Awareness and AGM Belief Revision*

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Abstract

We study revision of awareness and knowledge/beliefs in unawareness structures of Heifetz et al. [2006]. To this end, we introduce conditional possibility correspondences in unawareness structures. Beyond standard properties, we impose on the conditional possibility correspondences properties analogous to AGM belief revision and some new properties of awareness revision including a property that can be interpreted as a qualitative version of "reverse Bayesianism" [Karni and Vierø, 2013, Hayashi, 2012]. We define conditional knowledge and conditional awareness operators and derive their corresponding properties. In the special case of no unawareness, our semantics is equivalent to the Bonanno-Kripke-Lewis semantics for AGM belief revision [Bonanno, 2025a,b].

Keywords: Unawareness, belief revision, reverse Bayesianism.

JEL-Classification: D83.

1 Introduction

Unawareness has been studied in computer science, artificial intelligence, formal philosophy, game theory, decision theory, and economics. The two most commonly used approaches, awareness structures due to the seminal work by Fagin and Halpern [1988], which are Kripke structures augmented with syntactic awareness functions, and unawareness structures of Heifetz et al. [2006, 2008], Kripke structures with lattices of spaces, are best interpreted in a static multi-agent setting. Yet, the study of unawareness is especially interesting in dynamic settings when agents not only learn about issues they are aware of but also discover novel issues or strategically raised each other's awareness. So far changes of awareness have been mainly studied within dynamic games[1]. Another line of research focuses on updating probabilistic beliefs in light of new awareness à la "reverse" Bayesianism in decision theory[2] and philosophy[3]. Yet, another line of research studies changes of awareness within dynamic epistemic logic[4].

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[1] Rego and Halpern [2012], Halpern and Rego [2014], Heifetz et al. [2013b], Feinberg [2021], Grant and Quiggin [2013], Guarino [2020], Heifetz et al. [2021], Schipper [2021], Foo and Schipper [2025], Schipper [2026]

[2] Torres [2010], Karni and Vierø [2013], Hayashi [2012], Karni and Vierø [2015, 2017], Chambers and Hayashi [2018], Dominiak and Tserenjigmid [2018, 2022], Dietrich [2018], Karni et al. [2021], Vierø [2021], Piermont [2021], Chakravarty et al. [2022], Schipper [2024, 2025]

[3] Wenmackers and Romeijn [2016], Bradley [2017], Mahtani [2021], Steele and Stefansson [2021], Roussos [2024], de Canson [2024], Pettigrew [2024]

[4] van Benthem and Velazquez-Quesada [2010], Hill [2010], van Ditmarsch et al. [2012], van Ditmarsch and French [2014], Halpern and Piermont [2020], Tashiro [2022]

The dominant theory of belief revision is AGM belief revision [Alchourrón et al., 1985]. Suppose we accept the postulates of AGM belief revision, how can they be captured in and extended to unawareness structures? This is far from clear because AGM belief revision works with belief sets best interpreted for exogeneously fixed awareness. Moreover, it is only a theory of belief revision, not awareness revision. It does not contain postulates for changes of awareness. Further, the theory of AGM belief revision applies to a single agent and lacks higher-order reasoning while structures allowing for interactive multi-person reasoning are instrumental for applications of unawareness to strategic settings.

Recently Bonanno [2025a,b] presented a simple Kripke-Lewis semantics for AGM belief revision. It is a Kripke structure with a Lewis selection function which for each state singles out the states "closest" to it conditional on an event. It does not capture unawareness though. Inspired by this simple elegant way of thinking of AGM belief revision within Kripke structures, we extend unawareness structures to allow for both awareness revision and AGM belief revision. Yet, rather than augmenting unawareness structures with a Lewis selection function, an extra primitive, we extend the agents' possibility correspondences to *conditional* possibility correspondences. For each state and event, the agent's conditional possibility correspondence provides the set of states considered possible by the agent conditional on the event. We translate the frame properties of Bonanno [2025b] to our setting, suitably extended to unawareness, and prove in unawareness structures the event-based analogues to the modal axioms capturing the AGM axioms. We also postulate additional properties of the conditional possibility correspondence needed to capture awareness change. One such property can be interpreted as a qualitative analogue to "reverse" Bayesianism [Karni and Vierø, 2013, Hayashi, 2012] in situations of awareness changes without confounding changes of knowledge of events which the agent has been previously aware. However, when we consider beliefs rather than knowledge, a change of awareness may also change beliefs for events that the agent has been previously aware. This points to an assumption also implicitly made for reverse Bayesianism[5]: The agent is fully committed to her prior belief even in light of evidence that the foundation for her prior belief, the state space capturing her prior awareness, has been incomplete and defective. Knowledge featuring factivity or the axiom of truth captures such strong commitment to the prior epistemic attitude while belief can be false and hence may be revised when becoming aware.

Different from AGM belief revision, conditioning on vacuous and universal events becomes meaningful under unawareness. Recall that unawareness structures feature multiple vacuous and universal events, corresponding to differently expressive contradictions and tautologies. For instance, a tautology like "You suffer from cuppacupitis or you do not suffer from cuppacupitis" does not carry information in the sense of allowing the recipient to exclude some possible states from consideration. It does carry awareness though as it raises awareness of cuppacupitis, apparently a disease. Similarly, the contradiction "You suffer from cuppacupitis and you do not suffer from cuppacupitis" is confusing in terms of information but again it raises awareness of cuppacupities and the possibility of suffering from it. When considering knowledge rather than beliefs, conditioning on vacuous or universal events might lead to an awareness change without confounding changes of knowledge of what the agent has been previously aware while conditioning on other events may lead to changes of both awareness and knowledge of what the agent has been previously aware. When the agent has been aware of the conditioning event before, such an event may lead to changes of knowledge only, while when the agent has been previously unaware of the conditioning event, it will raise her awareness and may change her knowledge of what the agent has been previously aware. However, when considering beliefs rather than knowledge, conditioning on vacuous or universal events might not just lead to an awareness change but also a change in beliefs about

[5]See Schipper [2025] for a critique and extension of reverse Bayesian to lack of confidence in the prior.

events the agent has been previously aware. The reason is that under unawareness the agent may have had beliefs that are obviously false when becoming aware. For instance, an agent may have falsely believed that symptoms may only come from a common cold because common cold was the only illness she had been aware of. But after becoming aware of cuppacupities she realizes that just a conclusion had been premature and entertains the possibility that her symptoms come from cuppacupitis and not from a common cold. Of course, raising awareness of an event must always go hand in hand with some epistemic attitude for that newly discovered event, even it is "aware" ignorance of that event. In the theory presented in this paper, conditioning on an event has a slightly different connotation depending on whether beliefs/knowledge or awareness are conditioning on it. For belief/knowledge, conditioning on an event is best interpreted as supposition of the event, while for awareness conditioning is best interpreted as realizing the mere existence (but not necessarily occurrence) of such an event.

Further Related Literature: van Benthem and Velazquez-Quesada [2010] study how dynamic reasoning can let an agent transform implicit knowledge into explicit knowledge, thereby gaining awareness of her implicit knowledge. The motivation is grounded in the logical non-omniscience problem. This is slightly different from our motivation of studying changes of awareness (and beliefs/knowledge) conditional on exogenously given events. Under propositional awareness, agents are fully logical omniscient w.r.t. what they are aware of whereas van Benthem and Velazquez-Quesada [2010] consider a more "computational" notion of awareness. The unawareness structures considered in the present paper do not feature implicit knowledge as in economics the explicit knowledge is considered decision relevant and the primitive notion. Yet, unawareness structures can be easily extended to implicit knowledge as in Belardinelli and Schipper [2024].

Hill [2010] considers changes of awareness in an algebraic model of awareness and knowledge for a single agent. He distinguishes between pure changes of awareness (i.e., becoming aware but staying ignorant) and changes of awareness compounded by changes in knowledge. Beside a dynamic epistemic logic of change of awareness, [Hill, 2010, Appendix] presents briefly an AGM-style logic change of awareness.

van Ditmarsch and French [2014] present a quite comprehensive and interesting dynamic epistemic logic approach with bisimulation of awareness structures. One of the notions of becoming aware they study is conditionally becoming aware, i.e., an agent becomes aware of p conditionally on φ being true. Our understanding is that as long as φ is not a contradiction, the notion of conditionally becoming aware corresponds to our notion of conditional awareness in the present paper. Our event-based approach also allows for contradictions to raise awareness.

We are not the first one to employ a conditional possibility correspondence. Fukuda [2024] uses a conditional possibility correspondence to study the connection between conditional probability systems and priors (in structures without unawareness). We believe we can extend this analysis to our setting with unawareness when introducing a system of prior probability measures as in Heifetz et al. [2013a]. Using the conditional possibility correspondence one would then obtain conditional probability systems under unawareness as for instance used in Guarino [2020]. If the conditional possibility correspondences satisfy property AR1K introduced in the present paper, then the conditional probability systems derived from a system of priors and conditional possibility correspondences would automatically satisfy reverse Bayesianism à la Karni and Vierø [2013] and Hayashi [2012].

2 Conditional Unawareness Structures

A conditional unawareness structure $\langle I, (\mathcal{S}, \succeq), (r_S^{S'})_{S,S' \in \mathcal{S}, S' \succeq S}, ((\Pi_i(\cdot \mid E)_{E \in \Sigma})_{i \in I} \rangle$ consists of

- a non-empty set of agents I,
- a nonempty complete lattice $(\mathcal{S}, \succeq)$ of non-empty disjoint state spaces $S \in \mathcal{S}$. For $S', S \in \mathcal{S}$, $S' \succeq S$ is interpreted as S' being more expressive than S. Denote the set of all states in spaces of the lattice by $\Omega := \bigcup_{S \in \mathcal{S}} S$. Further, denote by $\underline{S}$ and $\overline{S}$ the meet and join of the complete lattice $\mathcal{S}$.
- Projections $(r_S^{S'})_{S,S' \in \mathcal{S}, S' \succeq S}$ such that, for any $S', S \in \mathcal{S}$ with $S' \succeq S$, $r_S^{S'} : S' \longrightarrow S$ is surjective; for any $S \in \mathcal{S}$, r_S^S is the identity on S, and for any $S'', S', S \in \mathcal{S}$ with $S'' \succeq S' \succeq S$, $r^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. For any $S \in \mathcal{S}$ and $D \subseteq S$, denote by $D^{\uparrow} := \bigcup_{S' \in \mathcal{S}, S' \succeq S} (r_S^{S'})^{-1}(D)$. An *event* $E \subseteq \Omega$ is defined by a space $S \in \mathcal{S}$ and a subset $D \subseteq S$ such that $E := D^{\uparrow}$. We call S the *base-space* of the event E (also denoted $S(E)$) and D the *base* of the event E. We denote by $\Sigma(S)$ the set of events with base space $S' \preceq S$.
- for each agent $i \in I$, a conditional possibility correspondence $\Pi_i : \bigcup_{E \in \Sigma} S^{\uparrow}(E) \times \{E\} \longrightarrow 2^{\Omega} \setminus \{\emptyset\}$ with $\Pi_i(\omega \mid E) \subseteq S^{\uparrow}(E)$ for all $E \in \Sigma$, and $\omega \in S^{\uparrow}(E)$,[6] where $\Pi_i(\omega \mid E)$ denotes the set of states that agent i considers possible conditional on considering event E.

Not every subset of the union of spaces is an event. Intuitively, $D^{\uparrow}$ collects all the "extensions of descriptions in D to at least as expressive vocabularies" [Heifetz et al., 2006]. Events are well defined by the above definition except for the case of vacuous events. Since the empty set is a subset of any space, we have as many vacuous events as there are state spaces. These vacuous events are distinguished by their base-space, so we denote them by $\emptyset^S$ for $S \in \mathcal{S}$. At a first glance, the existence of many vacuous events may seem puzzling. Note that vacuous events essentially represent contradictions, i.e., propositions that cannot hold at any state. Contradictions can be more or less complicated depending on the expressiveness of the underlying language describing states and hence are represented by different vacuous events.

The following notation will be convenient: For any $\omega \in \Omega$, we denote by S_ω the state space in $\mathcal{S}$ that contains state ω. For any $D \subseteq S'$, we denote by D_S the projection of D to S for $S \preceq S'$. Similarly, for any nonempty $D \subseteq S$, we denote by $D^{S'}$ the "elaboration" of D in the space S' with $S' \succeq S$, i.e., $D^{S'} := (r_S^{S'})^{-1}(D)$. The same applies to states, i.e., ω_S is the projection of $\omega \in S'$ to S with $S \preceq S'$. We say that an event E is expressible in S if $S(E) \preceq S$.

For any event $E \in \Sigma$ with base D and base space S, negation is defined by $\neg E := (S \setminus D)^{\uparrow}$. Note that typically $\neg E \neq \Omega \setminus E$ unless $S(E) = \underline{S}$. Conjunction of events is simply the intersection, i.e., for $E, F \in \Sigma$, $E \wedge F := E \cap F$. Disjunction is not the union of events, as this might collect states in which an event in the disjunction may not be expressible. Rather, it is defined by DeMorgan Law and our notion of negation by $E \vee F := \neg(\neg E \cap \neg F)$ for any $E, F \in \Sigma$, which collects only states in union of the events in spaces in which both E and F are expressible. See Heifetz et al. [2006, 2008] for further details and discussions.

For any individual $i \in I$, we require that the conditional possibility correspondence Π_i satisfies:

Conditional Confinement: If $\omega \in S'$ and $E \in \Sigma$, then $\Pi_i(\omega \mid E) \subseteq S$ with $S(E) \preceq S \preceq S'$.

We denote by $S_{\Pi_i(\omega \mid E)}$ the state space S for which $\Pi_i(\omega \mid E) \subseteq S$.

[6] Here and in what follows, we write $S(E)^{\uparrow}$ for $(S(E))^{\uparrow}$.

Conditional Stationarity: For any $E \in \Sigma$ and $\omega \in S^{\uparrow}(E)$, $\omega' \in \Pi_i(\omega \mid E)$ implies $\Pi_i(\omega' \mid E) = \Pi_i(\omega \mid E)$.

Projections Preserve Conditional Ignorance: If $\omega \in S'$ and $S(E) \preceq S \preceq S'$, then $\Pi_i^{\uparrow}(\omega \mid E) \subseteq \Pi_i^{\uparrow}(\omega_S \mid E)$.

Projections Preserve Conditional Knowledge: If $S'' \succeq S' \succeq S \succeq S(E)$, $\omega \in S''$ and $\Pi_i(\omega \mid E) \subseteq S'$, then $(\Pi_i(\omega \mid E))_S = \Pi_i(\omega_S \mid E)$.

When considering knowledge rather than belief, we also require the conditional possibility correspondence to satisfy

Generalized Conditional Reflexivity: For any $E \in \Sigma$ and $\omega \in S^{\uparrow}(E)$, $\omega \in \Pi_i^{\uparrow}(\omega \mid E)$.[7]

Note that the (unconditional) possibility correspondence in Heifetz et al. [2006, 2008] may be taken as $\Pi_i(\cdot) := \Pi_i(\cdot|\underline{S}^{\uparrow})$.

We allow for conditioning on universal events $S^{\uparrow}$ and vacuous events $\emptyset^S$ for $S \in \mathcal{S}$. While the universal event $S^{\uparrow}$ does not carry information, they might raise awareness if the agent's prior awareness level S' is such that $S' \not\succeq S$. Tautologies such as "You may suffer from cuppacupitis or not" do not carry information but can raise awareness of cuppacupitis. Similarly, vacuous events, interpreted as contradictions, may be discarded in terms of information but can raise nevertheless awareness. E.g., a contradiction "You suffer from cuppacupitis and you do not suffer from coppacupitis" seems nonsensical and incomprehensible in terms of information but can still raise awareness of cuppacupitis.

Similar to Aumann [1999] and Heifetz et al. [2006], define for each agent $i \in I$ a conditional knowledge/belief operator $K_i : \Sigma \times \Sigma \longrightarrow \Sigma$ by for any events $E, F \in \Sigma$,

$$K_i(F \mid E) := \{\omega \in \Omega : \Pi_i(\omega \mid E) \subseteq F\}$$

if there is a state $\omega \in \Omega$ such that $\Pi_i(\omega \mid E) \subseteq F$ and otherwise by $K_i(F \mid E) = \emptyset^{S(E) \vee S(F)}$. The intended interpretation of $K_i(F \mid E)$ is that is is the event that conditional on receiving information that event E occurred, agent i knows/believes the event F. The unconditional knowledge/belief operator of Heifetz et al. [2006, 2008] can be recovered by conditioning on the vacuous event of least expressive power, i.e., $K_i(\cdot) := K_i(\cdot \mid \underline{S}^{\uparrow})$.

When assuming Generalized Conditional Reflexivity, we call $K_i(\cdot \mid \cdot)$ the conditional knowledge operator of agent i. Without this assumption, we call $K_i(\cdot \mid \cdot)$ the conditional belief operator of agent i. When stating properties on these operators, we use the convention that these properties apply to both operators unless otherwise stated.

Lemma 1 *For any events $E, F \in \Sigma$, $K_i(F \mid E)$ is an $(S(E) \vee S(F))$-based event.*

The conditional knowledge/belief operator satisfies properties familiar from S5 except for (conditional) negative introspection, which is weakened to 6.

Proposition 1 *The operator $K_i(\cdot \mid \cdot)$ satisfies the following properties:*

1. *Conditional Necessitation: For any event $E \in \Sigma$,*

$$K_i(S^{\uparrow}(E) \mid E) = S^{\uparrow}(E).$$

[7]Here and in what follows, we write $\Pi_i^{\uparrow}(\omega)$ for $(\Pi_i(\omega))^{\uparrow}$.

2. *Conditional Monotonicity: For any events* $E, F, G \in \Sigma$,

$$F \subseteq G \text{ implies } K_i(F \mid E) \subseteq K_i(G \mid E).$$

3. *Conditional Conjunction: For any events* $E, F_j \in \Sigma$,

$$K_i\left(\bigcap_j F_j \mid E\right) = \bigcap_j K_i(F_j \mid E).$$

5. *Conditional Positive introspection: For any events* $E, F \in \Sigma$,

$$K_i(F \mid E)) \subseteq K_i(K_i(F \mid E) \mid E).$$

6. *For any events* $E, F \in \Sigma$,

$$\neg K_i(F \mid E) \cap \neg K_i(\neg K_i(F \mid E) \mid E) \subseteq \neg K_i(\neg K_i(\neg K_i(F \mid E) \mid E) \mid E).$$

The conditional knowledge operator $K_i(\cdot \mid \cdot)$ *satisfies additionally:*

4. *Conditional Truth: For any events* $E, F \in \Sigma$,

$$K_i(F \mid E) \subseteq F.$$

PROOF. The proof of Conditional Necessitation follows directly from Conditional Confinement. The proofs of the other properties are analogous to the proof of the corresponding properties of [Heifetz et al., 2006, Proposition 2]. □

Analogous to the (unconditional) awareness operator in Heifetz et al. [2008], for any agent $i \in I$, the conditional awareness operator is defined for any events $E, F \in \Sigma$ by

$$A_i(F \mid E) := \{\omega \in \Omega : S_{\Pi_i(\omega \mid E)} \succeq S(F)\}$$

if there is a $\omega \in \Omega$ such that $S_{\Pi_i(\omega \mid E)} \succeq S(F)$ and by $A_i(F \mid E) = \emptyset^{S(E) \vee S(F)}$. The interpretation is that $A_i(F \mid E)$ is the event that conditional on having received information about the occurrence/existence of the event E, the agent is aware of event F. For any agent $i \in I$, the conditional unawareness operator is defined for any event $E, F \in \Sigma$ by

$$U_i(F \mid E) := \neg A_i(F \mid E).$$

The unconditional awareness and unawareness operators can be recovered by $A_i(\cdot) := A_i(\cdot \mid \underline{S})$ and $U_i(\cdot) := U_i(\cdot \mid \underline{S})$, respectively.

Proposition 2 *The following properties of conditional knowledge/belief and conditional awareness obtain:*

1. *Conditional Awareness: For all* $E \in \Sigma$, $A_i(E \mid E) = S^{\uparrow}(E)$.

2. *Strong Conditional Awareness: For all* $E, F \in \Sigma$ *with* $S(E) \succeq S(F)$, $A_i(F \mid E) = S^{\uparrow}(E)$.

3. *Conditional KU-Introspection: For all* $E, F \in \Sigma$, $K_i(U_i(F \mid E) \mid E) = \emptyset^{S(E) \vee S(F)}$.

4. *Conditional AU-Introspection: For all* $E, F \in \Sigma$, $U_i(F \mid E) = U_i(U_i(F \mid E) \mid E)$.

5. *Weak Conditional Necessitation: For all* $E, F \in \Sigma$, $A_i(F \mid E) = K_i(S^{\uparrow}(E) \mid E)$.

6. *Conditional Plausibility: For all* $E, F \in \Sigma$, $U_i(F \mid E) \subseteq \neg K_i(F \mid E) \cap \neg K_i(\neg K_i(F \mid E) \mid E)$.

7. *Strong Conditional Plausibility: For all* $E, F \in \Sigma$, $U_i(F \mid E) \subseteq \bigcap_{n=1}^{\infty} (\neg K_i)^n (F \mid E)(\mid E)^n$.

8. *Weak Negative Introspection: For all* $E, F \in \Sigma$, $\neg K_i(F \mid E) \cap A_i(F \mid E) \subseteq K_i(\neg K_i(F \mid E) \mid E)$.

9. *Conditional Symmetry: For all* $E, F \in \Sigma$, $A_i(\neg F \mid E) = A_i(F \mid E)$.

10. *Conditional A-Conjunction: For all* $\{E_j\} \subseteq \Sigma$, $\bigcap_j A_i(F_j \mid E) = A_i\left(\bigcap_j F_j \mid E\right)$.

11. *Conditional AK-Selfreflection: For all* $E, F \in \Sigma$, $A_i(K_i(F \mid E) \mid E) = A_i(F \mid E)$.

12. *Conditional AA-Selfreflection: For all* $E, F \in \Sigma$, $A_i(A_i(F \mid E) \mid E) = A_i(F \mid E)$.

13. *Conditional A-Introspection: For all* $E, F \in \Sigma$, $K_i(A_i(F \mid E) \mid E) = A_i(F \mid E)$.

Proof. 1. is implied by 2.

2. By the definition of the conditional awareness operator, $\omega \in A_i(F \mid E)$ if and only if $S_{\Pi_i(\omega \mid E)} \succeq S(F)$. Since $S_{\Pi_i(\omega' \mid E)} \succeq S(E)$ for all $\omega' \in S^{\uparrow}(E)$ by the definition of the conditional possibility correspondence $\Pi_i(\cdot \mid E)$, and $S(E) \succeq S(F)$, we have $S_{\Pi_i(\omega' \mid E)} \succeq S(F)$ for all $\omega' \in S^{\uparrow}(E)$.

The other properties follow by arguments analogous to the proof of [Heifetz et al., 2006, Proposition 3]. □

In Appendix A.1 we briefly introduced mutual and common knowledge/belief/awareness.

3 AGM Belief Revision

To capture the theory of AGM-belief revision, we require the conditional possibility correspondence to satisfy the following additional properties:

P2: For any event $E \in \Sigma$ with $E \neq \emptyset^{S(E)}$ and $\omega \in S^{\uparrow}(E)$, $\Pi_i(\omega \mid E) \subseteq E$.

P3: For any event $E \in \Sigma$ with $E \neq \emptyset^{S(E)}$ and $\omega \in S^{\uparrow}(E)$, $S_{\Pi_i(\omega)} \succeq S(E)$ implies $\Pi_i(\omega) \cap E \subseteq \Pi_i(\omega \mid E)$.

P4: For any event $E \in \Sigma$ with $E \neq \emptyset^{S(E)}$ and $\omega \in S^{\uparrow}(E)$, if $\Pi_i(\omega) \cap E \neq \emptyset$, then $\Pi_i(\omega \mid E) \subseteq \Pi_i(\omega) \cap E$.

P5: For any event $E \in \Sigma$ with $E \neq \emptyset^{S(E)}$ and $\omega \in S^{\uparrow}(E)$, $\Pi_i(\omega \mid E) \neq \emptyset$.

P7: For any events $E, F, G \in \Sigma$ with $E \cap F \neq \emptyset$ and $\omega \in (S(E) \vee S(F))^{\uparrow}$, $\Pi_i(\omega \mid E \cap F) \subseteq G$ implies $\Pi_i(\omega \mid E) \cap F \subseteq G$.

P8: For any events $E, F \in \Sigma$ with $E \neq \emptyset^{S(E)}$ and $F \neq \emptyset^{S(F)}$ and $\omega \in (S(E) \vee S(F))^{\uparrow}$, $\Pi_i(\omega \mid E) \cap F \neq \emptyset$ implies $\Pi_i(\omega \mid E \cap F) \subseteq \Pi_i(\omega \mid E) \cap F$.

When the lattice of spaces is a singleton and the unawareness structure collapses to an Aumann structure of Kripke frame (for which the accessibility relations are captured by the possibility correspondences), then these properties are equivalent to the frame properties of the Bonanno-Kripke-Lewis semantics [Bonanno, 2025a,b] for AGM belief revision. Section A.2 discusses the connection in more detail. The numbering follows the analogous properties in Bonanno [2025b].

Conditional knowledge in our setting satisfies the event-based analogues to the modal axioms/rules of inferences of Bonanno [2025b] model logic for AGM belief revision. The numbering follows the corresponding AGM axioms even though properties are the event-based counterparts of the modal axioms/rules of inferences, see Bonanno [2025b], suitably adapted to our more general framework with unawareness.

Proposition 3 (AGM-Belief Revision) *The conditional knowledge operator satisfies the following:*

K1: For any events $E, F, G \in \Sigma$ with $E \neq \emptyset^{S(E)}$,

$$K_i(F \mid E) \cap K_i(\neg F \vee G \mid E) \subseteq K_i(G \mid E).$$

K2: P2 implies for any event $E \in \Sigma$ with $E \neq \emptyset^{S(E)}$,

$$K_i(E \mid E) = S^{\uparrow}(E).$$

K3: P3 implies for any events $E, F \in \Sigma$ with $E \neq \emptyset^{S(E)}$,

$$A_i(E) \cap K_i(F \mid E) \subseteq K_i(\neg E \vee F).$$

K4: P4 implies for any events $E, F \in \Sigma$ with $E \neq \emptyset^{S(E)}$,

$$A_i(E) \cap A_i(F) \cap \neg K_i(\neg E) \cap K_i(\neg E \vee F) \subseteq K_i(F \mid E).$$

K5: P5 implies for any events $E, F \in \Sigma$ with $E \neq \emptyset^{S(E)}$,

$$K_i(F \mid E) \subseteq \neg K_i(\neg F \mid E) \cap A_i(\neg F \mid E).$$

K6: For any events $E, F, G \in \Sigma$ with $E \neq \emptyset^{S(E)}$,

$$F = G \text{ implies } K_i(F \mid E) = K_i(G \mid E).$$

K7: P7 implies for any events $E, F, G \in \Sigma$ with $E \cap F \neq \emptyset^{S(E \cap F)}$,

$$K_i(G \mid E \cap F) \subseteq K_i(\neg F \vee G \mid E).$$

K8: P8 implies for any events $E, F, G \in \Sigma$ with $E \cap F \neq \emptyset^{S(E \cap F)}$,

$$\neg K_i(\neg F \mid E) \cap K_i(\neg F \vee G \mid E) \subseteq K_i(F \cap G \mid E \cap F).$$

4 Awareness Revision

To capture desirable properties of awareness revision, we require the conditional possibility correspondence to satisfy the following additional properties:

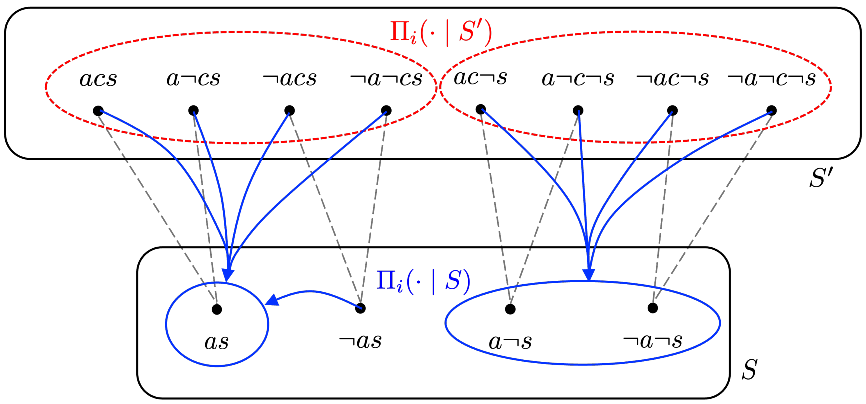
AR1: For all $E \in \Sigma$, $\omega \in S^{\uparrow}(E)$, $\Pi_i(\omega \mid S^{\uparrow}(E)) = \{\omega' \in S_{\Pi_i(\omega)} : \Pi_i(\omega') = \Pi_i(\omega)\}^{S(E) \vee S_{\Pi_i(\omega)}}$.

AR2: For all $E \in \Sigma$, $\omega \in S^{\uparrow}(E)$, $S_{\Pi_i(\omega | S^{\uparrow}(E))} = S_{\Pi_i(\omega|E)}$.

AR3: For all $E \in \Sigma$, $\omega \in S^{\uparrow}(E)$, $\Pi_i(\omega \mid S^{\uparrow}(E)) = \Pi_i(\omega \mid \emptyset^{S(E)})$.

To parse the properties, recall that for any set $D \subseteq S$ and space $S' \succeq S$, $D^{S'} = (r_S^{S'})^{-1}(D)$ denotes the "elaboration" of D at the more expressive awareness level S'. Also recall that $\Pi_i(\cdot) = \Pi_i(\cdot \mid \underline{S})$ is the unconditional possibility correspondence identified by conditioning on the least expressive space $\underline{S}$ of the lattice $\mathcal{S}$.

Figure 1: Example of Awareness Revision and Belief



AR1 means that when the agent's awareness is raised to $S(E)$, she considers possible states at least at awareness level $S(E)$. In particular, she considers possible elaborations of all the worlds she considered possible before. Moreover, she might also consider possible elaborations of worlds that she considered impossible before but in which she would have considered states possible that she did consider possible before. That is, she might realize that previously she had some false beliefs. An example may help to clarify such a case. Assume that the agent is only aware of artery decease a as well as the symptom chest pain s. She can conceive of four states of the world depicted in the lower space S of Figure 1. Her belief and awareness conditional on S is given by the blue possibility correspondence. It is such that when she has the symptom, then she must have artery decease. In particular, she thinks it is impossible to have the symptom and not artery decease. Note that in state $\neg as$, she believes in state as, violating (generalized) reflexivity. Moreover, whenever she has no symptom, she does not know whether she has artery decease. Once she is aware of S' and thus the existence of cuppacupitis c, she can conceive of all states also in space S'. Her belief and awareness conditional on S is given by the red dotted possibility sets. In particular, when she has symptoms, it is not the inverse image of $\{as\}$ but the inverse image of $\{as, \neg as\}$ in S', which is $\{acs, a\neg cs, \neg acs, \neg a\neg cs\}$. Notice that not only her awareness changed but also her belief with respect to the event $\{\neg as\}^{\uparrow} = \{\neg as, \neg acs, \neg a\neg cs\}$ has changed upon becoming aware of the existence of c. While previously she considered this event to be impossible, she now does not rule it out. This is presumably that she realizes that having symptoms does not imply having an artery decease. The symptoms may also be due to cuppacupitis. Moreover, she also realizes that having symptoms may not imply any decease she is aware of.

When (conditional) belief is strengthened to (conditional) knowledge by also imposing generalized (conditional) reflexivity on the (conditional) possibility correspondences, then AR1 is equivalent to

AR1K: For all $E \in \Sigma$, $\omega \in S^{\uparrow}(E)$, $\Pi_i(\omega \mid S^{\uparrow}(E)) = \Pi_i(\omega)^{S(E) \vee S_{\Pi_i(\omega)}}$.

Note that if the conditional possibility correspondences also satisfy generalized conditional reflexivity, then $\Pi_i(\omega \mid \cdot) = \{\omega' \in S_{\Pi_i(\omega|\cdot)} : \Pi_i(\omega' \mid \cdot) = \Pi_i(\omega \mid \cdot)\}$. To see this note that if $\omega' \in \Pi_i(\omega \mid \cdot)$, then by conditional confinement $\omega' \in S_{\Pi_i(\omega|\cdot)}$ and $\Pi_i(\omega' \mid \cdot) = \Pi_i(\omega \mid \cdot)$ by conditional stationarity. Conversely, if $\omega'' \in \{\omega' \in S_{\Pi_i(\omega|\cdot)} : \Pi_i(\omega' \mid \cdot) = \Pi_i(\omega \mid \cdot)\}$ then either $\omega'' \in \Pi_i(\omega \mid \cdot)$, in which case there is nothing to prove. Or $\omega'' \notin \Pi_i(\omega \mid \cdot)$, which contradicts generalized conditional reflexivity.

AR1K means that becoming aware does not carry information w.r.t. events the agent has been aware of. It preserves knowledge and ignorance of events the agent has been aware of. It is essentially the qualitative counterpart of "reverse Bayesianism" [Karni and Vierø, 2013, Hayashi, 2012].

AR2 means that receiving information that event E occurred does not change awareness when the agent has been aware of E before. AR3 means that contradictions are not informative or confusing but carry awareness.

Proposition 4 *The conditional belief/knowledge and awareness operators satisfy the following:*

0. *AR1 implies for all $E, F, G \in \Sigma$ with $S(G) \preceq S(F) \preceq S(E)$,*

$$A_i(G \mid S^{\uparrow}(F)) \cap S^{\uparrow}(E) \quad = \quad A_i(G \mid S^{\uparrow}(E))$$

1. *AR1K implies for all $E, F, G \in \Sigma$ with $S(G) \preceq S(F) \preceq S(E)$,*

$$\begin{aligned} K_i(G \mid S^{\uparrow}(F)) \cap S^{\uparrow}(E) &= K_i(G \mid S^{\uparrow}(E)) \\ \neg K_i(G \mid S^{\uparrow}(F)) \cap S^{\uparrow}(E) &= \neg K_i(G \mid S^{\uparrow}(E)) \end{aligned}$$

2. *AR2 implies for all $E, F \in \Sigma$,*

$$A_i(F \mid S^{\uparrow}(E)) \quad = \quad A_i(F \mid E).$$

   Moreover, AR1 and AR2 imply for all $E, F, G \in \Sigma$ with $S(E) \succeq S(F)$,

$$A_i(G \mid F) \cap S^{\uparrow}(E) \quad \subseteq \quad A_i(G \mid E)$$

3. *AR2 or AR3 implies for all $E, F \in \Sigma$,*

$$A_i(F \mid S^{\uparrow}(E)) \quad = \quad A_i(F \mid \emptyset^{S^{\uparrow}(E)})$$

   AR3 implies for all $E, F \in \Sigma$,

$$K_i(F \mid S^{\uparrow}(E)) \quad = \quad K_i(F \mid \emptyset^{S^{\uparrow}(E)})$$

Note that 1. can be interpreted as a qualitative version of "reverse Bayesianism". Raising awareness preserves knowledge and ignorance of events the agent has been previously aware of.

A Appendices

A.1 Common Conditional Belief/Knowledge/Awareness

Conditional mutual knowledge/belief and conditional common knowledge/belief operators can be defined analogous to their unconditional counterparts in Heifetz et al. [2006, 2008]. I.e., the conditional mutual knowledge/belief operator is defined by for all $E, F \in \Sigma$,

$$K(F \mid E) := \bigcap_{i \in I} K_i(F \mid E).$$

The interpretation is that this is the event that conditional on receiving information that E occurred, everybody knows/believes that F occurred. Similarly, the conditional common knowledge/belief operator is defined by for any events $E, F \in \Sigma$,

$$CK(F \mid E) := \bigcap_{n=1}^{\infty} K^n(F \mid E).$$

The unconditional versions can simply be recovered from $K(\cdot) := K(\cdot \mid \underline{S})$ and $CK(\cdot) := CK(\cdot \mid \underline{S})$, respectively.

For interactive awareness, note that analogous to unconditional awareness for any two agents $i, j \in I$, $A_i(A_j(F \mid E) \mid E) = A_i(F \mid E)$. Analogous to the unconditional versions (see [Heifetz et al., 2008, Definition 22]), we can define the conditional mutual awareness and conditional common awareness operators, respectively, by for all $E, F \in \Sigma$,

$$A(F \mid E) := \bigcap_{i \in I} A_i(F \mid E)$$

and

$$CA(F \mid E) := \bigcap_{n=1}^{\infty} A^n(F \mid E).$$

The unconditional versions can be recovered from $K(\cdot) := K(\cdot \mid \underline{S})$ and $CK(\cdot) := CK(\cdot \mid \underline{S})$, respectively. The properties of the unconditional mutual knowledge/belief, common knowledge/belief, mutual awareness, and common awareness (see [Heifetz et al., 2008, Proposition 11]) can be extended in a straightforward way to conditional mutual knowledge/belief, conditional common knowledge/belief, conditional mutual awareness, and conditional common awareness, respectively.

A.2 Connection to Bonanno-Kripke-Lewis semantics for AGM belief revision

Consider an Aumann structure $\langle I, S, (\Pi_i)_{i \in I} \rangle$ with

- a non-empty set of agents I,
- a nonempty space of states S, and
- for each agent $i \in I$, a possibility correspondence $\Pi_i : S \longrightarrow 2^S \setminus \{\emptyset\}$.

Clearly, such an Aumann structure is equivalent to a Kripke frame in which the accessibility relations are modeled equivalently by possibility correspondences. It is also equivalent to a degenerated (unconditional) unawareness structure with a singleton lattice of spaces. It is well known that such structures cannot model unawareness [Dekel et al., 1998].

An Aumann structure is a Bonanno-Kripke-Lewis frame when we add a selection function $f : S \times (2^S \setminus \{\emptyset\}) \longrightarrow 2^S$. For each state $\omega \in S$ and event $E \in 2^S \setminus \{\emptyset\}$, $f(\omega, E)$ is interpreted as the set of states "closest" to ω conditional on E. $\langle I, S, (\Pi_i)_{i \in I}, f \rangle$ is a Bonanno-Kripke-Lewis frame [Bonanno, 2025a,b] in which the accessibility relations are equivalently modeled with possibility correspondences.

Consider the special case of a conditional unawareness structure $\langle I, (\mathcal{S}, \succeq), (r_S^{S'})_{S,S' \in \mathcal{S} \succeq S}, ((\Pi_i(\cdot \mid E)_{E \in \Sigma})_{i \in I} \rangle$ with a singleton lattice $\mathcal{S} = \{S\}$. This boils down to what one could call a conditional Aumann structure or conditional Kripke frame (in which conditional accessibility relations are equivalently modeled by conditional possibility correspondences) $\langle I, S, ((\Pi_i(\cdot \mid E)_{E \in 2^S \setminus \{\emptyset\}})_{i \in I} \rangle$.

What is the connection between conditional Aumann structures and Bonanno-Kripke-Lewis frames? If $\Pi_i(\omega \mid E) := \bigcup_{\omega' \in \Pi_i(\omega)} f(\omega', E)$, then each of the properties of the conditional possibility correspondence imposed in Section 3 (suitably specialized to the single state space) has a corresponding frame property in the Bonanno-Kripke-Lewis frame listed in [Bonanno, 2025b, Figure 1]. From Bonanno [2025b] it is now clear how the properties of the conditional possibility correspondence relate to the AGM axioms, respectively, as well as to the corresponding the modal axiom/rule of inference listed in [Bonanno, 2025b, Figure 2] when Bonanno-Kripke-Lewis frames are extended to models.

B Proofs

Proof of Lemma 1

Note that for the proof of Lemma 1, we cannot use the arguments of the proof of [Heifetz et al., 2006, Proposition 1] because there the analogous result is proved for the unconditional possibility correspondence using the extra assumption of generalized reflexivity. Here we do not necessarily impose generalized conditional reflexivity.

Let $K_i(F \mid E)$ be nonempty. Then $\omega \in K_i(F \mid E)$ if and only if $\Pi_i(\omega \mid E) \subseteq F$. Since F is an event, it has a base space $S(F)$. By Conditional Confinement, $\Pi_i(\omega \mid E) \subseteq S$ for some $S \in \mathcal{S}$ with $S \succeq S(F) \vee S(F)$. Thus, $(\Pi_i(\omega \mid E))_{S(F) \vee S(E)}$ is defined. Moreover, $(\Pi_i(\omega \mid F))_{S(F) \vee S(E)} \subseteq F \cap S(F) \vee S(E)$. By Projections Preserve Conditional Knowledge, $\Pi_i(\omega_{S(F) \vee S(E)} \mid E) \subseteq F$.

Define $B := \{\omega \in S(F) \vee S(E) : \Pi_i(\omega \mid E) \subseteq F \cap S(F) \vee S(E)\}$. By the definition of the conditional knowledge/belief operator, $B = K_i(F \mid E) \cap S(F) \vee S(E)$.

We need to show that $B^\uparrow = K_i(F \mid E)$. $\omega \in B^\uparrow$ if and only if $\omega_{S(F) \vee S(E)} \in B$ if and only if $\Pi_i(\omega_{S(F)} \mid E) \subseteq F \cap S(F) \vee S(E)$. By Projections Preserve Conditional Knowledge, $(\Pi_i(\omega \mid E))_{S(F) \vee S(E)} \subseteq F \cap S(F) \vee S(E)$. By F being an event, $\Pi_i(\omega \mid E) \subseteq F$ if and only if $\omega \in K_i(F \mid E)$. For the converse, let $\omega \in K_i(F \mid E)$ if and only if $\Pi_i(\omega \mid E) \subseteq F$. By Conditional Confinement, $(\Pi_i(\omega \mid E)_{S(F) \vee S(E)}$ is defined, and F being an event implies $(\Pi_i(\omega \mid E)_{S(F) \vee S(E)} \subseteq F$. By Projections Preserve Conditional Knowledge, $\Pi_i(\omega_{S(F) \vee S(E)} \mid E) \subseteq F \cap S(F) \vee S(E)$ if and only if $\omega_{S(F) \vee S(E)} \in B$ if and only if $\omega \in B^\uparrow$. □

Proof of Proposition 3

K1: Let $\omega \in K_i(F \mid E) \cap K_i(\neg F \vee G \mid E)$. By the definition of the conditional knowledge operator, $\omega \in K_i(F \mid E)$ if and only if $\Pi_i(\omega \mid E) \subseteq F$ as well as $\omega \in K_i(\neg F \vee \mid E)$ if and only if $\Pi_i(\omega \mid E) \subseteq \neg F \vee G$. We conclude $\Pi_i(\omega \mid E) \subseteq F \cap (\neg F \vee G) = F \cap \neg F \vee F \cap G = \emptyset^{S(F)} \vee F \cap G \subseteq G$. Thus, by the definition of the conditional knowledge operator, $\omega \in K_i(G \mid E)$.

K2: By the definition of the conditional knowledge operator, $K_i(E \mid E) \subseteq S^\uparrow(E)$. Conversely, if $E \neq \emptyset^{S(E)}$, for all $\omega \in S^\uparrow(E)$, $\Pi_i(\omega \mid E) \subseteq E$ by P2. Thus, $\omega \in K_i(E \mid E)$ by the definition of the conditional knowledge operator.

K3: By the definition of the conditional knowledge operator, $\omega \in K_i(F \mid E)$ if and only if $\Pi_i(\omega \mid E) \subseteq F$. By the definition of the awareness operator, $\omega \in A_i(E)$ if and only if $S_{\Pi_i(\omega)} \succeq S(E)$. By P3, if $E \neq \emptyset^{S(E)}$, then $\Pi_i(\omega) \cap E \subseteq \Pi_i(\omega \mid E)$. It follows that $\Pi_i(\omega) \cap E \subseteq F$. This implies $\Pi_i(\omega \mid S^{\uparrow}(E)) \subseteq \neg E \vee F$. Hence, $\omega \in K_i(\neg E \vee F)$ by the definition of the knowledge operator.

K4: Let $\omega \in A_i(E) \cap A_i(F) \cap \neg K_i(\neg E) \cap K_i(\neg E \vee F)$ By the definition of the knowledge operator, $\omega \in \neg K(\neg E)$ if and only if $\Pi_i(\omega) \not\subseteq \neg E$. Since $\omega \in A_i(E)$, by the definition of the awareness operator, $S_{\Pi_i(\omega)} \succeq S(E)$. Together with assumption $E \neq \emptyset^{S(E)}$, we have $\Pi_i(\omega) \cap E \neq \emptyset$. By P4, $\Pi_i(\omega \mid E) \subseteq \Pi_i(\omega) \cap E$.

By the definition of the knowledge operator, $\omega \in K(\neg E \cup F)$ if and only if $\Pi_i(\omega) \subseteq \neg E \cup F$. Hence $\Pi_i(\omega) \cap E \subseteq F$. Earlier, we concluded $\Pi_i(\omega \mid E) \subseteq \Pi_i(\omega) \cap E$, thus $\Pi_i(\omega \mid E) \subseteq F$, and hence by the definition of the conditional knowledge operator $\omega \in K_i(F \mid E)$.

K5: By the definition of the conditional knowledge operator, $\omega \in K_i(F \mid E)$ if and only if $\Pi_i(\omega \mid E) \subseteq F$. By P5, $\Pi_i(\omega \mid E) \neq \emptyset$. Thus, $\Pi_i(\omega \mid E) \not\subseteq \neg F$. Hence, (1) by the definition of the conditional knowledge operator, $\omega \in \neg K_i(\neg F \mid E)$, and (2) $S_{\Pi_i(\omega \mid E)} \succeq S(F) \vee S(E)$ and thus by the definition of the conditional awareness operator, $\omega \in A_i(\neg F \mid E)$.

K6: By the definition of the conditional knowledge operator $\omega \in K_i(F \mid E)$ if and only if $\Pi_i(\omega \mid E) \subseteq F = G \supseteq \Pi_i(\omega \mid E)$ if and only if $K_i(G \mid E) \ni \omega$.

K7: By the definition of the conditional knowledge operator $\omega \in K_i(G \mid E \cap F)$ if and only if $\Pi_i(\omega \mid E \cap F) \subseteq G$. By P7, $\Pi_i(\omega \mid E) \cap F \subseteq G$. It implies $\Pi_i(\omega \mid E) \subseteq \neg F \vee G$. Thus, $\omega \in K_i(\neg F \vee G \mid E)$ by the definition of the conditional knowledge operator.

K8: Let $\omega \in \neg K_i(\neg F \mid E) \cap K_i(\neg F \vee G \mid E)$. By the definition of the conditional knowledge operator, $\omega \in \neg K_i(\neg F \mid E)$ if and only if $\Pi_i(\omega \mid E) \not\subseteq \neg F$. Assumption $E \cap F \neq \emptyset^{S(E \cap F)}$ implies that $F \neq \emptyset^{S(F)}$. Thus, $\Pi_i(\omega \mid E) \cap F \neq \emptyset$. Then by P8, $\Pi_i(\omega \mid E \cap F) \subseteq \Pi_i(\omega \mid E) \cap F$.

We conclude $\Pi_i(\omega \mid E \cap F) \subseteq F$.

By the definition of the conditional knowledge operator, $\omega \in K_i(\neg F \vee G \mid E)$ if and only if $\Pi_i(\omega \mid E) \subseteq \neg F \vee G$. It implies $\Pi_i(\omega \mid E) \cap F \subseteq G$. Earlier we observed $\Pi_i(\omega \mid E \cap F) \subseteq \Pi_i(\omega \mid E) \cap F$. Hence, $\Pi_i(\omega \mid E \cap F) \subseteq G$.

We conclude $\Pi_i(\omega \mid E \cap F) \subseteq F \cap G$. Hence, by the definition of the conditional knowledge operator $\omega \in K_i(F \cap G \mid E \cap F)$. □

Proof of Proposition 4

0. Let $\omega \in A_i(G \mid S^{\uparrow}(F)) \cap S^{\uparrow}(E)$. By the definition of the conditional awareness operator, $\omega \in A_i(G \mid S^{\uparrow}(F))$ if and only if $S_{\Pi_i(\omega \mid S^{\uparrow}(F))} \succeq S(F)$. By AR1, $S_{\Pi_i(\omega \mid S^{\uparrow}(F))} = S(F) \vee S_{\Pi_i(\omega)}$. Since $\omega \in S^{\uparrow}(E)$, $\Pi_i(\omega \mid S^{\uparrow}(E))$ is defined. Also by AR1, $S_{\Pi_i(\omega \mid S^{\uparrow}(E))} = S(E) \vee S_{\Pi_i(\omega)}$. Since $S(G) \preceq S(F) \preceq S(E)$, $S(G) \preceq S(F) \vee S_{\Pi_i(\omega)} \preceq S(E) \vee S_{\Pi_i(\omega)}$ if and only if $S(G) \preceq S_{\Pi_i(\omega \mid S^{\uparrow}(F))} \preceq S_{\Pi_i(\omega \mid S^{\uparrow}(E))}$. By the definition of the conditional awareness operator, $\omega \in A_i(G \mid S^{\uparrow}(E))$.

1. Let $\omega \in K_i(G \mid S^{\uparrow}(F)) \cap S^{\uparrow}(E)$. By the definition of the conditional awareness operator, $\omega \in K_i(G \mid S^{\uparrow}(F))$ if and only if $\Pi_i(\omega \mid S^{\uparrow}(F)) \subseteq G$. By AR1K, $\Pi_i(\omega \mid S^{\uparrow}(F)) = (\Pi_i(\omega))^{S(F) \vee S_{\Pi_i(\omega)}}$. Since $\omega \in S^{\uparrow}(E)$, $\Pi_i(\omega \mid S^{\uparrow}(E))$ is defined. Also by AR1K, $\Pi_i(\omega \mid S^{\uparrow}(E)) = (\Pi_i(\omega))^{S(E) \vee S_{\Pi_i(\omega)}}$. Since $S(G) \preceq S(F) \preceq S(E)$, $\Pi_i(\omega \mid S(F)) \subseteq G$ if and only if $\Pi_i(\omega \mid S(E)) \subseteq G$ if and only if $\omega \in K_i(G \mid S^{\uparrow}(E))$ by the definition of the conditional knowledge operator.

$K_i(G \mid S^{\uparrow}(F)) \cap S^{\uparrow}(E) = K_i(G \mid S^{\uparrow}(E))$ if and only if $\neg(K_i(G \mid S^{\uparrow}(F)) \cap S^{\uparrow}(E)) = \neg K_i(G \mid S^{\uparrow}(E))$. Note that left-hand-side, $\neg(K_i(G \mid S^{\uparrow}(F)) \cap S^{\uparrow}(E)) = \neg K_i(G \mid S^{\uparrow}(F)) \vee \emptyset^{\uparrow}(E)) = \neg K_i(G \mid S^{\uparrow}(F)) \wedge S^{\uparrow}(E))$, where the last equality follows from the definition of disjunction in unawareness structures. We conclude $x \neg K_i(G \mid S^{\uparrow}(F)) \wedge S^{\uparrow}(E)) = \neg K_i(G \mid S^{\uparrow}(E))$.

2. By the definition of the conditional awareness operator, $\omega \in A_i(F \mid S^{\uparrow}(E))$ if and only if $S_{\Pi_i(\omega|S^{\uparrow}(E)} \succeq S(F)$. By AR2, $S_{\Pi_i(\omega|S^{\uparrow}(E))} = S_{\Pi_i(\omega|E)}$. Thus, $S_{\Pi_i(\omega|E)} \succeq S(F)$ if and only if $\omega \in A_i(F \mid E)$ by the conditional awareness operator.

Let $\omega \in A_i(G \mid F) \cap S^{\uparrow}(E)$. By the definition of the conditional awareness operator, $\omega \in A_i(G \mid F)$ if and only if $S_{\Pi_i(\omega|F)} \succeq S(G)$. We have $S_{\Pi_i(\omega)} \vee S(F) = S_{\Pi_i(\omega|S(F))} = S_{\Pi_i(\omega|F)} \succeq S(G)$, where the first equality follows from AR1 and the second equality from AR2. Since $S(E) \succeq S(F)$ and $\omega \in S^{\uparrow}(E)$, $\Pi_i(\omega \mid E)$ is defined. $S_{\Pi_i(\omega)} \vee S(E) = S_{\Pi_i(\omega|S(E))} = S_{\Pi_i(\omega|E)}$ where the first equality follows from AR1 and the second equality from AR2. Since $S(E) \succeq S(F)$, $S_{\Pi_i(\omega)} \vee S(E) \succeq S_{\Pi_i(\omega)} \vee S(F)$. Thus, $S_{\Pi_i(\omega|E)} \succeq S(G)$ if and only if $\omega \in A_i(G \mid E)$ by the definition of the conditional awareness operator.

3. For vacuous events, AR3 implies AR2. Thus, $A_i(F \mid S^{\uparrow}(E)) = A_i(F \mid \emptyset^{S^{\uparrow}(E)})$ follows from 2.

The claim $K_i(F \mid S^{\uparrow}(E)) = K_i(F \mid \emptyset^{S^{\uparrow}(E)})$ follows directly from AR3 and the definition of the conditional knowledge operator. □

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