,t+\tau}}\middle|\omega_{i,T|t^*}\right]\end{aligned} \tag{25}$$

where $\hat{m}_{i,t+\tau}$ is consumer i's income deviation at period $t+\tau$. To see this, first replace $n^\ell_{i,t+\tau}$ for $s = 0, 1, \ldots$ in equation (22) using equation (24).

$$\sum_{\tau=0}^{\infty}\beta^\tau\hat{c}^\ell_{i,t+\tau} = \frac{s^\ell_{i,t-1}}{P_{ss}c^\ell_{ss}} + \sum_{\tau=0}^{\infty}\beta^\tau\left(L_{ss}\left(\hat{w}_{t+\tau} + \frac{1}{\psi}\hat{w}_{t+\tau} - \frac{1}{\psi\gamma}\hat{c}^\ell_{i,t+\tau}\right) + (1-L_{ss})\hat{d}_{i,t+\tau}\right)$$

Moving $\hat{c}^\ell_{i,t+\tau}$ to the left side,

$$\sum_{\tau=0}^{\infty}\beta^\tau\left(1 + \frac{L_{ss}}{\psi\gamma}\right)\hat{c}^\ell_{i,t+\tau} = \frac{s^\ell_{i,t-1}}{P_{ss}c^\ell_{ss}} + \sum_{\tau=0}^{\infty}\beta^\tau\left(L_{ss}\left(\frac{1+\psi}{\psi}\right)\hat{w}_{t+\tau} + (1-L_{ss})\hat{d}_{i,t+\tau}\right)$$

or equivalently,

$$\sum_{\tau=0}^{\infty}\beta^\tau\hat{c}^\ell_{i,t+\tau} = \frac{\psi\gamma}{\psi\gamma+L_{ss}}\frac{s^\ell_{i,t-1}}{P_{ss}c^\ell_{ss}} + \sum_{\tau=0}^{\infty}\beta^\tau\left(\frac{(1+\psi)\gamma L_{ss}}{\psi\gamma+L_{ss}}\hat{w}_{t+\tau} + \frac{\psi\gamma(1-L_{ss})}{\psi\gamma+L_{ss}}\hat{d}_{i,t+\tau}\right). \tag{26}$$

Further, from equation (23), we obtain

$$\sum_{\tau=1}^{\infty}\beta^\tau\hat{c}^\ell_{i,t} = -\gamma\sum_{\tau=1}^{\infty}\beta^\tau\left(\tilde{R}_{t+\tau} - \mathbb{E}_{\ell,t}[\pi_{t+1+\tau} \mid \omega_{i,T|t^*}]\right) + \sum_{\tau=1}^{\infty}\beta^\tau\mathbb{E}_{\ell,t}[\hat{c}^\ell_{i,t+1+\tau} \mid \omega_{i,T|t^*}] \tag{27}$$

by multiplying with β^t and summing from t to ∞. Equation (25) follows now by multiplying equation (26) with $1-\beta$ and adding equation (27).

As a next step, we show the income-production identity. Intermediate goods producers' surplus is distributed as a dividend,

$$d_{j,t} = \left(\frac{P_{j,t}}{P_t} - \frac{W_t}{P_t}\frac{1}{\exp(z_t)}\right)y_{j,t}$$

where $d_{j,t}$ is the real dividend from firm j. Recall that $y_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} Y_t$ is the factor demand. Also, recall the price aggregation, $P_t = \left(\int_0^1 (P_{j,t})^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}$. Integrating the dividend over intermediate goods producers yields the aggregate (real) dividend,

$$\begin{aligned} d_t &= \int_{j\in J} d_{j,t} dj = \left(\frac{\int_j \left(P_{j,t}^{1-\varepsilon}\right) dj}{(P_t)^{1-\varepsilon}}\right) Y_t - w_t \int_{j\in J} n_{j,t} dj \\ &= Y_t - w_t N_t \end{aligned}$$

We log-linearize this equation as follows: First, taking total derivatives evaluated at the steady state,

$$\Delta d_t = \Delta Y_t - \Delta w_t N_{ss} - w_{ss} \Delta N_t$$

Dividing both sides by $d_{ss} = Y_{ss} - w_{ss} N_{ss}$,

$$\hat{d}_t = \frac{Y_{ss}}{Y_{ss} - w_{ss} N_{ss}} \hat{Y}_t - \frac{w_{ss} N_{ss}}{Y_{ss} - w_{ss} N_{ss}} (\hat{w}_t + \hat{N}_t) \tag{28}$$

In this equation, $\frac{Y_{ss}}{Y_{ss}-w_{ss}N_{ss}} = \frac{1}{1-L_{ss}}$ is the inverse of the dividend share of income in the steady state. Thus, $\frac{w_{ss}N_{ss}}{Y_{ss}-w_{ss}N_{ss}} = \frac{L_{ss}}{1-L_{ss}}$.

Next, we aggregate the individual labor supply. Recall the labor supply condition in the consumers' problem (equation (9)).

$$\psi \ln n_{i,t}^{\ell} = \ln w_t - \frac{1}{\gamma} \ln c_{i,t}^{\ell}$$

By integrating both sides, we get

$$\begin{aligned} \psi \int_{i\in I} \ln n_{i,t}^{\ell} di &= \int_{i\in I} \ln w_t di - \frac{1}{\gamma} \int_{i\in I} \ln c_{i,t}^{\ell} di \\ \psi \ln N_t^u &= \ln w_t - \frac{1}{\gamma} \ln Y_t^u \\ \psi \hat{N}_t^u &= \hat{w}_t - \frac{1}{\gamma} \hat{Y}_t^u \end{aligned} \tag{29}$$

at the lower space, and similarly, we get

$$\psi \hat{N}_t^a = \hat{w}_t - \frac{1}{\gamma} \hat{Y}_t^a \tag{30}$$

at the upper space.[12] Then, plugging equation (28) into the definition of $\hat{m}_{i,t+\tau}$ of equation (25)

[12]We interchange the natural log and integral using the approximation result.

$$\int_i \ln x_i di \approx \int_i 1 + x_i di = 1 + \int_i x_i di = 1 + X_i \approx \ln X_i = \ln \int x_i di$$

and imposing market clearing gives,

$$
\begin{aligned}
&\mathbb{E}_{\ell,t}\left[\hat{m}_{i,t+\tau} \mid \omega_{i,T|t^*}\right] \\
&:= \mathbb{E}_{\ell,t}\left[\frac{(\psi+1)\gamma L_{ss}}{\psi\gamma+L_{ss}}\hat{w}_{t+\tau} + \frac{\psi\gamma(1-L_{ss})}{\psi\gamma+L_{ss}}\hat{d}_{t+\tau}\middle|\omega_{i,T|t^*}\right] \\
&= \mathbb{E}_{\ell,t}\left[\frac{(\psi+1)\gamma L_{ss}}{\psi\gamma+L_{ss}}\hat{w}_{t+\tau} + \frac{\psi\gamma(1-L_{ss})}{\psi\gamma+L_{ss}}\left(\frac{1}{1-L_{ss}}\hat{Y}^{\ell}_{t+\tau} - \frac{L_{ss}}{1-L_{ss}}(\hat{w}_{t+\tau}+\hat{N}^{\ell}_{t+\tau})\right)\middle|\omega_{i,T|t^*}\right] \\
&= \mathbb{E}_{\ell,t}\left[\frac{\gamma L_{ss}}{\psi\gamma+L_{ss}}\hat{w}_{t+\tau} - \frac{\psi\gamma L_{ss}}{\psi\gamma+L_{ss}}\hat{N}^{\ell}_{t+\tau} + \frac{\psi\gamma}{\psi\gamma+L_{ss}}\hat{Y}^{\ell}_{t+\tau}\middle|\omega_{i,T|t^*}\right] \\
&= \mathbb{E}_{\ell,t}\left[\frac{\gamma L_{ss}}{\psi\gamma+L_{ss}}(\hat{w}_{t+\tau} - \psi\hat{N}^{\ell}_{t+\tau}) + \frac{\psi\gamma}{\psi\gamma+L_{ss}}\hat{Y}^{\ell}_{t+\tau}\middle|\omega_{i,T|t^*}\right]
\end{aligned}
$$

Then,

$$\mathbb{E}_{\ell,t}\left[\hat{m}_{i,t+\tau} \mid \omega_{i,T|t^*}\right] = \mathbb{E}_{\ell,t}\left[\frac{L_{ss}}{\psi\gamma+L_{ss}}\hat{Y}^{\ell}_{t+\tau} + \frac{\psi\gamma}{\psi\gamma+L_{ss}}\hat{Y}^{\ell}_{t+\tau}\middle|\omega_{i,T|t^*}\right] = \mathbb{E}_{\ell,t}\left[\hat{Y}^{\ell}_{t+\tau} \mid \omega_{i,T|t^*}\right]$$

because of the aggregate labor supply ($\hat{w}_t - \psi\hat{N}^{\ell}_t = \frac{1}{\gamma}\hat{Y}^{\ell}_t$) in the equation (29) and equation (30).

So far we derived the individual reactions to shocks. Next, we aggregate individual reactions to the aggregate reaction of the economy. We begin by considering the economy in the lowest space from an unaware consumer's point of view. For such a consumer, every consumer is unaware.

First, take the average of the individual beauty contest (equation (25)) In the lower state space. We get,

$$\hat{C}^u_t = -\gamma\sum_{\tau=0}^{\infty}\beta^{\tau+1}\overline{\mathbb{E}}_{I,t}[\tilde{R}_{t+\tau} - \pi_{t+1+\tau}] + (1-\beta)\sum_{\tau=0}^{\infty}\beta^{\tau}\overline{\mathbb{E}}_{I,t}\left[\hat{Y}^u_{t+\tau}\right] \tag{31}$$

where $\overline{\mathbb{E}}_{I,t}[\cdot] := \int_{i\in I}\mathbb{E}_{u,t}[\cdot \mid \omega_{i,T|t^*}]di$ denotes the average expectation of the consumers. As Angeletos and Lian (2018), we use the fact that the aggregate saving $\int_I s^u_{i,t-1}di$ is zero in the aggregation.

Moving to the upper space, the aware type consumer $i \in I_a$, taking the average among the aware type consumers gives

$$\frac{1}{\mu}\int_{i\in I_a}\hat{c}^a_{i,t}di = -\gamma\sum_{\tau=0}^{\infty}\beta^{\tau+1}\overline{\mathbb{E}}_{I_a,t}[\tilde{R}_{t+\tau} - \pi_{t+1+\tau}] + (1-\beta)\sum_{\tau=0}^{\infty}\beta^{\tau}\overline{\mathbb{E}}_{I_a,t}\left[\hat{Y}^a_{t+\tau}\right] \tag{32}$$

where $\overline{\mathbb{E}}_{I_a,t}[\cdot] := \frac{1}{\mu}\int_{i\in I_a}\mathbb{E}_{u,t}[\cdot]di$ is the average expectation of the aware type consumers. Because the aggregate savings in the lower space is zero, the aggregate savings in the upper space also becomes zero ($\int_{I_a} s^a_{i,t-1}di = 0$).

Finally, recall that the aggregate reaction from all consumers is $\hat{C}^a_t$ because the aware type understands the market structure correctly, or equivalently, it is the weighted average of $\hat{C}^u_t$

and $\frac{1}{\mu}\int_{i\in I_a}\hat{c}^a_{i,t}di$ as follows:

$$\begin{aligned}\hat{Y}_t =& \int_{i\in I_a}\hat{c}^a_{i,t}di + (1-\mu)\hat{C}^u_t\\ =& -\gamma\sum_{\tau=0}^{\infty}\beta^{\tau+1}\left(\mu\overline{\mathbb{E}}_{I_a,t}\left[\tilde{R}_{t+\tau}-\pi_{t+1+\tau}\right]+(1-\mu)\overline{\mathbb{E}}_{I_u,t}\left[\tilde{R}_{t+\tau}-\pi_{t+1+\tau}\right]\right)\\ &+(1-\beta)\sum_{\tau=0}^{\infty}\beta^{\tau}\left(\mu\overline{\mathbb{E}}_{I_a,t}\left[\hat{Y}^a_{t+\tau}\right]+(1-\mu)\overline{\mathbb{E}}_{I_u,t}\left[\hat{Y}^u_{t+\tau}\right]\right)\end{aligned}$$

as in the proposition. □

Proof of Proposition 2

Step 1. We begin by considering the contemporaneous effect in the lower space. We first derive the output gap using the augmented IS relation from the Proposition 1. Recall from Proposition 1 that the unaware consumer's perceived IS curve is

$$\hat{Y}^u_t = -\gamma\sum_{\tau=0}^{\infty}\beta^{\tau+1}\overline{\mathbb{E}}_{I,t}\left[r_{t+\tau}+\ln\beta\right]+(1-\beta)\sum_{\tau=0}^{\infty}\beta^{\tau}\overline{\mathbb{E}}_{I,t}\left[\hat{Y}^u_{t+\tau}\right]$$

where $r_t := R_t - \pi_{t+1}$ is the real interest rate, and this relation also holds at the natural level of output,

$$\hat{Y}^n_t = -\gamma\sum_{\tau=0}^{\infty}\beta^{\tau+1}\overline{\mathbb{E}}_{I,t}\left[r^n_{t+\tau}+\ln\beta\right]+(1-\beta)\sum_{\tau=0}^{\infty}\beta^{\tau}\overline{\mathbb{E}}_{I,t}\left[\hat{Y}^n_{t+\tau}\right] \tag{33}$$

By taking the difference between the two equations, we obtain

$$\hat{X}^u_t = -\gamma\sum_{\tau=0}^{\infty}\beta^{\tau+1}\overline{\mathbb{E}}_{I,t}\left[r_{t+\tau}-r^n_{t+\tau}\right]+(1-\beta)\sum_{\tau=0}^{\infty}\beta^{\tau}\overline{\mathbb{E}}_{I,t}\left[\hat{X}^u_{t+\tau}\right] \tag{34}$$

where $\hat{X}^u_t$ is the output gap in the lower space. To get an expression for r^n_t, multiply the next period's counterpart of equation (33) by β. Then we take the difference with equation (33) to obtain

$$\hat{Y}^n_t - \hat{Y}^n_{t+1} = -\gamma\left(r^n_t+\ln\beta\right).$$

Recall that $\hat{Y}^n_t := \ln Y^n_t - \ln Y^n_{ss} = \frac{1+\psi}{\psi+\frac{1}{\gamma}}z_t$. Therefore,

$$\begin{aligned}\frac{1+\psi}{\psi+\frac{1}{\gamma}}\left(z_t-z_{t+1}\right) &= -\gamma\left(r^n_t+\ln\beta\right)\\ r^n_t &= -\ln\beta+\frac{1}{\gamma}\left(\frac{1+\psi}{\psi+\frac{1}{\gamma}}\right)\left(z_{t+1}-z_t\right).\end{aligned}$$

Plugging r^n_t into equation (34),

$$\hat{X}^u_t = -\gamma\sum_{\tau=0}^{\infty}\beta^{\tau+1}\overline{\mathbb{E}}_{I,t}\left[r_{t+\tau}+\ln\beta-\frac{1}{\gamma}\left(\frac{1+\psi}{\psi+\frac{1}{\gamma}}\right)\left(z_{t+1}-z_t\right)\right]+(1-\beta)\sum_{\tau=0}^{\infty}\beta^{\tau}\overline{\mathbb{E}}_{I,t}\left[\hat{X}^u_{t+\tau}\right] \tag{35}$$

where $\hat{X}_t^u$ indicates the output gap perceived by the unaware type consumers.

Now consider the contemporaneous effect in the lower space when an announcement on the nominal interest rate $\tilde{R}_{T|T}$ is made at period T. Given the announcement $\tilde{R}_{T|T}$, the aggregate reaction in the lower space is derived from equation (35) is as follows:

$$\begin{aligned}\hat{X}_T^u &= -\gamma\beta\overline{\mathbb{E}}_{I,T}\left[r_{T|T} - r_{T|T}^n\right] + (1-\beta)\hat{X}_T^u \\ &= -\gamma\beta\overline{\mathbb{E}}_{I,T}\left[\tilde{R}_{T|T} - \pi_{T+1}\right] - \left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\overline{\mathbb{E}}_{I,T}[z_{T|T}] + (1-\beta)\hat{X}_T^u \\ &= -\gamma\overline{\mathbb{E}}_{I,T}\left[\tilde{R}_{T|T}\right] - \left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\overline{\mathbb{E}}_{I,T}[z_{T|T}] \end{aligned} \tag{36}$$

To understand this equation, recall that the shock occurs only at period T and the output gap is zero thereafter.

Note that we drop the forward-looking terms beyond T in the first line by assuming that the economy is initially in the steady state, and especially at the natural output level. In the lower space, the Taylor rule is $\tilde{R}_T = \phi_y\hat{X}_T$ since there is no monetary policy shock. Hence the announcement follows this Taylor rule as well.

$$\begin{aligned}\hat{X}_T^u &= -\gamma\phi_y\overline{\mathbb{E}}_{I,T}\left[\hat{X}_T^u\right] - \left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\overline{\mathbb{E}}_{I,T}[z_{T|T}] \\ &= \frac{1}{1+\gamma\phi_y}\left(-\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\overline{\mathbb{E}}_{I,T}[z_{T|T}]\end{aligned} \tag{37}$$

Therefore, the Taylor rule perceived by unaware types as a function of the TFP shocks is

$$\begin{aligned}\overline{\mathbb{E}}_{I,T}\left[\tilde{R}_{T|T}\right] &= \overline{\mathbb{E}}_{I,T}\left[\phi_y\hat{X}_T^u\right] \\ &= \underbrace{\frac{\phi_y}{1+\gamma\phi_y}\left(-\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)}_{:=\xi_z}\overline{\mathbb{E}}_{I,T}[z_{T|T}]\end{aligned}$$

Recall the estimate of the shock (equation (2)). Aggregate over all consumers (in the lower space), we obtain

$$\begin{aligned}\overline{\mathbb{E}}_{I,T}[z_{T|T}] &= \int_{i\in I}\frac{\lambda_z^u}{\xi_z}\omega_{i,T|T}di \\ &= \int_{i\in I}\frac{1+\gamma\phi_y}{\phi_y}\left(-\frac{\frac{1}{\gamma}+\psi}{1+\psi}\right)\lambda_z^u\omega_{i,T|T}di\end{aligned}$$

Plugging this expression into equation (37) gives

$$\hat{X}_T^u = \frac{1}{\phi_y}\int_{i\in I}\lambda_z^u\omega_{i,T|T}di = \frac{1}{\phi_y}\lambda_z^u\tilde{R}_{T|T} = \frac{1}{\phi_y}\overline{\mathbb{E}}_{I,T}\left[\tilde{R}_{T|T}\right]$$

where the second equation follows from the law of large numbers, $\int_{i\in I_u}\eta_{i,T}di = 0$, and the last equation makes use of the Taylor rule.

To study forward guidance that goes beyond the contemporaneous effect, we now consider that the announcement time t^* differs from the time of the realization of the shock and nominal interest rate change. We show the effect of forward guidance by induction. To this end, we introduce some notation. Denote by Φ_t^u the reaction of the output gap in period t given the signals of unaware consumers in the lower space at t^*. Similarly, we use Ω_t^u for the reaction of inflation at period t to the signal at period t^* perceived by unaware consumers in the lower space.

$$\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}_t^u\right] = \Phi_t^u \ \overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right] \tag{38}$$

$$\overline{\mathbb{E}}_{I,t^*}\left[\pi_t\right] = \kappa\left(\psi+\frac{1}{\gamma}\right)\hat{X}_t^u + \Omega_t^u \ \overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right] \tag{39}$$

Using this notation, the contemporaneous reaction on the output gap we derived earlier can be stated as

$$\Phi_T^u = \frac{1}{\phi_y}. \tag{40}$$

Moreover, from the Phillips curve, equation (18), we observe that the contemporaneous reaction on inflation must be $\Omega_T^u = 0$.

Step 2. To get a reaction at period t, we will use a mathematical induction using period $T-2$ as a base case. As a preliminary work, we derive the $T-2$ output gap and introduce some notation to simplify our exposition. Assume now that the announcement on the nominal interest rate at period T is made one period ahead at $T-1$. The perceived output gap comes from equation (35) as follows:

$$\begin{aligned}\hat{X}_{T-1}^u = &-\gamma\beta^2\overline{\mathbb{E}}_{I,T-1}\left[r_{T|T-1}-r_{T|T-1}^n\right]-\gamma\beta\overline{\mathbb{E}}_{I,T-1}\left[r_{T-1|T-1}-r_{T-1|T-1}^n\right]\\ &+(1-\beta)\hat{X}_{T-1}^u+(1-\beta)\beta\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}_T^u\right]\end{aligned}$$

Moving $(1-\beta)\hat{X}_{T-1}^u$ to the left side and dividing β gives

$$\hat{X}_{T-1}^u = -\gamma\beta\overline{\mathbb{E}}_{I,T-1}\left[R_{T|T-1}-r_{T|T-1}^n\right]-\gamma\overline{\mathbb{E}}_{I,T-1}\left[-\pi_T\right]+(1-\beta)\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}_T^u\right]$$

because $\pi_{T+1}=0$, $\tilde{R}_{T-1}=0$, and $z_{T-1}=0$. Using the Phillips curve, equation (18), we get

$$\begin{aligned}\hat{X}_{T-1}^u &= -\gamma\beta\overline{\mathbb{E}}_{I,T-1}\left[R_{T|T-1}-r_{T|T-1}^n\right]+\gamma\kappa\left(\psi+\frac{1}{\gamma}\right)\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}_T^u\right]+(1-\beta)\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}_T^u\right]\\ &= -\gamma\beta\overline{\mathbb{E}}_{I,T-1}\left[R_{T|T-1}-r_{T|T-1}^n\right]+(1-\beta+\gamma\Xi)\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}_T^u\right] \qquad (41)\end{aligned}$$

where $\Xi := \kappa\left(\psi+\frac{1}{\gamma}\right)$. We now from the analysis of the contemporaneous effect, equation (36), that at period T we have

$$\hat{X}_T^u = -\gamma\overline{\mathbb{E}}_{I,T}\left[R_{T|T}-r_{T|T}^n\right]. \tag{42}$$

Considering now forward guidance at $T-1$ and taking expectations at $T-1$, we obtain

$$\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}_T^u\right] = -\gamma\overline{\mathbb{E}}_{I,T-1}\left[R_{T|T-1}-r_{T|T-1}^n\right].$$

Therefore, we can restate equation (41) using Φ_t^u as

$$\begin{aligned}
\hat{X}_{T-1}^u &= \beta\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}_T^u\right] + (1-\beta+\gamma\Xi)\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}_T^u\right] \\
&= (1+\gamma\Xi)\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}_T^u\right] \\
&= (1+\gamma\Xi)\Phi_T^u\ \overline{\mathbb{E}}_{I,T-1}\left[\tilde{R}_{T|T-1}\right]
\end{aligned} \tag{43}$$

For inflation, recall the Phillips curve, equation (18), which we can write

$$\begin{aligned}
\pi_{T-1} &= \beta\overline{\mathbb{E}}_{I,T-1}\left[\pi_T\right] + \Xi\hat{X}_{T-1}^u \\
&= \beta\overline{\mathbb{E}}_{I,T-1}\left[\beta\pi_{T+1} + \Xi X_T\right] + \Xi\hat{X}_{T-1}^u \\
&= \beta\Xi\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}_T^u\right] + \Xi\hat{X}_{T-1}^u && (44) \\
&= \beta\Xi\Phi_T^u\ \overline{\mathbb{E}}_{I,T-1}\left[\tilde{R}_{T|T-1}\right] + \Xi\hat{X}_{T-1}^u. && (45)
\end{aligned}$$

where the second equation follows from the next period Phillips curve, the third equation follows from the fact that steady inflation at $T+1$ is zero, and the last equation follows from the equation (38). Therefore, the two coefficients are

$$\Phi_{T-1}^u = (1+\gamma\Xi)\Phi_T^u, \quad \Omega_{T-1}^u = \beta\Xi\Phi_T^u$$

respectively.

For periods $t \leq T-2$, we show that the following relation is satisfied:

$$\begin{pmatrix}\Phi_t^u \\ \Omega_t^u\end{pmatrix} = M_u \cdot \begin{pmatrix}\Phi_{t+1}^u \\ \Omega_{t+1}^u\end{pmatrix}$$

where

$$M_u := \begin{pmatrix}\beta + (1-\beta+\gamma\Xi)\lambda_z^u & \gamma \\ \beta\Xi & \beta\end{pmatrix}$$

The proof uses mathematical induction, using the $T-2$ reaction as a base case. That is, we first show that the claim holds at period $T-2$, and then we show that the claim also holds for a general t^* with an assumption that the claim holds for every $\tau \in \{t^*+1, ..., T-2\}$.

From the perceived IS curve (equation (35)),

$$\begin{aligned}
\hat{X}_{T-2}^u = &-\gamma\beta^3\overline{\mathbb{E}}_{I,T-2}\left[r_{T|T-2} - r_{T|T-2}^n\right] - \gamma\beta^2\overline{\mathbb{E}}_{I,T-2}\left[r_{T-1|T-2} - r_{T-1|T-2}^n\right] \\
&- \gamma\beta\overline{\mathbb{E}}_{I,T-2}\left[r_{T-2|t} - r_{T-2|t}^n\right] + (1-\beta)\hat{X}_{T-2}^u \\
&+ (1-\beta)\beta\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}_{T-1}^u\right] + (1-\beta)\beta^2\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}_T^u\right]
\end{aligned}$$

Moving $(1-\beta)\hat{X}_{T-2}^u$ to the l.h.s. and dividing both sides by β yields

$$\begin{aligned}
\hat{X}_{T-2}^u = &-\gamma\beta^2\overline{\mathbb{E}}_{I,T-2}\left[R_{T|T-2} - r_{T|T-2}^n\right] - \gamma\beta\overline{\mathbb{E}}_{I,T-2}\left[-\pi_T\right] - \gamma\overline{\mathbb{E}}_{I,T-2}\left[-\pi_{T-1}\right] \\
&+ (1-\beta)\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}_{T-1}^u\right] + (1-\beta)\beta\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}_T^u\right]
\end{aligned}$$

using $\pi_{T+1} = 0$, $z_{T-1} = z_{T-2} = 0$, and $\tilde{R}_{T-1} = \tilde{R}_{T-2} = 0$. From the equations for inflation in periods T (equation (39)) and $T-1$ (equation (44)), we get

$$\begin{aligned}\hat{X}^u_{T-2} = &-\gamma\beta^2\overline{\mathbb{E}}_{I,T-2}\left[R_{T|T-2} - r^n_{T|T-2}\right] + \gamma\beta\Xi\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^u_T\right] + \gamma\Xi\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^u_{T-1} + \beta\hat{X}^u_T\right] \\ &+ (1-\beta)\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^u_{T-1}\right] + (1-\beta)\beta\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^u_T\right]\end{aligned}$$

Further, using the equation for the output gap in period T, equation (42), we replace the first interest rates in the above equation, and using equation (43), we replace the $T-1$ output gap with the T output gap as follows:

$$\begin{aligned}\hat{X}^u_{T-2} = &\beta^2\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^u_T\right] + \gamma\beta\Xi\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^u_T\right] + \gamma\Xi\overline{\mathbb{E}}_{I,T-2}\left[(1+\gamma\Xi)\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}^u_T\right] + \beta\hat{X}^u_T\right] \qquad (46) \\ &+ (1-\beta)\overline{\mathbb{E}}_{I,T-2}\left[(1+\gamma\Xi)\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}^u_T\right]\right] + (1-\beta)\beta\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^u_T\right] \\ = &\beta^2\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^u_T\right] + \gamma\beta\Xi\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^u_T\right] + \gamma\Xi(1+\gamma\Xi)\overline{\mathbb{E}}^2_{u,T-2}\left[\hat{X}^u_T\right] + \gamma\beta\Xi\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^u_T\right] \\ &+ (1-\beta)(1+\gamma\Xi)\overline{\mathbb{E}}^2_{I,T-2}\left[\hat{X}^u_T\right] + (1-\beta)\beta\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^u_T\right] \qquad (47)\end{aligned}$$

where $\overline{\mathbb{E}}^2_{I,T-2}[\cdot] := \frac{1}{\mu}\int_{i\in I}\mathbb{E}_{I,T-2}\left[\overline{\mathbb{E}}_{I,T-2}[\cdot] \mid \omega_{i,T|T-2}\right]di$ is the average second order expectation. We can see that the claim holds for the first row of M_u since the equation (47) can be rewritten using Φ and Ω as follows:

$$\begin{aligned}\Phi^u_{T-2} &= \left(\beta^2 + \gamma\beta\Xi + \gamma\Xi(1+\gamma\Xi)\lambda^u_z + (1-\beta)(1+\gamma\Xi)\lambda^u_z + (1-\beta)\beta\right)\Phi^u_T + \gamma\beta\Xi\Phi^u_T \\ &= (1+\gamma\Xi)(\beta + (1-\beta+\gamma\Xi)\lambda^u_z)\Phi^u_T + \gamma\beta\Xi\Phi^u_T \\ &= (1+\gamma\Xi)(\beta + (1-\beta+\gamma\Xi)\lambda^u_z)\Phi^u_T + \gamma\Omega^u_{T-1} \\ &= (\beta + (1-\beta+\gamma\Xi)\lambda^u_z)\Phi^u_{T-1} + \gamma\Omega^u_{T-1}\end{aligned}$$

The inflation at period $T-2$ is,

$$\begin{aligned}\pi_{T-2} &= \beta\overline{\mathbb{E}}_{I,T-2}\left[\pi_{T-1}\right] + \Xi\hat{X}^u_{T-2} \\ &= \beta\overline{\mathbb{E}}_{I,T-2}\left[\Xi\hat{X}^u_{T-1} + \Omega_{T-1}R_{T|T-2}\right] + \Xi\hat{X}^u_{T-2} \\ &= \beta\left(\Xi\Phi^u_{T-1} + \Omega_{T-1}\right)\overline{\mathbb{E}}_{I,T-2}\left[R_{T|T-2}\right] + \Xi\hat{X}^u_{T-2}\end{aligned}$$

where the second and the third lines come from the definition of Φ and Ω (equations (38) and (39)). This can be equivalently stated

$$\Omega_{T-2} = \beta\Xi\Phi^u_{T-1} + \beta\Omega_{T-1}$$

which proves the second row of M_u.

Now assume as an induction hypothesis that the claim holds for every $\tau \in \{t^*+1, t^*+2, ..., T-2\}$. We would like to show that the claim also holds for t^*. The period t^* reaction at the lower space is

$$\hat{X}^u_{t^*} = -\gamma\sum_{\tau=0}^{T-t^*}\beta^\tau\overline{\mathbb{E}}_{I,t^*}\left[r_{t^*+\tau|t^*} - r^n_{t^*+\tau|t^*}\right] + (1-\beta)\sum_{\tau=1}^{T-t^*}\beta^{\tau-1}\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^u_{t^*+\tau}\right].$$

From the assumption that the claim holds for every $\tau \in \{t^*+1, t^*+2, ..., T-2\}$, we rewrite above equation in terms of Φ and Ω. First, because $\pi_{T+1} = 0$ and $R_{t^*+\tau|t^*} - r^n_{t^*+\tau|t^*} = 0$ for all $\tau \geq 0$ except $\tau = T - t^*$, the above equation is equivalent to

$$\begin{aligned}
\hat{X}^u_{t^*} =& -\gamma\beta^{T-t^*}\overline{\mathbb{E}}_{I,t^*}\left[R_{T|t^*} - r^n_{T|t^*}\right] + \gamma\sum_{\tau=0}^{T-t^*-1}\beta^\tau\overline{\mathbb{E}}_{I,t^*}\left[\pi_{t^*+\tau+1|t^*}\right] \\
&+ (1-\beta)\sum_{\tau=1}^{T-t^*}\beta^{\tau-1}\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^u_{t^*+\tau}\right].
\end{aligned}$$

Using the induction hypothesis, we replace the inflation and get

$$\begin{aligned}
\hat{X}^u_{t^*} =& -\gamma\beta^{T-t^*}\overline{\mathbb{E}}_{I,t^*}\left[R_{T|t^*} - r^n_{T|t^*}\right] + \gamma\Xi\sum_{\tau=0}^{T-t^*-1}\beta^\tau\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^u_{t^*+\tau+1}\right] \\
&+ \gamma\sum_{\tau=0}^{T-t^*-1}\beta^\tau\Omega_{t^*+\tau+1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right] + (1-\beta)\sum_{\tau=1}^{T-t^*}\beta^{\tau-1}\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^u_{t+\tau}\right].
\end{aligned}$$

Using the result for period T (equation (42)) and collecting $\hat{X}^u_{t+\tau+1}$ gives

$$\begin{aligned}
\hat{X}^u_{t^*} =& \beta^{T-t^*}\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^u_T\right] + \sum_{\tau=0}^{T-t^*-1}\beta^\tau(1-\beta+\gamma\Xi)\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^u_{t^*+\tau+1}\right] \\
&+ \gamma\sum_{\tau=0}^{T-t^*-1}\beta^\tau\Omega_{t^*+\tau+1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right].
\end{aligned}$$

Again, we use the induction hypothesis to replace $\hat{X}^u_{t^*+\tau+1}$,

$$\begin{aligned}
\hat{X}^u_{t^*} =& \beta^{T-t^*}\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^u_T\right] + \sum_{\tau=0}^{T-t^*-1}\beta^\tau(1-\beta+\gamma\Xi)\overline{\mathbb{E}}_{I,t^*}\left[\Phi^u_{t^*+\tau+1}\overline{\mathbb{E}}_{I,t^*+\tau+1}\left[\tilde{R}_{T|t^*}\right]\right] \\
&+ \gamma\sum_{\tau=0}^{T-t^*-1}\beta^\tau\Omega_{t^*+\tau+1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right] \\
=& \beta^{T-t^*}\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^u_T\right] + \sum_{\tau=0}^{T-t^*-1}\beta^\tau(1-\beta+\gamma\Xi)\lambda^u_z\Phi^u_{t^*+\tau+1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right] \\
&+ \gamma\sum_{\tau=0}^{T-t^*-1}\beta^\tau\Omega_{t^*+\tau+1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right]
\end{aligned}$$

Moving the above equation one period forward (t^*+1), multiplying by β, and taking the difference with the equation for period t^*,

$$\Phi^u_{t^*} = \beta\Phi^u_{t^*+1} + (1-\beta+\gamma\Xi)\lambda^u_z\Phi^u_{t^*+1} + \gamma\Omega^u_{t^*+1},$$

which proves the first row of the M_u in the claim. The second row is straightforward from the New Keynesian Phillips Curve and the definition of Ω and Φ.

Step 3. We now move to the upper space. First, note that the unaware consumers' reaction in the lower space (step 2) is perceived reaction that may be different when realized market clearing is taken into account. To see this, recall the IS relation in the lower space (equation (34)):

$$\hat{X}_t^u = -\gamma \sum_{\tau=0}^{\infty} \beta^{\tau+1} \overline{\mathbb{E}}_{I,t} \left[r_{t+\tau} + \ln \beta - \frac{1}{\gamma} \left(\frac{1+\psi}{\psi + \frac{1}{\gamma}} \right) (z_{t+1} - z_t) \right] + (1-\beta) \sum_{\tau=0}^{\infty} \beta^{\tau} \overline{\mathbb{E}}_{I,t} \left[\hat{X}_{t+\tau}^u \right].$$

$\hat{X}_{t+\tau}^u$ at the right-hand side comes from the market clearing as perceived by unaware consumers. For example, the unaware consumer may expect

$$\hat{X}_T^u = -\gamma\beta \overline{\mathbb{E}}_{I,T} \left[R_{T|T} - r_{T|T}^n \right] + (1-\beta)\hat{X}_T^u$$

to hold at period T, but the realized output gap is

$$\hat{X}_T^u = -\gamma\beta \overline{\mathbb{E}}_{I,T} \left[R_{T|T} - r_{T|T}^n \right] + (1-\beta)\hat{X}_T^a$$

because the unaware consumers observe the actual market clearing price at period T. To deal with this perceived-realized reaction difference of the unaware consumers, we introduce the following notation. While we keep $\hat{X}_t^u$ for the perceived reaction of the unaware consumers at period t, we denote $\hat{X}_t^{u*}$ as the realized reaction of the unaware consumers when they observe the current market clearing price at period t. In line with this new notation, we also introduce Φ_t^{u*} to denote the realized reaction of the output gap in period t given the signals:

$$\overline{\mathbb{E}}_{I,t^*} \left[\hat{X}_t^{u*} \right] = \Phi_t^{u*} \; \overline{\mathbb{E}}_{I,t^*} \left[\tilde{R}_{T|t^*} \right] \tag{48}$$

Then, we can write the realized IS relation for the unaware consumers as follows:

$$\begin{aligned} \hat{X}_t^{u*} = & -\gamma \sum_{\tau=0}^{\infty} \beta^{\tau+1} \overline{\mathbb{E}}_{I,t} \left[r_{t+\tau} + \ln \beta - \frac{1}{\gamma} \left(\frac{1+\psi}{\psi + \frac{1}{\gamma}} \right) (z_{t+1} - z_t) \right] \\ & + (1-\beta)\hat{X}_t^a + (1-\beta) \sum_{\tau=1}^{\infty} \beta^{\tau} \overline{\mathbb{E}}_{I,t} \left[\hat{X}_{t+\tau}^u \right]. \end{aligned} \tag{49}$$

That is, unaware consumers react $\hat{X}_t^{u*}$ given the belief $(\hat{X}_\tau^u)_{\tau \geq t+1}$, and the current market clearing $(\hat{X}_t^a)$. By taking the difference between equation (49) and equation (34), we get the following relation which will be handy later:

$$\begin{aligned} \hat{X}_t^{u*} - \hat{X}_t^u &= (1-\beta) \left(\hat{X}_t^a - \hat{X}_t^u \right) \\ &= (1-\beta) \left(\mu \frac{\hat{X}_t^a - (1-\mu)\hat{X}_t^{u*}}{\mu} + (1-\mu)\hat{X}_t^{u*} - \hat{X}_t^u \right) \\ &= (1-\beta)\mu \left(\frac{\hat{X}_t^a - \hat{X}_t^{u*}}{\mu} \right) + (1-\beta) \left(\hat{X}_t^{u*} - \hat{X}_t^u \right) \\ &= \frac{(1-\beta)\mu}{\beta} \left(\frac{\hat{X}_t^a - \hat{X}_t^{u*}}{\mu} \right) \end{aligned}$$

and equivalently,

$$\hat{X}_t^a - \hat{X}_t^u = \frac{\mu}{\beta}\left(\frac{\hat{X}_t^a - \hat{X}_t^{u*}}{\mu}\right). \tag{50}$$

Step 4. Consider now the problem of aware consumers. Converting output ($\hat{Y}_t^a$) to the output gap ($\hat{X}_t^a$) is analogous to Step 1 except that we now use equation (32). We obtain:

$$\begin{aligned}
&\frac{1}{\mu}\int_{i\in I_a} \ln c_{i,t}^a di - \ln Y_t^n = \frac{\ln C_t^a - (1-\mu)\ln C_t^{u*}}{\mu} - \ln Y_t^n = \frac{\hat{X}_t^a - (1-\mu)\hat{X}_t^{u*}}{\mu} \\
&= -\gamma \sum_{\tau=0}^{\infty} \beta^{\tau+1}\overline{\mathbb{E}}_{I_a,t}\left[r_{t+\tau} + \ln\beta - \frac{1}{\gamma}\left(\frac{1+\psi}{\psi+\frac{1}{\gamma}}\right)(z_{t+1}-z_t)\right] + (1-\beta)\sum_{\tau=0}^{\infty}\beta^{\tau}\overline{\mathbb{E}}_{I_a,t}\left[\hat{X}_{t+\tau}^a\right]
\end{aligned} \tag{51}$$

where $\hat{X}_t^a$ is the average output gap perceived by aware consumers, which is also the realized one. Note that $\frac{\hat{X}_t^a-(1-\mu)\hat{X}_t^{u*}}{\mu}$ is the contribution to the output gap of aware consumers only.

Next, we derive the contemporaneous reaction of the aware consumers. We can write the average reaction among $i \in I_a$ as follows:

$$\begin{aligned}
\frac{\hat{X}_T^a - (1-\mu)\hat{X}_T^{u*}}{\mu} =& - \gamma\beta\overline{\mathbb{E}}_{I_a,T}\left[\tilde{R}_{T|T}\right] - \beta\left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\overline{\mathbb{E}}_{I_a,T}\left[z_{T|T}\right] \\
&+ (1-\beta)\left((1-\mu)\hat{X}_T^{u*} + \mu\frac{\hat{X}_T^a - (1-\mu)\hat{X}_t^{u*}}{\mu}\right)
\end{aligned}$$

because, $z_{t+1} = 0$ and $\pi_{T+1} = 0$. Then, since the aware consumer anticipates the lower space Taylor rule ($\tilde{R}_{T|T} = \phi_y \hat{X}_T^{u*}$), we can replace $\hat{X}_T^{u*}$ using the Taylor rule:

$$\begin{aligned}
\frac{\hat{X}_T^a - (1-\mu)\hat{X}_T^{u*}}{\mu} =& \left(-\gamma\beta + \frac{(1-\beta)(1-\mu)}{\phi_y}\right)\overline{\mathbb{E}}_{I_a,T}\left[\tilde{R}_{T|T}\right] - \beta\left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\overline{\mathbb{E}}_{I_a,T}\left[z_{T|T}\right] \\
&+ (1-\beta)\mu\left(\frac{\hat{X}_T^a - (1-\mu)\hat{X}_t^{u*}}{\mu}\right)
\end{aligned}$$

Collecting the average output gap of aware consumers, $\frac{\hat{X}_T^a-(1-\mu)\hat{X}_T^{u*}}{\mu}$, gives,

$$\begin{aligned}
(1-(1-\beta)\mu)\left(\frac{\hat{X}_T^a - (1-\mu)\hat{X}_T^{u*}}{\mu}\right) =& \left(-\gamma\beta + \frac{(1-\beta)(1-\mu)}{\phi_y}\right)\overline{\mathbb{E}}_{I_a,T}\left[\tilde{R}_{T|T}\right] \\
&- \beta\left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\overline{\mathbb{E}}_{I_a,T}\left[z_{T|T}\right].
\end{aligned} \tag{52}$$

Note that the Taylor rule in the upper space ($\tilde{R}_T = \phi_y \hat{X}_T^a + v_T$) can be written as

$$\begin{aligned}
\overline{\mathbb{E}}_{I_a,T}\left[\tilde{R}_{T|T}\right] &= (1-\mu)\phi_y\hat{X}_T^{u*} + \mu\left(\phi_y\frac{\hat{X}_T^a - (1-\mu)\hat{X}_T^{u*}}{\mu}\right) + \overline{\mathbb{E}}_{I_a,T}\left[v_{T|T}\right] \\
&= (1-\mu)\tilde{R}_{T|T} + \mu\left(\phi_y\frac{\hat{X}_T^a - (1-\mu)\hat{X}_T^{u*}}{\mu}\right) + \overline{\mathbb{E}}_{I_a,t}\left[v_{T|T}\right] \\
&= \phi_y\left(\frac{\hat{X}_T^a - (1-\mu)\hat{X}_T^{u*}}{\mu}\right) + \frac{\overline{\mathbb{E}}_{I_a,T}\left[v_{T|T}\right]}{\mu}.
\end{aligned} \tag{53}$$

Substituting the last equation into equation (52), we get

$$
\begin{aligned}
&(1-(1-\beta)\mu)\left(\frac{\hat{X}^a_T-(1-\mu)\hat{X}^{u*}_T}{\mu}\right)\\
&=\left(\frac{-\gamma\beta\phi_y(1-\beta)(1-\mu)}{\phi_y}\right)\left(\phi_y\left(\frac{\hat{X}^a_T-(1-\mu)\hat{X}^{u*}_T}{\mu}\right)+\frac{\overline{\mathbb{E}}_{I_a,t}\left[v_{T|T}\right]}{\mu}\right)\\
&-\beta\left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\overline{\mathbb{E}}_{I_a,T}\left[z_{T|T}\right]
\end{aligned}
$$

We collect the term $\frac{\hat{X}^a_T-(1-\mu)\hat{X}^{u*}_T}{\mu}$ once again and obtain

$$
\begin{aligned}
\frac{\hat{X}^a_T-(1-\mu)\hat{X}^{u*}_T}{\mu}=&\underbrace{\frac{\beta}{\gamma\beta\phi_y+\beta}\left(-\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)}_{:=\Lambda_{11}}\overline{\mathbb{E}}_{I_a,T}\left[z_{T|T}\right]\\
&+\underbrace{\frac{-\gamma\beta\phi_y+(1-\beta)(1-\mu)}{\gamma\beta\phi_y+\beta}\frac{1}{\phi_y\mu}}_{:=\Lambda_{12}}\overline{\mathbb{E}}_{I_a,T}\left[v_{T|T}\right],
\end{aligned}
$$

We substitute the last equation into the equation (53) to obtain the actual Taylor rule, which is now represented as a function of the two shocks:

$$
\overline{\mathbb{E}}_{I_a,T}\left[\tilde{R}_{T|T}\right]=\underbrace{\phi_y\Lambda_{11}}_{:=\xi_z=\Lambda_{21}}\overline{\mathbb{E}}_{I_a,T}\left[z_{T|T}\right]+\underbrace{\frac{1-(1-\beta)\mu}{\gamma\beta\phi_y+\beta}\frac{1}{\mu}}_{:=\xi_v=\Lambda_{22}}\overline{\mathbb{E}}_{I_a,T}\left[v_{T|T}\right]
$$

Replacing the shocks with the inference by aware consumers, i.e., equation (1), gives us the contemporaneous reaction (output gap) of the aware consumers:

$$
\begin{aligned}
\frac{\hat{X}^a_T-(1-\mu)\hat{X}^{u*}_T}{\mu}&=\frac{1}{\phi_y}\left(\lambda_z+\lambda_v\left(\frac{-\gamma\beta\phi_y+(1-\beta)(1-\mu)}{1-(1-\beta)\mu}\right)\right)\tilde{R}_{T|T}\\
&=\frac{1}{\phi_y}\left(\lambda_z+\lambda_v\left(\frac{-\gamma\beta\phi_y+(1-\beta)(1-\mu)}{1-(1-\beta)\mu}\right)\right)\left(\frac{1}{\lambda_z+\lambda_v}\right)\overline{\mathbb{E}}_{I_a,T}\left[\tilde{R}_{T|T}\right]
\end{aligned}
$$

We define Φ^a and Ω^a similar to their analogues in the lower space:

$$
\overline{\mathbb{E}}_{I_a,t^*}\left[\frac{\hat{X}^a_t-(1-\mu)\hat{X}^{u*}_t}{\mu}\right]=\Phi^a_t\,\overline{\mathbb{E}}_{I_a,t^*}\left[\tilde{R}_{T|t^*}\right] \tag{54}
$$

$$
\overline{\mathbb{E}}_{I_a,t^*}\left[\pi_t\right]=\Xi\hat{X}^a_t+\Omega^a_t\,\overline{\mathbb{E}}_{I_a,t^*}\left[\tilde{R}_{T|t^*}\right] \tag{55}
$$

Therefore, we get

$$
\Phi^a_T=\frac{1}{\phi_y}\left(\frac{\lambda_z}{\lambda_z+\lambda_v}+\left(\frac{\lambda_v}{\lambda_z+\lambda_v}\right)\left(\frac{-\gamma\beta\phi_y+(1-\beta)(1-\mu)}{1-(1-\beta)\mu}\right)\right) \tag{56}
$$

and $\Omega^a_T=0$.

Step 5. Recall from Step 2 that the base case for the inductive argument concerns period $T-2$. Before we can prove the base case, we need to state the reaction for $T-1$. In order to invoke backward recursion, we first write the IS curve in the upper space as a recursive formula. To this end, we first take the difference to the contemporaneous reactions. Recall that the unaware consumers' (realized) reaction at the upper space is

$$\hat{X}_T^{u*} = -\gamma\beta\overline{\mathbb{E}}_{I,T}\left[R_{T|T} - r_{T|T}^n\right] + (1-\beta)\hat{X}_T^a$$

and the aware consumers' reaction is

$$\frac{\hat{X}_T^a - (1-\mu)\hat{X}_T^{u*}}{\mu} = -\gamma\beta\overline{\mathbb{E}}_{I_a,T}\left[R_{T|T} - r_{T|T}^n\right] + (1-\beta)\hat{X}_T^a.$$

Therefore, the difference between the contemporaneous reactions is

$$\frac{\hat{X}_T^a - \hat{X}_T^{u*}}{\mu} = -\gamma\beta\left(\overline{\mathbb{E}}_{I_a,T}\left[R_{T|T} - r_{T|T}^n\right] - \overline{\mathbb{E}}_{I,T}\left[R_{T|T} - r_{T|T}^n\right]\right) \tag{57}$$

Using equations (56) and (40), we can write an expression for the difference in perceived output gaps:

$$\begin{aligned}
&\frac{\hat{X}_T^a - (1-\mu)\hat{X}_T^{u*} - \mu\hat{X}_T^u}{\mu} \\
&= \frac{1}{\phi_y}\left(\lambda_z + \lambda_v\left(\frac{-\gamma\beta\phi_y + (1-\beta)(1-\mu)}{1-(1-\beta)\mu}\right)\right)\tilde{R}_{T|T} - \frac{1}{\phi_y}\lambda_z^u\tilde{R}_{T|T} \\
&= \Phi_T^a\,\overline{\mathbb{E}}_{I_a,T}\left[\tilde{R}_{T|T}\right] - \Phi_T^u\,\overline{\mathbb{E}}_{I,T}\left[\tilde{R}_{T|T}\right] \\
&= -\gamma\beta\left(\overline{\mathbb{E}}_{I_a,T}\left[R_{T|T} - r_{T|T}^n\right] - \overline{\mathbb{E}}_{I,T}\left[R_{T|T} - r_{T|T}^n\right]\right) + (1-\beta)\left(\hat{X}_T^a - \hat{X}_T^u\right)
\end{aligned}$$

where the last line comes from the perceived IS relation. What we want to get is the difference in the realized output gaps. To this end, we use the perceived-realized relation in the previous step (equation (50)) to rewrite the last line of the above equation as follows:

$$\begin{aligned}
&\frac{\hat{X}_T^a - (1-\mu)\hat{X}_T^{u*} - \mu\hat{X}_T^u}{\mu} \\
&= -\gamma\beta\left(\overline{\mathbb{E}}_{I_a,T}\left[R_{T|T} - r_{T|T}^n\right] - \overline{\mathbb{E}}_{I,T}\left[R_{T|T} - r_{T|T}^n\right]\right) + (1-\beta)\frac{\mu}{\beta}\left(\frac{\hat{X}_T^a - \hat{X}_T^{u*}}{\mu}\right) \\
&= \left(1 + \frac{(1-\beta)\mu}{\beta}\right)\left(\frac{\hat{X}_T^a - \hat{X}_T^{u*}}{\mu}\right)
\end{aligned}$$

Therefore, the actual contemporaneous reaction is

$$\frac{\hat{X}_T^a - \hat{X}_T^{u*}}{\mu} = \left(\frac{\beta}{\beta + (1-\beta)\mu}\right)\left(\Phi_T^a\,\overline{\mathbb{E}}_{I_a,T}\left[\tilde{R}_{T|T}\right] - \Phi_T^u\,\overline{\mathbb{E}}_{I,T}\left[\tilde{R}_{T|T}\right]\right)$$

Since $\hat{X}_t^a$ in equation (51) contains $\hat{X}_t^{u*}$, taking the difference with the unaware consumers' reaction allows us to obtain a relation between $\frac{\hat{X}_t^a - \hat{X}_t^{u*}}{\mu}$ and $\frac{\hat{X}_{t+1}^a - \hat{X}_{t+1}^{u*}}{\mu}$. At $T-1$, the difference is

$$\left(\frac{\hat{X}^a_{T-1}-\hat{X}^{u*}_{T-1}}{\mu}\right) \tag{58}$$
$$= -\gamma\beta^2\left(\overline{\mathbb{E}}_{I_a,T-1}\left[R_{T|T-1}-r^n_{T|T-1}\right]-\overline{\mathbb{E}}_{I,T-1}\left[R_{T|T-1}-r^n_{T|T-1}\right]\right)$$
$$+\gamma\beta\left(\overline{\mathbb{E}}_{I_a,T-1}\left[\pi_T\right]-\overline{\mathbb{E}}_{I,T-1}\left[\pi_T\right]\right)+(1-\beta)\beta\overline{\mathbb{E}}_{I_a,T-1}\left[\hat{X}^a_T-\hat{X}^u_T\right].$$

Using equation (57) and (50), we write the right-hand side as

$$\beta\overline{\mathbb{E}}_{I_a,T-1}\left[\frac{\hat{X}^a_T-\hat{X}^{u*}_T}{\mu}\right]+\gamma\beta\left(\overline{\mathbb{E}}_{I_a,T-1}\left[\pi_T\right]-\overline{\mathbb{E}}_{I,T-1}\left[\pi_T\right]\right)+(1-\beta)\mu\overline{\mathbb{E}}_{I_a,T-1}\left[\frac{\hat{X}^a_T-\hat{X}^{u*}_T}{\mu}\right].$$

Further, the difference of the inflation in the above equation is

$$\begin{aligned}\overline{\mathbb{E}}_{I_a,T-1}\left[\pi_T\right]-\overline{\mathbb{E}}_{I,T-1}\left[\pi_T\right] =&\overline{\mathbb{E}}_{I_a,T-1}\left[\Xi\hat{X}^a_T\right]-\overline{\mathbb{E}}_{I,T-1}\left[\Xi\hat{X}^u_T\right]\\ =&\Xi\frac{\mu}{\beta}\overline{\mathbb{E}}_{I_a,T-1}\left[\frac{\hat{X}^a_T-\hat{X}^{u*}_T}{\mu}\right]\end{aligned}$$

if we use the notation in equation (55). Combining these two observations, equation (58) becomes

$$\begin{aligned}\frac{\hat{X}^a_{T-1}-\hat{X}^{u*}_{T-1}}{\mu} &= \beta\overline{\mathbb{E}}_{I_a,T-1}\left[\frac{\hat{X}^a_T-\hat{X}^{u*}_T}{\mu}\right]+(\gamma\Xi\mu+(1-\beta)\mu)\overline{\mathbb{E}}_{I_a,T-1}\left[\frac{\hat{X}^a_T-\hat{X}^{u*}_T}{\mu}\right]\\ &= (\beta+(1-\beta)\mu+\gamma\Xi\mu)\,\overline{\mathbb{E}}_{I_a,T-1}\left[\frac{\hat{X}^a_T-\hat{X}^{u*}_T}{\mu}\right].\end{aligned} \tag{59}$$

Therefore, we can get the expression for Φ^a_{T-1} as follows:

$$\begin{aligned}&\Phi^a_{T-1}\ \overline{\mathbb{E}}_{I_a,T-1}\left[\tilde{R}_{T|T-1}\right]-\Phi^{u*}_{T-1}\ \overline{\mathbb{E}}_{I,T-1}\left[\tilde{R}_{T|T-1}\right]\\ &=(\beta+(1-\beta)\mu+\gamma\Xi\mu)\left(\Phi^a_T\ \overline{\mathbb{E}}_{I_a,T-1}\left[\tilde{R}_{T|T-1}\right]-\Phi^{u*}_T\ \overline{\mathbb{E}}_{I,T-1}\left[\tilde{R}_{T|T-1}\right]\right)\end{aligned}$$

For inflation, recall the New Keynesian Phillips Curve (equation (18)),

$$\begin{aligned}\overline{\mathbb{E}}_{I_a,T-1}\left[\pi_{T-1}\right]-\overline{\mathbb{E}}_{I,T-1}\left[\pi_{T-1}\right] &= \Xi\hat{X}^a_{T-1}+\beta\overline{\mathbb{E}}_{I_a,T-1}\left[\pi_T\right]-\Xi\hat{X}^a_{T-1}-\beta\overline{\mathbb{E}}_{I,T-1}\left[\pi_T\right]\\ &= \beta\overline{\mathbb{E}}_{I_a,T-1}\left[\Xi\hat{X}^a_T+\beta\overline{\mathbb{E}}_{I,T}\left[\pi_{T+1}\right]\right]-\beta\overline{\mathbb{E}}_{I,T-1}\left[\Xi\hat{X}^u_T+\beta\overline{\mathbb{E}}_{I_a,T}\left[\pi_{T+1}\right]\right]\\ &= \beta\Xi\overline{\mathbb{E}}_{I_a,T-1}\left[\hat{X}^a_T-\hat{X}^u_T\right]\\ &= \Xi\mu\overline{\mathbb{E}}_{I_a,T-1}\left[\frac{\hat{X}^a_T-\hat{X}^{u*}_T}{\mu}\right]\\ &= \Xi\mu\left(\Phi^a_T\ \overline{\mathbb{E}}_{I_a,T-1}\left[\tilde{R}_{T|T-1}\right]-\Phi^{u*}_T\ \overline{\mathbb{E}}_{I,T-1}\left[\tilde{R}_{T|T-1}\right]\right)\end{aligned}$$

which implies $\Omega^a_{T-1}-\Omega^{u*}_{T-1}=\Xi\mu\left(\Phi^a_T-\Phi^{u*}_T\right)$. The second line comes from the next period (T) New Keynesian Phillips Curve, the third line is from $\pi_{T+1}=0$, the fourth line uses equation (50), and the last line uses the definition of Φ^a (equation (54)).

Next we claim that $\Phi_t^a - \Phi_t^{u*}$ and $\Omega_t^a - \Omega_t^{u*}$ follow the recursive description below for any period $t \leq T-2$

$$\begin{pmatrix} \Phi_t^a \\ \Phi_t^{u*} \\ \Omega_t^a \\ \Omega_t^{u*} \end{pmatrix} = M_a \begin{pmatrix} \Phi_{t+1}^a \\ \Phi_{t+1}^{u*} \\ \Omega_{t+1}^a \\ \Omega_{t+1}^{u*} \end{pmatrix}$$

where the transition matrix M_a is defined as

$$M_a := \begin{pmatrix} \beta + ((1-\beta)\mu + \gamma\Xi\mu)(\lambda_z + \lambda_v) & 0 & \gamma\beta & 0 \\ 0 & \beta + ((1-\beta)\mu + \gamma\Xi\mu)\lambda_z^u & 0 & \gamma\beta \\ \Xi\mu & 0 & \beta & 0 \\ 0 & \Xi\mu & 0 & \beta \end{pmatrix}$$

We prove the claim by induction starting with proving the base case, i.e., the reaction for $T-2$. Recall that the reaction in $T-2$ can be written as follows using equation (51):

$$\begin{aligned} &\frac{\hat{X}_{T-2}^a - \hat{X}_{T-2}^{u*}}{\mu} \\ &= -\gamma\beta^3 \left(\overline{\mathbb{E}}_{I_a,T-2}\left[R_{T|t^*} - r_{T|t^*}^n\right] - \overline{\mathbb{E}}_{I,T-2}\left[R_{T|t^*} - r_{T|t^*}^n\right] \right) + \gamma\beta^2 \left(\overline{\mathbb{E}}_{I_a,T-2}\left[\pi_T\right] - \overline{\mathbb{E}}_{I,T-2}\left[\pi_T\right] \right) \\ &\quad + \gamma\beta \left(\overline{\mathbb{E}}_{I_a,T-2}\left[\pi_{T-1}\right] - \overline{\mathbb{E}}_{I,T-2}\left[\pi_{T-1}\right] \right) + (1-\beta)\beta\mu \left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}_T^a\right] - \overline{\mathbb{E}}_{I,T-2}\left[\hat{X}_T^{u*}\right]}{\mu} \right) \\ &\quad + (1-\beta)\mu \left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}_{T-1}^a\right] - \overline{\mathbb{E}}_{I,T-2}\left[\hat{X}_{T-1}^{u*}\right]}{\mu} \right) \end{aligned} \tag{60}$$

We replace the first term in the right-hand side (the difference of the interest rates) using period T result (equation (57)):

$$\begin{aligned} &\beta^2 \left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}_T^a\right] - \overline{\mathbb{E}}_{I,T-2}\left[\hat{X}_T^{u*}\right]}{\mu} \right) + \gamma\beta^2 \left(\overline{\mathbb{E}}_{I_a,T-2}\left[\pi_T\right] - \overline{\mathbb{E}}_{I,T-2}\left[\pi_T\right] \right) \\ &+ \gamma\beta \left(\overline{\mathbb{E}}_{I_a,T-2}\left[\pi_{T-1}\right] - \overline{\mathbb{E}}_{I,T-2}\left[\pi_{T-1}\right] \right) + (1-\beta)\beta\mu \left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}_T^a\right] - \overline{\mathbb{E}}_{I,T-2}\left[\hat{X}_T^{u*}\right]}{\mu} \right) \\ &+ (1-\beta)\mu \left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}_{T-1}^a\right] - \overline{\mathbb{E}}_{I,T-2}\left[\hat{X}_{T-1}^{u*}\right]}{\mu} \right) \end{aligned}$$

The above expression features inflation differences. Using the New Keynesian Phillips Curve (equation (18)), we can replace both inflation differences with corresponding output gaps:

$$\gamma\beta^2\left(\overline{\mathbb{E}}_{I_a,T-2}\left[\pi_T\right]-\overline{\mathbb{E}}_{I,T-2}\left[\pi_T\right]\right)=\gamma\beta\Xi\mu\left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}^a_T\right]-\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^{u*}_T\right]}{\mu}\right)$$

$$\begin{aligned}&\gamma\beta\left(\overline{\mathbb{E}}_{I_a,T-2}\left[\pi_{T-1}\right]-\overline{\mathbb{E}}_{I,T-2}\left[\pi_{T-1}\right]\right)\\&\quad=\gamma\Xi\mu\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}^a_{T-1}\right]-\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^{u*}_{T-1}\right]}{\mu}+\gamma\beta^2\left(\overline{\mathbb{E}}_{I_a,T-2}\left[\pi_T\right]-\overline{\mathbb{E}}_{I,T-2}\left[\pi_T\right]\right)\end{aligned}$$

Applying these observations to the right-hand side of equation (60) yields

$$\begin{aligned}&\beta(\beta+\gamma\Xi\mu+(1-\beta)\mu)\left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}^a_T\right]-\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^{u*}_T\right]}{\mu}\right)\\&\quad+((1-\beta)\mu+\gamma\Xi\mu)\left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}^a_{T-1}\right]-\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^{u*}_{T-1}\right]}{\mu}\right)\\&\quad+\gamma\beta\Xi\mu\left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}^a_T\right]-\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^{u*}_T\right]}{\mu}\right)\end{aligned}$$

Using the result for period $T-1$, equation (59), we can write the above as

$$\begin{aligned}&\beta(\beta+(1-\beta)\mu+\gamma\Xi\mu)\left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}^a_T\right]-\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^{u*}_T\right]}{\mu}\right)\\&+((1-\beta)\mu+\gamma\Xi\mu)\times\\&\left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[(\beta+(1-\beta)\mu+\gamma\Xi\mu)\overline{\mathbb{E}}_{I_a,T-1}\left[\hat{X}^a_T\right]\right]-\overline{\mathbb{E}}_{I,T-2}\left[(\beta+(1-\beta)\mu+\gamma\Xi\mu)\overline{\mathbb{E}}_{I,T-1}\left[\hat{X}^{u*}_T\right]\right]}{\mu}\right)\\&+\gamma\beta\Xi\mu\left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}^a_T\right]-\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^{u*}_T\right]}{\mu}\right)\end{aligned}$$

Using the higher-order (average) expectations for each of the aware and unaware consumers, we rewrite the above as

$$\begin{aligned}&\frac{\hat{X}^a_{T-2}-\hat{X}^{u*}_{T-2}}{\mu}\\&=\beta(\beta+(1-\beta)\mu+\gamma\Xi\mu)\left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}^a_T\right]-\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^{u*}_T\right]}{\mu}\right)\\&\quad+((1-\beta)\mu+\gamma\Xi\mu)(\beta+(1-\beta)\mu+\gamma\Xi\mu)\left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[(\lambda_z+\lambda_v)\hat{X}^a_T\right]-\overline{\mathbb{E}}_{I,T-2}\left[\lambda^u_z\hat{X}^{u*}_T\right]}{\mu}\right)\\&\quad+\gamma\beta\Xi\mu\left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}^a_T\right]-\overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^{u*}_T\right]}{\mu}\right)\end{aligned}$$

and again using the result for period $T-1$ of equation (59) we obtain

$$
\begin{aligned}
&\frac{\hat{X}^a_{T-2} - \hat{X}^{u*}_{T-2}}{\mu} \\
&= \beta \left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}^a_{T-1}\right] - \overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^{u*}_{T-1}\right]}{\mu} \right) \\
&\quad + ((1-\beta)\mu + \gamma\Xi\mu) \left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[(\lambda_z+\lambda_v)\hat{X}^a_{T-1}\right] - \overline{\mathbb{E}}_{I,T-2}\left[\lambda^u_z \hat{X}^{u*}_{T-1}\right]}{\mu} \right) \\
&\quad + \gamma\beta\Xi\mu \left(\frac{\overline{\mathbb{E}}_{I_a,T-2}\left[\hat{X}^a_{T}\right] - \overline{\mathbb{E}}_{I,T-2}\left[\hat{X}^{u*}_{T}\right]}{\mu} \right).
\end{aligned}
$$

Now we are ready to restate the above equation using Φ and Ω as follows. This proves that our claim holds at period $T-2$ for the first and second rows of M_a:

$$
\begin{aligned}
&\Phi^a_{T-2}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right] - \Phi^{u*}_{T-2}\overline{\mathbb{E}}_{I,T-2}\left[\tilde{R}_{T|T-2}\right] \\
&= \left(\beta + ((1-\beta)\mu + \gamma\Xi\mu)(\lambda_z+\lambda_v)\right)\Phi^a_{T-1}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right] \\
&\quad - \left(\beta + ((1-\beta)\mu + \gamma\Xi\mu)\lambda^u_z\right)\Phi^{u*}_{T-1}\overline{\mathbb{E}}_{I,T-2}\left[\tilde{R}_{T|T-2}\right] \\
&\quad + \gamma\beta\left(\Omega^a_{T-1}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right] - \Omega^{u*}_{T-1}\overline{\mathbb{E}}_{I,T-2}\left[\tilde{R}_{T|T-2}\right]\right)
\end{aligned}
$$

For the inflation,

$$
\begin{aligned}
&\overline{\mathbb{E}}_{I_a,T-2}\left[\pi_{T-2}\right] - \overline{\mathbb{E}}_{I,T-2}\left[\pi_{T-2}\right] \\
&= \beta\overline{\mathbb{E}}_{I_a,T-2}\left[\pi_{T-1}\right] - \beta\overline{\mathbb{E}}_{I,T-2}\left[\pi_{T-1}\right] \\
&= \beta\left(\overline{\mathbb{E}}_{I_a,T-2}\left[\Xi\hat{X}^a_{T-1}\right] - \overline{\mathbb{E}}_{I,T-2}\left[\Xi\hat{X}^u_{T-1}\right] + \Omega^a_{T-1}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right] - \Omega^{u*}_{T-1}\overline{\mathbb{E}}_{I,T-2}\left[\tilde{R}_{T|T-2}\right]\right) \\
&= \Xi\mu\overline{\mathbb{E}}_{I_a,T-2}\left[\frac{\hat{X}^a_{T-1} - \hat{X}^{u*}_{T-1}}{\mu}\right] + \beta\left(\Omega^a_{T-1}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right] - \Omega^{u*}_{T-1}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right]\right) \\
&= \Xi\mu\left(\Phi^a_{T-1}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right] - \Phi^{u*}_{T-1}\overline{\mathbb{E}}_{I,T-2}\left[\tilde{R}_{T|T-2}\right]\right) \\
&\quad + \beta\left(\Omega^a_{T-1}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right] - \Omega^{u*}_{T-1}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right]\right)
\end{aligned}
$$

which implies $\Omega^a_{T-2} - \Omega^{u*}_{T-2} = \Xi\mu\left(\Phi^a_{T-1} - \Phi^{u*}_{T-1}\right) + \beta\left(\Omega^a_{T-1} - \Omega^{u*}_{T-1}\right)$, and this proves our claim for the third and fourth rows of M_a.

Assume as the induction hypothesis that for any $\tau \in \{t^*+1, t^*+2, ..., T-2\}$, the above claim on the transition matrix holds. We like to show that the claim holds for t^*. Again, the difference of the period t^* reactions can be written as follows:

$$
\begin{aligned}
\frac{\hat{X}^a_{t^*} - \hat{X}^{u*}_{t^*}}{\mu} &= -\gamma\sum_{\tau=0}^{T-t^*}\beta^{\tau+1}\left(\overline{\mathbb{E}}_{I_a,t^*}\left[r_{t^*+\tau|t^*} - r^n_{t^*+\tau|t^*}\right] - \overline{\mathbb{E}}_{I,t^*}\left[r_{t^*+\tau|t^*} - r^n_{t^*+\tau|t^*}\right]\right) \\
&\quad + (1-\beta)\mu\sum_{\tau=0}^{T-t^*}\beta^{\tau}\left(\frac{\overline{\mathbb{E}}_{I_a,t^*}\left[\hat{X}^a_{t^*+\tau}\right] - \overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^{u*}_{t^*+\tau}\right]}{\mu}\right) \qquad (61)
\end{aligned}
$$

Since $\pi_{T+1}=0$, $\tilde{R}_{t^*+\tau}=0$, and $z_{t^*+\tau}=0$ for any $\tau<T-t^*$, the first term in the right hand side is

$$\gamma\sum_{\tau=0}^{T-t^*}\beta^{\tau+1}\left(\overline{\mathbb{E}}_{I_a,t^*}\left[\pi_{t^*+\tau+1}\right]-\overline{\mathbb{E}}_{I,t^*}\left[\pi_{t^*+\tau+1}\right]\right).$$

Considering above equation one step forward at t^*+1, multiplying it with β, and taking difference to equation (61) gives,

$$\begin{aligned}
&\left(\frac{\hat{X}^a_{t^*}-\hat{X}^{u*}_{t^*}}{\mu}\right)-\beta\left(\frac{\overline{\mathbb{E}}_{I_a,t^*}\left[\hat{X}^a_{t^*+1}\right]-\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^{u*}_{t^*+1}\right]}{\mu}\right)\\
&=\gamma\beta\left(\overline{\mathbb{E}}_{I_a,t^*}\left[\pi_{t^*+1}\right]-\overline{\mathbb{E}}_{I,t^*}\left[\pi_{t^*+1}\right]\right)+(1-\beta)\mu\left(\frac{\overline{\mathbb{E}}_{I_a,t^*}\left[\hat{X}^a_{t^*+1}\right]-\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^{u*}_{t^*+1}\right]}{\mu}\right)
\end{aligned}$$

By replacing π with $\hat{X}$ and $\tilde{R}$ using equation (55), we obtain the following expression for the right-hand side of the above equation.

$$\begin{aligned}
&=\gamma\beta\left(\overline{\mathbb{E}}_{I_a,t^*}\left[\Xi\hat{X}^a_{t^*+1}\right]+\Omega^a_{t^*+1}\overline{\mathbb{E}}_{I_a,t^*}\left[\tilde{R}_{T|t^*}\right]-\overline{\mathbb{E}}_{I,t^*}\left[\Xi\hat{X}^u_{t^*+1}\right]-\Omega^{u*}_{t^*+1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right]\right)\\
&\quad+(1-\beta)\mu\left(\frac{\overline{\mathbb{E}}_{I_a,t^*}\left[\hat{X}^a_{t^*+1}\right]-\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^{u*}_{t^*+1}\right]}{\mu}\right)
\end{aligned}$$

Then we rearrange the expression as

$$\begin{aligned}
&=(\gamma\Xi\mu+(1-\beta)\mu)\left(\frac{\overline{\mathbb{E}}_{I_a,t^*}\left[\hat{X}^a_{t^*+1}\right]-\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^{u*}_{t^*+1}\right]}{\mu}\right)\\
&\quad+\gamma\beta\left(\Omega^a_{t^*+1}\overline{\mathbb{E}}_{I_a,t^*}\left[\tilde{R}_{T|t^*}\right]-\Omega^{u*}_{t^*+1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right]\right)\\
&=(\gamma\Xi\mu+(1-\beta)\mu)\left(\overline{\mathbb{E}}_{I_a,t^*}\left[\frac{\hat{X}^a_{t^*+1}-(1-\mu)\hat{X}^{u*}_{t^*+1}}{\mu}\right]-\overline{\mathbb{E}}_{I,t^*}\left[\hat{X}^u_{t^*+1}\right]\right)\\
&\quad+\gamma\beta\left(\Omega^a_{t^*+1}\overline{\mathbb{E}}_{I_a,t^*}\left[\tilde{R}_{T|t^*}\right]-\Omega^{u*}_{t^*+1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right]\right)\\
&=(\gamma\Xi\mu+(1-\beta)\mu)\left(\overline{\mathbb{E}}_{I_a,t^*}\left[\Phi^a_{t^*+1}\overline{\mathbb{E}}_{I_a,t^*+1}\left[\tilde{R}_{T|t^*}\right]\right]-\overline{\mathbb{E}}_{I,t^*}\left[\Phi^{u*}_{t^*+1}\overline{\mathbb{E}}_{I,t^*+1}\left[\tilde{R}_{T|t^*}\right]\right]\right)\\
&\quad+\gamma\beta\left(\Omega^a_{t^*+1}\overline{\mathbb{E}}_{I_a,t^*}\left[\tilde{R}_{T|t^*}\right]-\Omega^{u*}_{t^*+1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right]\right)
\end{aligned}$$

where the last line uses the definition of Φ^a (equation (54)). Therefore, we can rewrite equation (61) using Φ and Ω as follows:

$$\begin{aligned}
&\Phi^a_{t^*}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right]-\Phi^{u*}_{t^*}\overline{\mathbb{E}}_{I,T-2}\left[\tilde{R}_{T|T-2}\right]\\
&=\beta\left(\Phi^a_{t^*+1}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right]-\Phi^{u*}_{t^*+1}\overline{\mathbb{E}}_{I,T-2}\left[\tilde{R}_{T|T-2}\right]\right)\\
&\quad+((1-\beta)\mu+\gamma\Xi\mu)\left((\lambda_z+\lambda_v)\Phi^a_{t^*+1}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right]-\lambda^u_z\Phi^{u*}_{t^*+1}\overline{\mathbb{E}}_{I,T-2}\left[\tilde{R}_{T|T-2}\right]\right)\\
&\quad+\gamma\beta\left(\Omega^a_{t^*+1}\overline{\mathbb{E}}_{I_a,T-2}\left[\tilde{R}_{T|T-2}\right]-\Omega^{u*}_{t^*+1}\overline{\mathbb{E}}_{I,T-2}\left[\tilde{R}_{T|T-2}\right]\right),
\end{aligned}$$

and this concludes the claim.

Step 6. From equation (50), we know that the overall reaction of the current output gap $\hat{X}^a_{t^*}$ is as follows:

$$\hat{X}^a_{t^*} = \frac{\mu}{\beta}\left(\frac{\hat{X}^a_{t^*} - \hat{X}^{u*}_{t^*}}{\mu}\right) + \hat{X}^u_{t^*}.$$

Also, from Step 4, we get

$$\frac{\hat{X}^a_{t^*} - \hat{X}^{u*}_{t^*}}{\mu} = \begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix} (M_a)^{T-t^*-1} \cdot \begin{pmatrix} \Phi^a_{T-1}\overline{\mathbb{E}}_{I_a,t^*}\left[\tilde{R}_{T|t^*}\right] \\ \Phi^{u*}_{T-1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right] \\ \Omega^a_{T-1}\overline{\mathbb{E}}_{I_a,t^*}\left[\tilde{R}_{T|t^*}\right] \\ \Omega^{u^*}_{T-1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right] \end{pmatrix}$$

From Step 2, we get

$$\hat{X}^u_{t^*} = \begin{pmatrix} 1 & 0 \end{pmatrix} (M_u)^{T-t^*-1} \cdot \begin{pmatrix} \Phi^u_{T-1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right] \\ \Omega^u_{T-1}\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right] \end{pmatrix}$$

Recall that the $T-1$ results in the lower space and upper space are

$$\begin{aligned}
\Phi^a_{T-1} &= (\beta + (1-\beta)\mu + \gamma\Xi\mu)\,\Phi^a_T \\
\Phi^{u*}_{T-1} &= (\beta + (1-\beta)\mu + \gamma\Xi\mu)\,\Phi^{u*}_T \\
\Omega^a_{T-1} &= \Xi\mu\Phi^a_T \\
\Omega^{u*}_{T-1} &= \Xi\mu\Phi^{u*}_T \\
\Phi^u_{T-1} &= (1+\gamma\Xi)\Phi^u_T \\
\Omega^u_{T-1} &= \beta\Xi\Phi^u_T
\end{aligned}$$

and the period T results are

$$\begin{aligned}
\Phi^a_T\overline{\mathbb{E}}_{I_a,t^*}\left[\tilde{R}_{T|t^*}\right] &= \frac{1}{\phi_y}\left(\lambda_z + \lambda_v\left(\frac{-\gamma\beta\phi_y + (1-\beta)(1-\mu)}{1-(1-\beta)\mu}\right)\right)\tilde{R}_{T|t^*} \\
&= \left(\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z + \frac{\Lambda_{12}}{\Lambda_{22}}\lambda_v\right)\tilde{R}_{T|t^*}
\end{aligned}$$

and

$$\Phi^u_T\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right] = \frac{1}{\phi_y}\lambda^u_z\tilde{R}_{T|t^*} = \frac{\Lambda_{11}}{\Lambda_{21}}\lambda^u_z\tilde{R}_{T|t^*}$$

$$\Phi^{u*}_T\overline{\mathbb{E}}_{I,t^*}\left[\tilde{R}_{T|t^*}\right] = \frac{(1-\beta)\mu}{\beta+(1-\beta)\mu}\Phi^a_T + \frac{\beta}{\beta+(1-\beta)\mu}\Phi^u_T$$

Substitution allows us now to derive $\hat{X}^a_{t^*}$ as a function of $\tilde{R}_{T|t^*}$ as follows:

$$\hat{X}^a_{t^*}(\tilde{R}_{T|t^*}) =$$

$$\frac{\mu}{\beta}\begin{pmatrix}1 & -1 & 0 & 0\end{pmatrix}(M_a)^{T-t^*-1}\cdot\begin{pmatrix}(\beta+(1-\beta)\mu+\gamma\Xi\mu)\left(\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z+\frac{\Lambda_{12}}{\Lambda_{22}}\lambda_v\right)\\(\beta+(1-\beta)\mu+\gamma\Xi\mu)\left(\frac{\Lambda_{11}}{\Lambda_{21}}\frac{\beta\lambda^u_z+(1-\beta)\mu\lambda_z}{\beta+(1-\beta)\mu}+\frac{\Lambda_{12}}{\Lambda_{22}}\frac{(1-\beta)\mu\lambda_v}{\beta+(1-\beta)\mu}\right)\\\Xi\mu\left(\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z+\frac{\Lambda_{12}}{\Lambda_{22}}\lambda_v\right)\\\Xi\mu\left(\frac{\Lambda_{11}}{\Lambda_{21}}\frac{\beta\lambda^u_z+(1-\beta)\mu\lambda_z}{\beta+(1-\beta)\mu}+\frac{\Lambda_{12}}{\Lambda_{22}}\frac{(1-\beta)\mu\lambda_v}{\beta+(1-\beta)\mu}\right)\end{pmatrix}\tilde{R}_{T|t^*}$$

$$+\begin{pmatrix}1 & 0\end{pmatrix}(M_u)^{T-t^*-1}\cdot\begin{pmatrix}(1+\gamma\Xi)\frac{\Lambda_{11}}{\Lambda_{21}}\lambda^u_z\\\beta\Xi\frac{\Lambda_{11}}{\Lambda_{21}}\lambda^u_z\end{pmatrix}\tilde{R}_{T|t^*}$$

This proves the proposition. □

Proof of Corollary 1

By setting $\theta = 1$, we can see $\kappa = \frac{(1-\beta\theta)(1-\theta)}{\theta} = 0$ hence $\Xi = \kappa(\psi + \frac{1}{\gamma}) = 0$. Then, the transition matrices in the Proposition 2 are reduced as follows:

$$M_a = \begin{pmatrix}\beta+((1-\beta)\mu)(\lambda_z+\lambda_v) & 0 & \gamma\beta & 0\\0 & \beta+((1-\beta)\mu)\lambda^u_z & 0 & \gamma\beta\\0 & 0 & \beta & 0\\0 & 0 & 0 & \beta\end{pmatrix}$$

$$M_u = \begin{pmatrix}\beta+(1-\beta)\lambda^u_z & \gamma\\0 & \beta\end{pmatrix}$$

Note that the transition matrices are upper triangular hence $M_a^{T-t^*-1}$ and $M_u^{T-t^*-1}$ are also upper triangular. Further, the $T-1$ results are condensed as

$$\begin{pmatrix}\Phi^a_{T-1}\\\Phi^{u*}_{T-1}\\\Omega^a_{T-1}\\\Omega^{u*}_{T-1}\end{pmatrix} = (\beta+(1-\beta)\mu)\begin{pmatrix}\left(\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z+\frac{\Lambda_{12}}{\Lambda_{22}}\lambda_v\right)\\\left(\frac{\Lambda_{11}}{\Lambda_{21}}\frac{\beta\lambda^u_z+(1-\beta)\mu\lambda_z}{\beta+(1-\beta)\mu}+\frac{\Lambda_{12}}{\Lambda_{22}}\frac{(1-\beta)\mu\lambda_v}{\beta+(1-\beta)\mu}\right)\\0\\0\end{pmatrix}\tilde{R}_{T|t^*}$$

$$\begin{pmatrix}\Phi^u_{T-1}\\\Omega^u_{T-1}\end{pmatrix} = \begin{pmatrix}\frac{\Lambda_{11}}{\Lambda_{21}}\lambda^u_z\\0\end{pmatrix}\tilde{R}_{T|t^*}$$

Therefore, the overall reaction when we fix the inflation at 0 is,

$$\begin{aligned}\hat{X}^{IS}_{t^*} =& (\beta+(1-\beta)\lambda^u_z)^{T-t^*-1}\frac{\Lambda_{11}}{\Lambda_{21}}\lambda^u_z\tilde{R}_{T|t^*}\\&+\frac{\mu}{\beta}\left((\beta+(1-\beta)\mu)(\lambda_z+\lambda_v)\right)^{T-t^*-1}(\beta+(1-\beta)\mu)\left(\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z+\frac{\Lambda_{12}}{\Lambda_{22}}\lambda_v\right)\tilde{R}_{T|t^*}\\&-\frac{\mu}{\beta}\left((\beta+(1-\beta)\mu)(\lambda^u_z)\right)^{T-t^*-1}\left(\frac{\Lambda_{11}}{\Lambda_{21}}(\beta\lambda^u_z+(1-\beta)\mu\lambda_z)+\frac{\Lambda_{12}}{\Lambda_{22}}(1-\beta)\mu\lambda_v\right)\tilde{R}_{T|t^*}\end{aligned}$$

Under complete information, i.e., $\sigma^2_\eta = 0$, we have:

$$\hat{X}^{IS}_{t^*}|_{\sigma^2_\eta=0} = \frac{\Lambda_{11}}{\Lambda_{21}}\tilde{R}_{T|t^*}+\frac{\mu}{\beta}\left((\beta+(1-\beta)\mu)\right)^{T-t-1}\beta(1-\lambda)\left(\frac{\Lambda_{12}}{\Lambda_{22}}-\frac{\Lambda_{11}}{\Lambda_{21}}\right)\tilde{R}_{T|t^*}$$

□

Proof of Proposition 5

Recall the perceived market clearing in the baseline equilibrium:

$$Y_t = C_t^u = \int_{i \in I} c_{i,t}^u di$$

Since in self-confirming equilibrium, unaware consumers take actual market clearing prices into account, market clearing must now satisfy

$$\begin{aligned} C_t^u = Y_t = C_t^a &= (1-\mu)C_t^u + \int_{i \in I_a} c_{i,t} di \\ &= \frac{1}{\mu} \int_{i \in I_a} c_{i,t} di \end{aligned}$$

which implies $\hat{X}_t^u = \frac{\hat{X}_t^a - (1-\mu)\hat{X}_t^u}{\mu}$. We need to find an inference rule, $\mathbb{E}_{u,t}^{sc}[z_{T|t^*} \mid \omega_{i,T|t^*}]$, such that the above equation is satisfied. Consider an inference rule of the form $\mathbb{E}_{u,t}^{sc}[z_{T|t^*} \mid \omega_{i,T|t^*}] = \lambda_z^u \frac{\omega_{i,T|t^*}}{\xi_z} + \delta_{i,t}$ and a Taylor rule perceived by unaware consumers of the form $\tilde{R}_t = \phi \hat{X}_t^u + e_t$, where $\delta_{i,t}$ and e_t are to be determined.

Let $\overline{\mathbb{E}}_{I,T}^{sc}[\cdot] := \int_{i \in I} \mathbb{E}_{u,T}^{sc}[\cdot \mid \omega_{i,T|t^*}]di$. That is, unaware consumers still expect everyone to be like them. Rewrite the contemporaneous best response at period T for unaware consumers of equation (36) using now self-confirming expectations:

$$\begin{aligned} \hat{X}_T^u &= -\gamma \overline{\mathbb{E}}_{I,T}^{sc}\left[\tilde{R}_{T|T} - \pi_{T+1}\right] - \left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right) \overline{\mathbb{E}}_{I,T}^{sc}\left[z_{T|T}\right] \\ &= -\gamma \overline{\mathbb{E}}_{I,T}^{sc}\left[\phi_y \hat{X}_T^u + e_T\right] - \left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right) \overline{\mathbb{E}}_{I,T}^{sc}\left[z_{T|T}\right] \\ &= -\frac{\gamma}{1+\gamma\phi_y} \overline{\mathbb{E}}_{I,T}^{sc}\left[e_T\right] - \frac{1}{1+\gamma\phi_y} \left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right) \overline{\mathbb{E}}_{I,T}^{sc}\left[z_{T|T}\right] \end{aligned}$$

We rewrite the perceived Taylor rule as a function of shocks using the above result:

$$\overline{\mathbb{E}}_{I,T}^{sc}\left[\tilde{R}_{T|T}\right] = \frac{1}{1+\gamma\phi_y} \overline{\mathbb{E}}_{I,T}^{sc}\left[e_T\right] \underbrace{- \frac{\phi_y}{1+\gamma\phi_y} \left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)}_{=:\xi_z} \overline{\mathbb{E}}_{I,T}^{sc}\left[z_{T|T}\right]$$

Given this perceive Taylor rule, the unaware consumer's inference rule is

$$\begin{aligned} \overline{\mathbb{E}}_{I,T}^{sc}\left[z_{T|T}\right] &= \frac{\lambda_z^u \tilde{R}_{T|T} - \frac{1}{1+\gamma\phi_y} \overline{\mathbb{E}}_{I,T}^{sc}\left[e_T\right]}{\xi_z} \\ &= \frac{\lambda_z^u}{\xi_z} \tilde{R}_{T|T} + \frac{1}{\phi_y} \left(\frac{\frac{1}{\gamma}+\psi}{1+\psi}\right) \overline{\mathbb{E}}_{I,T}^{sc}\left[e_T\right] \end{aligned}$$

Given the form of the self-confirming inference rule, $\mathbb{E}_{u,T}^{sc}\left[z_{T|T}\right] = \lambda_z^u \frac{\omega_{i,T|T}}{\xi_z} + \delta_{i,T}$, we identify the parameter $\delta_{i,T}$ by

$$\delta_{i,T} = \frac{\lambda_z^u}{\xi_z} \left(\tilde{R}_{T|T} - \omega_{i,T|T}\right) + \frac{1}{\phi_y} \left(\frac{\frac{1}{\gamma}+\psi}{1+\psi}\right) \overline{\mathbb{E}}_{I,T}^{sc}\left[e_T\right]$$

and

$$\overline{\mathbb{E}}^{sc}_{I,T}\left[\hat{X}^u_T\right] = -\frac{\gamma}{1+\gamma\phi_y}\overline{\mathbb{E}}^{sc}_{I,T}\left[e_T\right] - \frac{1}{1+\gamma\phi_y}\left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\overline{\mathbb{E}}^{sc}_{I,T}\left[z_{T|T}\right]$$
$$= \frac{1}{\phi_y}\lambda^u_z\tilde{R}_{T|T} - \frac{1}{\phi_y}\overline{\mathbb{E}}^{sc}_{I,T}\left[e_T\right].$$

Given above contemporaneous reaction, we use Proposition 2 to derive the reaction at period t,

$$\hat{X}^u_t = \begin{pmatrix}1 & 0\end{pmatrix}(M_u)^{T-t-1}\begin{pmatrix}(1+\gamma\Xi)\\ \beta\Xi\end{pmatrix}\frac{\Lambda_{11}}{\Lambda_{21}}\left(\lambda^u_z\tilde{R}_{T|t^*} - \overline{\mathbb{E}}^{sc}_{I,t}\left[e_T\right]\right) \tag{62}$$

Next we consider aware consumer. Market clearing in self-confirming equilibrium implies that there exist $\overline{\mathbb{E}}^{sc}_{I,T}\left[e_T\right]$ such that $\hat{X}^a_T = \hat{X}^u_T$ with $\overline{\mathbb{E}}^{sc}_{I,T}\left[e_T\right]$.

$$\hat{X}^a_T = -\gamma\overline{\mathbb{E}}_{I_a,T}\left[\tilde{R}_{T|T} - \pi_{T+1}\right] - \left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\mathbb{E}_{I_a,T}\left[z_{T|T}\right]$$
$$= -\gamma\overline{\mathbb{E}}_{I_a,T}\left[\phi_y\hat{X}^a_T + v_{T|T}\right] - \left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\overline{\mathbb{E}}_{I_a,T}\left[z_{T|T}\right]$$
$$= \frac{-\gamma}{1+\gamma\phi_y}\overline{\mathbb{E}}_{I_a,T}\left[v_{T|T}\right] - \frac{1}{1+\gamma\phi_y}\left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\overline{\mathbb{E}}_{I_a,T}\left[z_{T|T}\right]$$

Then, the aware consumers' perceived and actual Taylor rule is,

$$\overline{\mathbb{E}}_{I_a,T}\left[\tilde{R}_{T|T}\right] = \underbrace{\frac{1}{1+\gamma\phi_y}}_{=:\xi'_v}\overline{\mathbb{E}}_{I_a,T}\left[v_{T|T}\right] \underbrace{-\frac{\phi}{1+\gamma\phi_y}\left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)}_{=\xi_z}\overline{\mathbb{E}}_{I_a,T}\left[z_{T|T}\right]$$

Using the inference rule of aware consumers, equation (1), we can rewrite the output reaction by

$$\hat{X}^a_T = \frac{-\gamma}{1+\gamma\phi_y}\frac{\tilde{R}_{T|T}}{\xi'_v}\lambda_v - \frac{1}{1+\gamma\phi_y}\left(\frac{1+\psi}{\frac{1}{\gamma}+\psi}\right)\frac{\tilde{R}_{T|T}}{\xi_z}\lambda_z$$
$$= \frac{1}{\phi_y}\lambda_z R_{T|T} - \gamma\lambda_v\tilde{R}_{T|T}$$
$$= \left(\lambda_z\frac{\Lambda_{11}}{\Lambda_{21}} + \lambda_v\left.\frac{\Lambda_{12}}{\Lambda_{22}}\right|_{\mu\to 1}\right)\tilde{R}_{T|T}$$

where the second equation comes from the definition of ξ'_v and ξ_z, and the last equation comes from the definition of Λs and the fact that $\xi'_v = \lim_{\mu\to 1}\xi_v$.

Now consider any period t between t^* and T. In the self-confirming equilibrium, $\hat{X}^a_t = \hat{X}^u_t$ for all $t \in \{t^*, ..., T\}$. Therefore, $\Phi^a_t = \Phi^{u^*}_t = \Phi^u_t$, $\Omega^a_t = \Omega^{u^*}_t = \Omega^u_t$, and the transition matrix M_a reduces to

$$\tilde{M}_a := \begin{pmatrix}\beta + (1-\beta+\gamma\Xi)(\lambda_z+\lambda_v) & \gamma\beta\\ \Xi & \beta\end{pmatrix}.$$

Therefore, the aware consumer's reaction at period t is

$$\hat{X}_t^a = \begin{pmatrix}1 & 0\end{pmatrix} (\tilde{M}_a)^{T-t-1} \begin{pmatrix}1+\gamma\Xi\mu \\ \beta\Xi\end{pmatrix} \left(\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z + \left.\frac{\Lambda_{12}}{\Lambda_{22}}\right|_{\mu\to 1} \lambda_v \right) \tilde{R}_{T|t^*} \tag{63}$$

$$\tag{64}$$

Finally, consider the average e_T. In the self-confirming equilibrium we have $\hat{X}_t^a = \hat{X}_t^u$. Equating the r.h.s. of equations (62) and (63), we obtain

$$\overline{\mathbb{E}}_{I,t}^{sc}\left[e_T\right] = \left(\lambda_z^u - \frac{\begin{pmatrix}1 & 0\end{pmatrix} (M_a)^{T-t-1} \begin{pmatrix}1+\gamma\Xi\mu \\ \beta\Xi\end{pmatrix} \left(\frac{\Lambda_{11}}{\Lambda_{21}}\lambda_z + \left.\frac{\Lambda_{12}}{\Lambda_{22}}\right|_{\mu\to 1} \lambda_v \right)}{\begin{pmatrix}1 & 0\end{pmatrix} (M_u)^{T-t-1} \begin{pmatrix}(1+\gamma\Xi) \\ \beta\Xi\end{pmatrix} \frac{\Lambda_{11}}{\Lambda_{21}}} \right) \tilde{R}_{T|t^*}$$

□

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