Tropical Expected Utility with Subjective Ranking Functions

Burkhard C. Schipper*

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Abstract

Courts, scientists, physicians, modern AI, etc. routinely accept hypotheses at graded thresholds. Ranking functions (Spohn, 1988, 2012) model such graded disbelief and are standard in formal epistemology and artificial intelligence. We characterize the decision maker's preferences over integer-valued acts for which there exist a subjective ranking function ρ on states of nature, an integer-valued utility function u, and a pessimistic or optimistic orientation σ (sign variable) such that for any two acts X and Y,

$$X \succsim Y \text{ if and only if } \sigma \cdot \min_{\omega\in\Omega} \big\{\sigma \cdot u(X(\omega)) + \rho(\omega)\big\} \geq \sigma \cdot \min_{\omega\in\Omega} \big\{\sigma \cdot u(Y(\omega)) + \rho(\omega)\big\}.$$

In the language of tropical mathematics, a ranking function is an idempotent measure and a point of a tropical projective space or of its boundary, our representation is an idempotent integral of utility, and our characterization (in the special case of linear utility) is a Riesz-type representation for preferences. Tropical expected utility satisfies a tropical independence axiom and thus does not accommodate preference reversals in Allais and Ellsberg examples, while featuring threshold effects not captured by subjective expected utility.

Keywords: Ranking functions, tropical mathematics, subjective expected utility, idempotent analysis, tropical independence, pessimism, optimism.

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*Department of Economics, University of California, Davis, bcschipper@ucdavis.edu. I thank Gabriele Kern-Isberner for teaching me about ranking functions. Moreover, I thank Wolfgang Spohn and Franz Huber for discussions about ranking theory and avenues for decision theory with ranking functions.

1 Introduction

People and institutions routinely accept hypotheses at graded thresholds and act on them.[1] Courts distinguish standards of proof, from a preponderance of the evidence through clear and convincing evidence to proof beyond reasonable doubt. Physicians rule out diagnoses, engineers certify components, and scientists accept hypotheses at various levels of confidence. In modern artificial intelligence, classifiers in machine learning assign a label once its score exceeds those of the alternatives by a margin. In each of these cases, the verdict is not a probability that the hypothesis is true. It is a hypothesis that is accepted, together with how firmly it is accepted, or equivalently, how surprised one would be if it turned out false. Such graded acceptance has a structure of its own. Hypotheses accepted at a given standard can be combined, so that accepting each of two hypotheses at a standard means accepting both at that standard. When accepted hypotheses turn out false, beliefs are revised rather than abandoned. Can we construct a decision theory under uncertainty that takes graded disbelief directly as the representation of personal beliefs, that reveals it from choices, that is as tractable as subjective expected utility, that generates novel explanations for existing behavioral phenomena, and that allows us to explore novel behavioral implications of choice with graded disbeliefs?

Spohn (1988, 2012) developed an extensive theory that represents a decision maker's beliefs by degrees of disbelief, called ranks. Each state receives a rank in $\mathbb{N} \cup \{\infty\}$, the rank of an event is the minimum rank of its states, and rank 0 means that the decision maker does not disbelieve the event at all. A hypothesis is accepted at level k if its negation has a rank weakly larger than k. The negation of a conjunction of hypotheses is the union of their negations, and the rank of a union of events is the minimum of their ranks. Hence the hypotheses accepted at any level are closed under conjunction, avoiding problems like the lottery paradox (Kyburg, 1961). The theory of ranking functions has been further developed in artificial intelligence (Goldszmidt and Pearl, 1992a, b, 1996). Earlier, in economics, Shackle (1949) proposed a theory of degrees of potential surprise and used it to analyze investment and speculation. Ranking functions obey laws that parallel those of probability, with minimum in place of sum and sum in place of product. Ranking functions have become a standard model of belief and belief revision in philosophy and artificial intelligence (see Huber, 2019, and Kern-Isberner, Skovgaard-Olsen, and Spohn, 2021, for surveys). Several applications have been developed. Schipper and Kern-Isberner (2026) develop the theory of games with ranking functions in both normal and extensive form, with rationalizability notions that use ranking functions rather than probabilities as (dis)beliefs, and they demonstrate sophisticated behavior such as forward induction and infections and contagion of behavior through higher-order (dis)beliefs. Schipper (2026a) extends Aumann's famous no-agreeing-to-disagree theorem to ranking functions, studies the possibility of speculative trade, characterizes common prior ranking functions by the absence of speculation, and shows that learning and belief revision with ranking functions lead to agreement. Schipper (2026b) defines the analog of the Arrow–Pratt coefficient of risk aversion and extends Pratt's (1964) results to ranking functions, which yields comparative statics with respect to risk aversion for decision makers who use ranking functions. He also analyzes investment and insurance with ranking functions. Ehrig and Schmidt (2026) use ranking functions to model how entrepreneurs persuade investors and other stakeholders through belief revision. When these applications deliver known results, they demonstrate the robustness of

[1] Our notion of accepting a hypothesis differs from the hypothesis testing model of Ortoleva (2012). There, a hypothesis is a prior, i.e., a probability measure over states, which the decision maker uses until news to which it assigns a probability below a threshold leads her to switch to another prior. Here, a hypothesis is an event, and accepting it means disbelieving its negation, without reference to probabilities. Decision making with reaction to surprising news is the subject of a companion paper (Schipper, 2026c).

standard methods. When they generate novel results, such as portfolio inertia, endowment effects, and bid–ask spreads, they show that some degree of insensitivity to changes in beliefs or valuations is consistent with rational behavior. Moreover, as we observe in this paper, ranking functions fit naturally into tropical geometry.[2] This new tool set may allow us to derive results that have proved intractable for standard methods or may facilitate novel applications in the future. In any case, such developments presuppose a solid behavioral foundation for subjective ranking functions, which is the goal of this paper.

We follow the approach of subjective expected utility theory (Savage, 1954). The decision maker's choices are described by a preference relation over acts, i.e., over assignments of outcomes to states. We look for properties of this preference relation under which the decision maker behaves as if she had personal beliefs given by a ranking function and a utility function over outcomes, and chose acts that maximize the tropical expectation of utility with respect to her ranking function. Neither the ranking function nor the utility function is exogenously given to the analyst. Both are derived endogenously from the decision maker's choices, and the behavioral properties identify them uniquely up to the transformations that leave the representation unchanged. This has three benefits. First, it gives subjective ranking functions a behavioral meaning. The behavior of the decision maker provides a window into subjective disbeliefs represented by ranking functions. In artificial intelligence, ranks are commonly interpreted as orders of magnitude of infinitesimal probabilities (Pearl, 1993; Wilson, 1995; Goldszmidt and Pearl, 1996). Our representation theorem shows that ranking theory does not need this interpretation for its foundation. Ranking functions are derived directly from behavior, without the detour through probabilities, and neither the laws of ranking functions nor our properties refer to probabilities. Second, it makes our tropical expected utility with subjective ranking functions testable with behavioral experiments and facilitates the elicitation of subjective ranking functions from behavior. An analyst interested in finding out whether or not a decision maker's beliefs can be represented by ranking functions can simply test our behavioral properties, most directly tropical independence, P3. Third, the behavioral properties make explicit what a decision maker commits to when she decides on the basis of a ranking function, so that the decision rule can be judged normatively. They describe what it means to act rationally on graded disbelief.

Our starting point is an observation about the mathematics of ranking functions, which to our surprise we could not find in the literature on ranking functions. The laws of ranks are the laws of probability over the tropical semiring, in which addition is replaced by the minimum and multiplication by addition, $a \oplus b := \min\{a, b\}$ and $a \odot b := a + b$ (Speyer and Sturmfels, 2009). The rank of an event is the tropical sum of the ranks of its states, and conditioning divides tropically, i.e., subtracts. In the language of idempotent analysis, which develops measure, integration, and probability theory over idempotent semirings such as the tropical one (Kolokoltsov and Maslov, 1997; Akian, 1999; Del Moral and Doisy, 1999; Kolokoltsov, 2001), a ranking function is a normalized idempotent measure in the sense of Maslov (Kolokoltsov and Maslov, 1997, Section 1.4). Tropical geometry and idempotent analysis are two sides of mathematics over the tropical semiring, the former studying its geometry (e.g., Develin and Sturmfels, 2004), the latter measure and integration. Ranking functions belong to both, as points of tropical projective space (see Section 2) and as idempotent measures. The natural analog of an expectation is then the idempotent integral, $\bigoplus\{\rho(\omega) \odot X(\omega) : \omega \in \Omega\} = \min_{\omega\in\Omega}\{X(\omega) + \rho(\omega)\}$, and its counterpart in the max-plus semiring is the $(\max, +)$-expectation of idempotent probability theory. The two are related by the isomorphism $x \mapsto -x$ between the min-plus and the max-plus semiring. A decision theory for ranking functions

[2]Tropical geometry has recently been applied to economics by Baldwin and Klemperer (2019) and Tran and Yu (2019).

should therefore be an expected utility theory over the tropical semiring, a tropical expected utility theory.[3] This viewpoint guides the paper throughout. Our properties are preference counterparts of the operations of the tropical semiring. Our representation theorem with linear utility is a preference counterpart of the idempotent analog of the Riesz representation of linear functionals by integrals (Kolokoltsov and Maslov, 1987; Kolokoltsov and Maslov, 1997, Theorem 1.9; Kolokoltsov, 2001), and our general representation theorem extends it to possibly nonlinear utility. Our measures of risk attitude are tropical versions of the classical ones. The certainty equivalent is a tropical power mean, and the coefficient of risk aversion is a tropical quotient (see Schipper, 2026b).

Prior approaches to an expected utility theory with ranking functions are dissatisfactory: First, they take ranking functions as exogenously given and something objective rather than derive them from choices. Both Giang and Shenoy (2000) and Spohn (2025) translate the von Neumann–Morgenstern axioms to ranking functions. Their objects of choice are ranking function lotteries whose uncertainty is already described by ranks, the analog of objective lotteries in von Neumann and Morgenstern (1947). Yet, we are not aware of randomization devices that would generate objective ranking function lotteries. Without such devices, their theories are not testable with choice experiments. Moreover, they cannot provide a behavioral foundation for subjective ranking functions. What is missing is the analog of subjective expected utility (Savage, 1954), i.e., a theory in which the decision maker has preferences over acts, and her personal ranking function is derived from her choices together with her utility.

Second, each existing approach commits exogenously to a disposition toward risk. The criterion of Spohn (2025) is optimistic (also used by Fargier and Sabbadin, 2005). Giang and Shenoy (2000) let the utility of prizes vary, but their criterion is pessimistic over prizes better than a neutral prize and optimistic over prizes worse than it, so the orientation is tied to whether a prize is a gain or a loss. Other authors use opposite conventions for their criteria. Pearl (1993) and Wilson (1995) assume the pattern opposite to Giang and Shenoy (2000) (see Table 1 in Section 2). In each of these approaches, the orientation, or the pattern of orientations over gains and losses, is the same for every decision maker who decides by its criterion. In contrast, our properties admit decision makers of either orientation, and choices identify which one a decision maker has, just as expected utility admits both risk-averse and risk-loving decision makers.

We address both gaps. The decision maker has preferences over acts that assign integer outcomes to finitely many states. Integer outcomes reflect that in reality every quantity can be measured only with finite precision, and they let us measure utility and disbelief in a common unit. Our main result, Theorem 1, characterizes preferences represented by a functional that takes every integer value and has the form

$$U_{\sigma,u}(X) = \Big(\bigoplus_{\omega\in\Omega} u(X(\omega))^{\odot\sigma} \odot \rho(\omega) \Big)^{\odot\sigma} = \sigma \cdot \min_{\omega\in\Omega} \big\{ \sigma \cdot u(X(\omega)) + \rho(\omega) \big\},$$

where $\sigma \in \{-1, +1\}$ is the orientation, u is an integer-valued utility function, ρ is a ranking function, and $a^{\odot\sigma} := \sigma a$ is the tropical power. For $\sigma = +1$, the decision maker evaluates an act by its worst utility, where the utility in a state is raised the more she disbelieves the state. For $\sigma = -1$, she evaluates an act by its best utility, where the utility in a state is lowered the more she disbelieves it. The disposition toward risk, the orientation σ, is captured by a tropical independence property. In tropical arithmetic, the state-wise minimum (maximum) of two acts is their tropical sum in the min-plus (max-plus) semiring, and adding a sure amount is tropical scalar multiplication. Tropical

[3] We speak of tropical expected utility rather than ranking expected utility to avoid confusion with rank-dependent expected utility, in which probabilities of outcomes are weighted according to the rank of the outcomes from best to worst.

independence requires that combining two acts with a common third act by the tropical sum does not reverse their ranking. In the min-plus semiring, this yields pessimism, and in the max-plus semiring, optimism. The orientation of the decision maker's disposition is thus the choice of semiring, and since the two semirings are isomorphic, both orientations follow from one proof. In form, tropical independence parallels the independence axiom of von Neumann and Morgenstern, with tropical sums of acts in place of mixtures. Mixtures, however, combine probabilities, whereas tropical sums combine outcomes state by state. The counterpart of the linearity of expected utility in probabilities is the tropical linearity of our representation in the ranking function, which holds for every utility function (see Section 5). Utility is identified by a tropical analog of tradeoff consistency, which requires that utility differences, measured by the number of distinguishable preference steps, are the same in every state. The resulting system of properties parallels Wakker's (1989) axioms for subjective expected utility, weak order, monotonicity, continuity, and tradeoff consistency, with one additional property for the aggregation over states. The tropical structure enters mainly through the aggregation property, the tropical independence axiom, and in addition through the non-compensatory form of the Archimedean property and the additive form of tradeoff consistency. Both the ranking function and the utility function are derived from preferences over acts. Under a normalization that measures utility and disbelief on the finest common grid, the ranking function is unique, and the utility function is unique up to an additive constant. That is, utility and disbelief are measured in the same units.

Tropical expected utility satisfies a weak form of the sure-thing principle, so, like subjective expected utility, it does not accommodate the paradoxa of Allais and Ellsberg (Section 4.1). It captures, however, behavior that subjective expected utility, Choquet expected utility, α-maxmin expected utility, and pessimistic qualitative utility (Dubois, Prade, and Sabbadin, 2001) cannot. A pessimist ignores a loss in a state she disbelieves as long as it does not exceed her degree of disbelief in that state, while every additional unit of a larger loss counts fully, and an optimist treats gains symmetrically (Section 4.2).

Because utility and disbelief are measured in the same units, risk attitudes take a different form than in expected utility. We study them in a companion paper (Schipper, 2026b). Every utility function is tropically concave in the min-plus semiring and tropically convex in the max-plus semiring, so the orientation alone, and no utility function, decides whether the decision maker is risk averse or risk loving, while utility carries the intensity of the risk attitude. The companion paper develops the tropical counterparts of the Arrow–Pratt coefficient, of Pratt's (1964) theorem, and of decreasing absolute risk aversion, and it applies the theory to investment and insurance.

The general theorem nests the special cases. Under constant additivity, utility is linear, $u(x) = |t|x$, and acts are evaluated by $U_t(X) = \operatorname{sgn}(t) \min_{\omega\in\Omega}\{tX(\omega) + \rho(\omega)\}$ for an integer $t \neq 0$, a tropical power, whose sign is the orientation (Theorem 2). Under solvability, i.e., if every act has a sure-money equivalent, $t = \sigma$, and the ranks of states are directly revealed in money (Corollary 1 and Remark 3). The case $t = -1$ is the criterion of Spohn (2025), and the case $t = 1$ is the pessimistic criterion used for the no-betting results of Schipper (2026a). With $t = \sigma$, the decision maker is risk neutral, evaluating acts by the tropical expectation of her own semiring, and $t = 1$ and $t = -1$ differ only in the orientation (see Schipper, 2026b).

In a companion paper (Schipper, 2026c), we take preferences conditional on every event as primitive. We introduce conditional ranking systems, which extend Spohn's conditional ranking functions to events that the decision maker considers impossible, and we characterize them, together with conditional tropical expected utility, by properties of conditional preferences. In yet another companion paper (Schipper, 2026d), we propose behavioral procedures for eliciting ranking functions.

The paper is organized as follows. Section 2 introduces the model. Section 3 states the properties

and discusses their interpretation and their relation to subjective expected utility. Section 4 states the representation theorem, relates tropical independence to the sure-thing principle, and illustrates behavior that alternative decision theories cannot capture. Section 5 characterizes tropical expected utility with linear utility. Section 6 relates our results to the literature, including variational preferences and risk measures. Proofs are in the Appendix.

2 Model

Let Ω be a nonempty finite set of states. An *act* is a function $X : \Omega \longrightarrow \mathbb{Z}$, interpreted as integer-valued monetary outcomes. We call $X(\omega)$ the *outcome* of X in state ω. Outcomes are amounts of money, not utilities as often in game theory. Let $\mathcal{X} := \mathbb{Z}^\Omega$ denote the set of acts. For $k \in \mathbb{Z}$, we also write k for the constant act with outcome k in every state. For acts X, Y, let $X \wedge Y$ and $X \vee Y$ denote the state-wise minimum and maximum, and write $X \geq Y$ if $X(\omega) \geq Y(\omega)$ for all $\omega \in \Omega$. For an event $E \subseteq \Omega$ and acts X, Y, let $X_E Y$ denote the act that agrees with X on E and with Y off E, and write $X_\omega Y := X_{\{\omega\}} Y$ for $\omega \in \Omega$. In particular, for constant acts $x, y \in \mathbb{Z}$, $x_\omega y$ pays x at ω and y elsewhere.

We make use of tropical notation (Speyer and Sturmfels, 2009). In the min-plus semiring, tropical addition is $a \oplus b := \min\{a, b\}$ and tropical multiplication is $a \odot b := a + b$, for $a, b \in \mathbb{Z} \cup \{-\infty, \infty\}$ with $z + \infty = \infty$ and $z - \infty = -\infty$ for $z \in \mathbb{Z}$. For finitely many a_ω, we write $\bigoplus_{\omega\in\Omega} a_\omega := \min_{\omega\in\Omega} a_\omega$. Tropical powers are $a^{\odot n} := n \cdot a$ for $n \in \mathbb{Z}$, so that $a^{\odot(-1)} = -a$ is the tropical inverse of a. On acts, these operations apply state-wise. So $X \oplus Y = X \wedge Y$, and tropical scalar multiplication adds a sure amount, $c \odot X = X + c$ for $c \in \mathbb{Z}$.

We assume that outcomes, ranks, and utilities are integers. We think that this is more realistic than a continuum, since every quantity can be measured only with finite precision. Money comes in smallest units, such as cents, and choices reveal preferences only up to such a unit. Spohn's ranking theory grades disbelief in discrete steps rather than measuring it on a continuous scale. Integers also make utility and disbelief commensurable. Since the representation takes integer values, an outcome increment offsets a whole number of ranks of disbelief, and preferences can be measured by counting indifference classes rather than by continuity arguments. This is why finitely many states suffice and no rich set of consequences is needed (see Section 6). Real-valued outcomes would not simplify the theory. The coefficient t that governs the intensity of risk attitudes (see Section 5) could then be absorbed into the scale of ranks, and integer ranks could not be identified.

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$. A *ranking function* is a function $\rho : 2^\Omega \longrightarrow \mathbb{N} \cup \{\infty\}$ such that $\rho(\Omega) = 0$, $\rho(\emptyset) = \infty$, and $\rho(E \cup F) = \min\{\rho(E), \rho(F)\}$ for all events $E, F \subseteq \Omega$ (Spohn, 1988, 2012). Since Ω is finite, $\rho(E) = \min_{\omega\in E} \rho(\{\omega\})$ for every nonempty event E, so ρ is determined by the ranks of states, and we write $\rho(\omega) := \rho(\{\omega\})$. In particular, $\min_{\omega\in\Omega} \rho(\omega) = 0$. We interpret $\rho(E)$ as the decision maker's degree of disbelief in the event E. It can also be interpreted as the potential surprise, in the sense of Shackle (1949), that she would experience if she learned that E occurred.[4] States of rank 0 would not surprise her at all, and states of rank ∞ are those she regards as impossible.[5]

Geometrically, ranking functions play the role in the tropical world that probability measures play in the classical one. A probability measure on the finite set Ω is a vector of nonnegative numbers

[4] Shackle (1949, p. 11) writes, "It is only a man who feels very sure of a given outcome who can be greatly *surprised* by its non-occurrence. A degree of belief is not in itself a sensation or an emotion; but a high degree of belief is a condition of our being able to feel a high degree of surprise."

[5] Shackle (1949, p. 131) requires that "[a]t least one member of an exhaustive set of rival hypotheses must carry zero potential surprise", which corresponds to our normalization $\min_{\omega\in\Omega} \rho(\omega) = 0$, and he assigns the highest degree of potential surprise to the "absolute disbelief in the truth of the suggested answer to a question or the possibility of the suggested outcome of an 'experiment' " (p. 130).

that sum to one. Any nonnegative nonzero vector v defines a probability measure, $v/\sum_\omega v(\omega)$, and two such vectors define the same probability measure if and only if one is a positive multiple of the other. The normalization $\sum_\omega \mu(\omega) = 1$ selects one of these vectors. For instance, with three states, $(1,1,2)$ and $(2,2,4)$ define the same probability measure, $(\frac{1}{4},\frac{1}{4},\frac{1}{2})$. The set of probability measures is the probability simplex. Tropically, multiplying a vector by a number means multiplying each entry tropically by it, i.e., adding the same constant to each entry. Two vectors of ranks that differ by a constant disbelieve the same states more than others and by the same amounts, and they define the same ranking function. The normalization $\min_{\omega\in\Omega} \rho(\omega) = 0$, i.e., that the tropical sum of the ranks is the tropical one, selects one of these vectors. For instance, $(1,4,6)$ and $(-2,1,3)$ define the same ranking function, $(0,3,5)$. In tropical geometry, the set of real vectors up to adding a constant is called tropical projective space (Develin and Sturmfels, 2004). It is the tropical analog of the interior of the probability simplex, and ranking functions with finite ranks are its integer points. States of rank ∞, which the decision maker rules out, play the role of states of probability zero, and ranking functions with such states correspond to points on the boundary of the simplex.

For $\sigma \in \{-1,+1\}$, a strictly increasing *utility function* $u : \mathbb{Z} \longrightarrow \mathbb{Z}$, and a ranking function ρ, define the *tropical expected utility* of the act X by

$$U_{\sigma,u}(X) := \Big(\bigoplus_{\omega\in\Omega} u(X(\omega))^{\odot\sigma} \odot \rho(\omega) \Big)^{\odot\sigma} = \sigma \cdot \min_{\omega\in\Omega} \big\{ \sigma \cdot u(X(\omega)) + \rho(\omega) \big\}.$$

The minimum is finite, since $\rho(\omega) = 0$ for some state ω. Thus

$$U_{+1,u}(X) = \min_{\omega\in\Omega} \big\{ u(X(\omega)) + \rho(\omega) \big\} \qquad \text{and} \qquad U_{-1,u}(X) = \max_{\omega\in\Omega} \big\{ u(X(\omega)) - \rho(\omega) \big\}.$$

For $\sigma = -1$, the tropical inversions inside and outside turn the minimum into a maximum. For convenience, we therefore also use the max-plus semiring. For $\sigma \in \{-1,+1\}$, let

$$a \oplus^\sigma b := \big(a^{\odot\sigma} \oplus b^{\odot\sigma} \big)^{\odot\sigma} = \sigma \cdot \min\{\sigma a, \sigma b\} = \begin{cases} \min\{a,b\} & \text{if } \sigma = +1, \\ \max\{a,b\} & \text{if } \sigma = -1, \end{cases}$$

Thus $\oplus^{+1} = \oplus$ and $\odot$ form the min-plus semiring, and $\oplus^{-1}$ and $\odot$ form the max-plus semiring. On acts, $X \oplus^{+1} Y = X \wedge Y$ and $X \oplus^{-1} Y = X \vee Y$, and the set of acts is closed under both tropical sums and under tropical scalar multiplication by integers. For tropical sums of more than two terms, we keep the tropical power and write $\big(\bigoplus_{\omega\in\Omega} a_\omega^{\odot\sigma} \big)^{\odot\sigma}$, which is the minimum of the a_ω if $\sigma = +1$ and their maximum if $\sigma = -1$. In both cases, $U_{\sigma,u}$ is to be maximized. For $\sigma = +1$, the decision maker is *pessimistic*, i.e., she evaluates an act by the minimum over states of utility plus rank. A state with a bad outcome thus counts less the more it is disbelieved. For $\sigma = -1$, the decision maker is *optimistic*, i.e., she evaluates an act by the maximum over states of utility minus rank. A state with a good outcome thus counts less the more it is disbelieved.[6] We call σ the *orientation* of the decision maker's disposition. It is positive for pessimism (easy to remember, since both start with "p"), as for risk aversion, and negative for optimism. The utility function u carries the intensity of the risk attitude and its change with wealth (Schipper, 2026b).

In tropical terms, a ranking function is additive with respect to the tropical sum, $\rho(E) = \bigoplus\{\rho(\omega) : \omega \in E\}$ for nonempty E, and normalized by $\rho(\emptyset) = \infty$, the tropical zero, and $\rho(\Omega) = 0$, the tropical one. It is thus a normalized idempotent measure in the sense of Maslov (Kolokoltsov

[6] Shackle (1949, p. 18) describes the same dependence for the anticipation of outcomes. The intensity of enjoyment of an outcome "will plainly be an *increasing* function of the *desirability* of the outcome in question, and a *decreasing* function of the degree of potential surprise associated with it", and the intensity of "distress by anticipation" increases "with the degree of hurtfulness or disadvantage" and decreases "with the degree of potential surprise."

and Maslov, 1997; Akian, 1999; Del Moral and Doisy, 1999; Kolokoltsov, 2001). The pessimistic criterion $U_{+1,u}$ is the idempotent integral of the utility act $u \circ X$ with respect to this measure, the tropical analog of the subjective expected utility $\sum_{\omega\in\Omega} \mu(\omega)u(X(\omega))$ with respect to a probability measure μ. The optimistic criterion $U_{-1,u}$ is the $(\max, +)$-expectation of idempotent probability theory with respect to the idempotent measure $-\rho$ in the max-plus semiring (Del Moral and Doisy, 1999, Definition 4). Pessimism and optimism thus correspond to evaluating acts in the min-plus and in the max-plus semiring.

Other ways of writing ranking-based criteria reduce to these two. Since multiplying by -1 reverses the order of the real numbers and hence turns a maximum into a minimum, we can write

$$\max_{\omega\in\Omega}\{u(X(\omega)) - \rho(\omega)\} = -\min_{\omega\in\Omega}\{-u(X(\omega)) + \rho(\omega)\}.$$

Hence the four criteria in Table 1 represent only two preferences, $U_{+1,u}$ and $U_{-1,u}$.

Attitude	Objective	Integral	Applied to	Used by
pessimistic	maximize	$\min_{\omega\in\Omega}\{u(X(\omega)) + \rho(\omega)\}$	utility	Schipper (2026a, 2026b); Giang and Shenoy (2000) over gains; Pearl (1993) and Wilson (1995) over losses
pessimistic	minimize	$\max_{\omega\in\Omega}\{-u(X(\omega)) - \rho(\omega)\}$	disutility	
optimistic	maximize	$\max_{\omega\in\Omega}\{u(X(\omega)) - \rho(\omega)\}$	utility	Schipper (2026b); Del Moral and Doisy (1999)*; Giang and Shenoy (2000) over losses; Pearl (1993) and Wilson (1995) over gains
optimistic	minimize	$\min_{\omega\in\Omega}\{-u(X(\omega)) + \rho(\omega)\}$	disutility	Spohn (2012, 2025), Fargier and Sabbadin (2005), Schipper and Kern-Isberner (2026)

Table 1: Ranking-based criteria. Rows with the same attitude represent the same preference. *As an expectation, not as a decision criterion.

We model the preferences of the decision maker with a binary relation $\succsim$ on the set of all acts $\mathcal{X}$, and denote by $\succ$ the asymmetric part and by $\sim$ the symmetric part of $\succsim$. We say that $\succsim$ is *represented* by a functional $F : \mathcal{X} \longrightarrow \mathbb{R}$ if, for all $X, Y \in \mathcal{X}$, $X \succsim Y$ if and only if $F(X) \geq F(Y)$.

A state ω is *null* if $X \sim Y$ for all acts X, Y with $X(\omega') = Y(\omega')$ for all $\omega' \neq \omega$. That is, changing the outcome in a null state alone never changes the decision maker's evaluation of an act. She acts as if a null state could not occur, just as null events in Savage (1954) are those of subjective probability zero. Let Ω_+ denote the set of non-null states. We will see that the null states are exactly the states that she disbelieves with rank ∞ (Theorem 1 (ii)).

3 Behavioral Properties

We impose five properties on preference that capture behavior of tropical expected utility maximizers. Each of the properties has a counterpart in subjective expected utility.

P1 (Weak order) $\succsim$ is complete and transitive.

Completeness means that the decision maker is able to compare any two acts. This may be less of a heroic assumption in our model compared to models of subjective expected utility that assume an uncountable set of acts. As usual, transitivity rules out cyclic choice over acts.

P2 (Monotonicity) If $X \geq Y$, then $X \succsim Y$. Moreover, $k+1 \succ k$ for all $k \in \mathbb{Z}$.

If an act pays weakly more than another in every state, it is weakly preferred, and a larger sure amount is strictly preferred to a smaller one. By transitivity (P1), the second part of P2 implies that, for all integers j and k,

$$j > k \quad \text{implies} \quad j \succ k. \tag{1}$$

A violation would mean choosing an act that is dominated, i.e., that yields less in every state. Our decision maker is assumed to like more money, not less. The property corresponds to state-wise monotonicity in subjective expected utility. Note that P2 requires strict preference only for sure amounts. Raising the outcome in one state may leave the evaluation unchanged. Under pessimism, this happens if the state is not the one that determines the evaluation of the act, i.e., if another state determines the act's worst outcome, adjusted for disbelief.

P3 (Tropical independence) There is $\sigma \in \{-1, +1\}$ such that, for all $X, Y, Z \in \mathcal{X}$, if $X \succsim Y$, then $X \oplus^\sigma Z \succsim Y \oplus^\sigma Z$.

When $\sigma = +1$, we write P3$^+$ for the case of min-plus independence or pessimism, i.e., for all $X, Y, Z \in \mathcal{X}$, if $X \succsim Y$, then $X \wedge Z \succsim Y \wedge Z$. Analogously, when $\sigma = -1$, we write P3$^-$ for the case of max-plus independence or optimism, i.e., for all $X, Y, Z \in \mathcal{X}$, if $X \succsim Y$, then $X \vee Z \succsim Y \vee Z$.

Under P3$^+$, if the decision maker weakly prefers X to Y, she also weakly prefers $X \wedge Z$ to $Y \wedge Z$ for any act Z, i.e., when both acts are capped from above state by state by the same act. Such a decision maker judges an act by what can go wrong. Capping both acts by the same act imposes the same ceiling $Z(\omega)$ on both of them in every state. It can create bad scenarios that both acts share, and so make them equally good, but it cannot reverse their ranking. Under P3$^-$, she judges an act by what can go right. Raising both acts to the same floor $Z(\omega)$ in every state can create good scenarios that both acts share, but again cannot reverse their ranking. P3$^+$ describes a decision maker who reasons qualitatively about which scenarios are possible and focuses on the least disbelieved bad scenario, as in precautionary or worst-case reasoning. P3$^-$ describes a decision maker who focuses on the least disbelieved good scenario, as in some entrepreneurial or gambling behavior. Recall that σ is the orientation of the decision maker's disposition, so her disposition is pessimistic under P3$^+$ and optimistic under P3$^-$.

Tropical independence has an equivalent form, which makes its implications transparent. Under P1 and P2, P3$^+$ holds if and only if $X \precsim Y$ implies $X \wedge Y \sim X$, i.e., the state-wise minimum of two acts is exactly as good as the worse of the two, and P3$^-$ holds if and only if $X \precsim Y$ implies $X \vee Y \sim Y$, i.e., the state-wise maximum is exactly as good as the better of the two. For instance, if $X = (10, 2)$ and $Y = (3, 8)$ and $X \precsim Y$, then the pessimistic decision maker is indifferent between X and $X \wedge Y = (3, 2)$. This extends to finitely many acts: under P3$^+$, the state-wise minimum of finitely many acts is indifferent to the worst of them, and under P3$^-$, the state-wise maximum is indifferent to the best of them (see Lemma 7 in the Appendix). Every act is such a minimum of acts, one for each state, each of which pays the act's outcome in that state and an outcome weakly higher than $\max_{\omega \in \Omega} X(\omega)$ in every other state. Hence the evaluation of an act reduces to the evaluations of its outcomes in the individual states, and the act is as good as the worst of them

under pessimism. Under optimism, every act is the state-wise maximum of the analogous acts with an outcome weakly lower than $\min_{\omega\in\Omega} X(\omega)$ in every other state, and the act is as good as the best of them.[7] In tropical terms, the evaluation of an act is the tropical sum of the evaluations of its parts. Property P3 thus takes over the role that additivity across states plays in subjective expected utility (see Section 4.1). Together with P4 below, this yields the decomposition of acts into component acts on which our representation rests.

In subjective expected utility with objective randomization, as in Anscombe and Aumann (1963), the independence axiom requires that mixing two acts with a common third act Z does not affect their ranking, i.e., $X \succsim Y$ if and only if $\alpha X + (1-\alpha)Z \succsim \alpha Y + (1-\alpha)Z$ for $\alpha \in (0,1]$. In tropical arithmetic, mixtures are replaced by tropical linear combinations $(c\odot X)\oplus^\sigma (d\odot Z)$, and P3 is the independence property for the tropical sum. Its counterpart for tropical scalar multiplication is introduced in Section 5. Unlike its classical counterpart, tropical independence is stated as an implication only. Tropical addition is idempotent, $a \oplus^\sigma a = a$, and has no inverse, so a common third act can mask a difference between two acts. For instance, if Z is low enough in every state, then $X \wedge Z = Y \wedge Z = Z$ even if $X \succ Y$. Strict preference is therefore not preserved. We return to the comparison with independence and the sure-thing principle after the representation theorem, in Section 4.1, where we also discuss ranking function versions of the Allais paradoxon and the Ellsberg paradoxon.

For an act X, the *indifference class* of X is

$$[X] := \{Y \in \mathcal{X} : Y \sim X\}.$$

Under P1, the indifference classes partition $\mathcal{X}$, and $[X] \succsim [Y]$ if and only if $X \succsim Y$ defines a linear order on the set of indifference classes. We say that $\succsim$ satisfies P1 to P5 if it satisfies P1, P2, P3 for some σ, P4, and P5.

P4 (Archimedean property)

(i) (Discreteness) For all acts $X \precsim Y$, there are only finitely many indifference classes of acts Z with $X \precsim Z \precsim Y$.

(ii) (Bounded disbelief) For all $\omega \in \Omega_+$ and $x \in \mathbb{Z}$, there is $\bar{y} \in \mathbb{Z}$ such that $x_\omega y \sim x_\omega \bar{y}$ for all $y \geq \bar{y}$, or for all $y \leq \bar{y}$.

For $\omega \in \Omega_+$ and $x \in \mathbb{Z}$, let $\bar{y} \in \mathbb{Z}$ be as in P4(ii), which may depend on x and ω. We let $c(x,\omega) := x_\omega \bar{y}$ denote the act that pays x in state ω and $\bar{y}$ in every other state, and call it a *component act* of outcome x in state ω. The component act is the tropical analog of a component of a vector. Under pessimism, the tropical zero is $+\infty$, and $x_\omega \bar{y}$ behaves like the act that pays x at ω and $+\infty$ elsewhere. Under optimism, the tropical zero is $-\infty$. With the properties imposed on preferences, every act is indifferent to the tropical sum of its component acts, i.e.,

$$X \sim \Big(\bigoplus_{\omega\in\Omega_+} c(X(\omega),\omega)^{\odot\sigma} \Big)^{\odot\sigma},$$

i.e., $X \sim \bigwedge_{\omega\in\Omega_+} c(X(\omega),\omega)$ under pessimism and $X \sim \bigvee_{\omega\in\Omega_+} c(X(\omega),\omega)$ under optimism, where the minimum and the maximum of acts are taken state by state (see the proof of Theorem 1).

[7] Shackle (1949, p. 17) postulates the same non-additivity: "the power of *mutually exclusive* hypotheses of success to afford enjoyment by imagination is *not additive*, and ... therefore the power of the entire set of hypotheses of success associated with any course of action to afford enjoyment by imagination is simply that of one alone amongst these hypotheses, whichever has this power in higher degree than any of the others", and similarly for distress. He compares it to "the strength of a chain is the strength of its weakest link" (p. 17).

The indifference class of a component act does not depend on the choice of $\bar{y}$ within each of the two alternatives in (ii). Indeed, if $x_\omega y \sim x_\omega \bar{y}$ for all $y \geq \bar{y}$ and $x_\omega y \sim x_\omega \bar{y}'$ for all $y \geq \bar{y}'$, then $x_\omega \bar{y} \sim x_\omega \max\{\bar{y}, \bar{y}'\} \sim x_\omega \bar{y}'$, and analogously, with $\min\{\bar{y}, \bar{y}'\}$, for the second alternative.

Part (i) requires that between any two acts, the decision maker distinguishes only finitely many levels of preference. Together with monotonicity, it implies that the levels of preference can be numbered by the integers. Behaviorally, this coarseness does not come from imprecise perception of outcomes, since by P2 she strictly prefers one more unit of a sure outcome. It comes from her coarse perception of uncertainty. She grades disbelief in whole steps, and she does not weigh states against each other. An act is as good as its outcome in a single state, the one that determines her evaluation. In contrast, a decision maker in subjective expected utility can weight states against each other and therefore value an act at any level between its outcomes. Thus, Part (i) excludes subjective expected utility with generic probabilities. Nevertheless, like in expected utility, Part (i) takes the place of continuity. As in expected utility, the Archimedean property rules out lexicographic preferences. For instance, under leximin on two states, which compares acts first by their worse and then by their better outcome, every act $(0, k)$ with $k \geq 1$ lies strictly between $(0, 0)$ and $(1, 1)$, so there are infinitely many indifference classes between these two acts. The same holds for lexicographic refinements of tropical expected utility by a strictly monotone criterion, such as those discussed in Section 4.1.

For part (ii), consider the act $x_\omega y$ that pays x in state ω and y in every other state. Part (ii) requires one of two alternatives: Either raising y eventually stops making this act better, or lowering y eventually stops making the act worse. Under pessimism, the first alternative holds, i.e., raising y eventually stops making the act better because an act is evaluated by its value in its worst state including disbelief. That state cannot be compensated by arbitrarily high outcomes elsewhere. This is the opposite of what continuity delivers in subjective expected utility, where any loss in one state can be compensated by large enough gains in other states that have positive probability. Bounded disbelief is therefore not merely technical. It expresses the non-compensatory character of ranking-based evaluation and is plausible to the same extent as worst-case reasoning. Under optimism, the second alternative holds, so that a state that matters cannot be spoiled by arbitrarily low outcomes elsewhere. Which of the two holds is not part of the property. Given (ii), P1 to P3, P4(i), and P5 imply that the first alternative holds under P3$^+$ and the second under P3$^-$ (see the proof of Theorem 1). If at least two states are non-null, the other alternative fails.

P5 (State-independent utility differences) For all $\omega, \omega' \in \Omega_+$, all $x < x'$, and all component acts $c(x, \omega)$, $c(x', \omega)$, $c(x, \omega')$, and $c(x', \omega')$, the number of indifference classes of acts Z with $c(x, \omega) \precsim Z \precsim c(x', \omega)$ equals the number of indifference classes of acts Z with $c(x, \omega') \precsim Z \precsim c(x', \omega')$.

The component act $c(x, \omega)$ captures the evaluation of outcome x in state ω. The number of indifference classes between $c(x, \omega)$ and $c(x', \omega)$ measures how much better x' is than x as revealed in state ω, in units of the finest preference step. That is, it measures the strength of preference of x' over x, or the tradeoff between x and x', in state ω. Property P5 requires that tradeoffs are consistent across states, i.e., in every state that matters, x' is to x what it is in any other state that matters. Behaviorally, the decision maker's valuation of outcomes does not depend on the state in which they occur. The property thus expresses a form of state-independent utility. It suits monetary outcomes whose value does not depend on the state, and fails when it does, for instance when states differ in health or wealth outside the model. Property P5 is a counterpart of state independence in subjective expected utility, for utility differences rather than utilities, and it is related to tradeoff consistency in Wakker (1989). We compare these properties in Section 6.

The following table summarizes the correspondence.

Subjective expected utility	Tropical expected utility
weak order	weak order (P1)
monotonicity	monotonicity (P2)
sure-thing principle	tropical independence (P3)
continuity or Archimedean axiom	Archimedean property (P4)
state independence	state-independent utility differences (P5)

Property P3 determines the form of the representation, i.e., whether states are aggregated by the minimum or by the maximum. P4 and P5 do not refer to the orientation at all. They make the measurement possible. P4 says there are finitely many preference levels and finite disbelief in non-null states. P5 says utility differences are the same in every state.

Although P4 and P5 do not refer to the orientation, they make it a well-defined trait of the decision maker. Once two states matter, she cannot be both pessimistic and optimistic.

Lemma 1 (Uniqueness of orientation)**.** *Let $\succsim$ satisfy P1, P2, P4, and P5, and let at least two states be non-null. Then $\succsim$ does not satisfy both P3$^+$ and P3$^-$.*

To see why, let ω_1 and ω_2 be non-null, and consider the acts $0_{\omega_1}y$ and $0_{\omega_2}y$, each of which pays 0 in one of the two states and y elsewhere. Their state-wise maximum is the sure amount y. Under P3$^+$, raising y eventually stops making $0_{\omega_i}y$ better, by P4(ii), while the sure amount y keeps improving, by P2. So for large y, the sure amount is strictly better than both acts, whereas P3$^-$ requires the state-wise maximum to be indifferent to the better of the two. The proof is in the Appendix and uses the representation of Section 4. If only one state is non-null, an act is as good as its outcome in that state, and P3 holds for both orientations.

4 Representation Theorem

We are ready to state our main result.

Theorem 1. *A binary relation $\succsim$ on $\mathcal{X}$ satisfies P1 to P5 if and only if there exist $\sigma \in \{-1, +1\}$, a strictly increasing $u : \mathbb{Z} \longrightarrow \mathbb{Z}$, and a ranking function ρ such that, for all acts $X, Y \in \mathcal{X}$,*

$$X \succsim Y \quad \textit{if and only if} \quad \Big(\bigoplus_{\omega \in \Omega} u(X(\omega))^{\odot\sigma} \odot \rho(\omega) \Big)^{\odot\sigma} \geq \Big(\bigoplus_{\omega \in \Omega} u(Y(\omega))^{\odot\sigma} \odot \rho(\omega) \Big)^{\odot\sigma},$$

and $U_{\sigma,u}$ takes every integer value, i.e., $U_{\sigma,u}(\mathcal{X}) = \mathbb{Z}$. Moreover, the following hold.

(i) If $\succsim$ satisfies P3 for σ, then the representation holds with this σ. If at least two states are non-null, σ is unique.

(ii) A state ω is null if and only if $\rho(\omega) = \infty$.

(iii) For given σ, ρ is unique, and u is unique up to an additive constant.

It might be instructive to write out the representation result with the min-plus notation for pessimism and max-plus for optimism, respectively. For $\sigma = +1$, the decision maker is pessimistic, and for all acts $X, Y \in \mathcal{X}$,

$$X \succsim Y \quad \text{if and only if} \quad \min_{\omega \in \Omega} \big\{ u(X(\omega)) + \rho(\omega) \big\} \geq \min_{\omega \in \Omega} \big\{ u(Y(\omega)) + \rho(\omega) \big\}.$$

For $\sigma = -1$, the decision maker is optimistic, and for all acts $X, Y \in \mathcal{X}$,

$$X \succsim Y \quad \text{if and only if} \quad \max_{\omega\in\Omega} \big\{u(X(\omega)) - \rho(\omega)\big\} \geq \max_{\omega\in\Omega} \big\{u(Y(\omega)) - \rho(\omega)\big\}.$$

Theorem 1 characterizes the behavior of a decision maker who maximizes the tropical expectation of her utility with respect to a ranking function. Under pessimism, she evaluates an act by the minimum over states of utility plus rank, and under optimism, by the maximum over states of utility minus rank. Both the utility function and the ranking function are derived from her preferences over acts. This distinguishes our characterization from the few earlier decision theories with ranking functions, which take the ranking function as a primitive. Giang and Shenoy (2000) and Spohn (2025) attach objective ranking functions to lotteries, analogous to the objective probabilities of von Neumann and Morgenstern (1947). This is problematic as the literature has not suggested any objective device generating ranks. Typically, coins, dice, lotteries etc. generate probabilities and not ranks. We instead *derive subjective ranking functions* from behavior, as Savage (1954) derives subjective probabilities. In this sense, Theorem 1 is the counterpart for ranking functions of Savage's theorem for subjective probabilities.

The proof in the appendix establishes first the pessimistic case and then obtains the optimistic case by duality (see Lemma 6). Negation of outcomes, $x \mapsto -x$, is an isomorphism between the min-plus and the max-plus semiring, so an optimistic preference is a pessimistic preference over the negated acts. In the pessimistic case, the steps of the proof use the properties as follows. P1, P2, and P4(i) number the indifference classes by the integers. P3 writes the evaluation of an act as the minimum of one function per state. P4(ii) and P5 show that these functions are finite on non-null states, so that component acts exist. P5 then shows that these functions differ across states only by a constant, the rank, so that each has the form $u(x) + \rho(\omega)$. The utility function is thus identified jointly with the ranking function.

The requirement $U_{\sigma,u}(\mathcal{X}) = \mathbb{Z}$ is a normalization of units. It says that $U_{\sigma,u}$ enumerates the indifference classes, so that utility and disbelief are measured in units of the finest preference step. It fixes the unit but not the origin. For instance, multiplying both u and ρ by 2 does not change the represented relation, but the values are then even integers only, which the normalization excludes. Adding a constant to u leaves the values in $\mathbb{Z}$, which is why u is unique only up to an additive constant. The normalization plays the role of solvability in Wakker (1991), which requires that changing the outcome in one state can attain every intermediate level of preference exactly. Here, every level is attained by some act, and P5 requires that the levels between component acts are filled in the same way in every state. The normalization is therefore not an additional assumption but follows from P1 to P5. It restricts the relations covered by Theorem 1, since some relations represented by some $U_{\sigma,u}$ admit no representation that takes every integer value (Example 1).

The difference from subjective expected utility is best seen in tropical terms. There, utility is unique up to positive affine transformations $\alpha u + \beta$ with $\alpha > 0$. The tropical analog of an affine transformation of utility is $x \mapsto (b \odot x) \oplus^{\sigma} c$, i.e., $\min\{x + b, c\}$ under pessimism and $\max\{x + b, c\}$ under optimism, with $b \in \mathbb{Z}$ and a constant c that may be the tropical zero. Tropical multiplication by b is a shift, like β. Tropical addition of c, however, is a truncation rather than a shift. Under pessimism, for instance, a finite c identifies all utilities weakly above $c-b$ with c, so the transformed function cannot represent strict preferences among acts with such values. In subjective expected utility, every positive affine transformation is strictly increasing, so both parameters survive. In tropical expected utility, only tropical multiplication survives, i.e., c must be the tropical zero ($+\infty$ under pessimism and $-\infty$ under optimism) to avoid truncation, and u is unique up to $b \odot u = u + b$. The unit corresponds to the tropical power $u^{\odot\alpha} = \alpha u$. It must be applied to u and ρ jointly, and the normalization fixes it to $\alpha = 1$, i.e., to the unit of the finest preference step.

The uniqueness of σ in (i) follows from Lemma 1, since a relation represented by $U_{\sigma,u}$ satisfies P3 for σ. If only one state ω_0 is non-null, then $U_{\sigma,u}(X) = u(X(\omega_0))$ for both signs of σ, so σ is not unique, as stated in (i).

Some relations represented by $U_{+1,u}$ violate P5. By Theorem 1, they admit no representation that takes every integer value. This is demonstrated in the following example.

Example 1 (Scope of the normalization)**.** Consider $\Omega = \{\omega_1, \omega_2\}$, $\rho = (0, 2)$, and

$$u(x) = x \ \text{ for } x \leq 0, \qquad u(1) = 2, \qquad u(x) = x + 2 \ \text{ for } x \geq 2.$$

Then $U_{+1,u}(X) = \min\{u(X(\omega_1)), u(X(\omega_2)) + 2\}$, so every value of $U_{+1,u}$ is of the form $u(x)$ or $u(x) + 2$. Conversely, the component act $c(x, \omega_1)$ has the value $u(x)$, and $c(x, \omega_2)$ has the value $u(x)+2$. Hence, as X ranges over all acts, $U_{+1,u}(X)$ takes exactly the values in $u(\mathbb{Z}) \cup (u(\mathbb{Z})+2) = \mathbb{Z}\backslash\{3\}$. The component acts $c(0, \omega_1)$ and $c(1, \omega_1)$ have the values 0 and 2, and there are 3 indifference classes between them, with the values 0, 1, and 2. The component acts $c(0, \omega_2)$ and $c(1, \omega_2)$ have the values 2 and 4, and there are only 2 indifference classes between them, with the values 2 and 4, since no act has the value 3. Thus, P5 fails.

4.1 Independence and the Sure-Thing Principle

In this subsection, we demonstrate that the well-known violations of subjective expected utility by the Allais and Ellsberg paradoxa also apply to tropical expected utility.

In subjective expected utility, independence makes the representation additive. With objective randomization,

$$\mathrm{EU}(\alpha X + (1-\alpha)Z) = \alpha\,\mathrm{EU}(X) + (1-\alpha)\,\mathrm{EU}(Z),$$

so that

$$\alpha X+(1-\alpha)Z \succsim \alpha Y+(1-\alpha)Z \quad \text{if and only if} \quad \alpha\mathrm{EU}(X)+(1-\alpha)\mathrm{EU}(Z) \geq \alpha\mathrm{EU}(Y)+(1-\alpha)\mathrm{EU}(Z).$$

The common part $(1-\alpha)\,\mathrm{EU}(Z)$ appears on both sides and cancels. Tropical independence similarly forces the representation to be tropically additive, i.e., minimitive under pessimism and maximitive under optimism. Since u is strictly increasing, $u(\min\{a, b\}) = \min\{u(a), u(b)\}$ and $u(\max\{a, b\}) = \max\{u(a), u(b)\}$. Interchanging the minimum or maximum over states with the one over the two acts yields

$$U_{\sigma,u}(X \oplus^{\sigma} Z) = U_{\sigma,u}(X) \oplus^{\sigma} U_{\sigma,u}(Z). \tag{2}$$

Unlike in subjective expected utility, the common part does not cancel. Under pessimism, for instance,

$$\min\{U_{+1,u}(X), U_{+1,u}(Z)\} \geq \min\{U_{+1,u}(Y), U_{+1,u}(Z)\}$$

does not imply $U_{+1,u}(X) \geq U_{+1,u}(Y)$, since the common part $U_{+1,u}(Z)$ may attain both minima.

In Savage's (1954) framework without objective randomization, the role of independence is played by the sure-thing principle. The sure-thing principle requires that $X_E H \succsim Y_E H$ implies $X_E H' \succsim Y_E H'$ for all acts H'. Tropical expected utility satisfies only a weak form, namely, $X_E H \succ Y_E H$ implies $X_E H' \succsim Y_E H'$. To see this under pessimism, let $U_E(X) := \min_{\omega \in E}\{u(X(\omega)) + \rho(\omega)\}$. Then

$$U_{+1,u}(X_E H) = \min\{U_E(X), U_{\Omega\backslash E}(H)\}.$$

Hence $X_E H \succ Y_E H$ implies $U_E(X) > U_E(Y)$, which implies $X_E H' \succsim Y_E H'$ for every H'. The argument under optimism is the same with max in place of min. The full principle fails because a low common part masks differences on E. For instance, if H is low enough off E, then $X_E H \sim Y_E H$

even if $X_E H' \succ Y_E H'$ for a high common part H'. Since subjective expected utility satisfies the weak form as well, the weak sure-thing principle cannot distinguish aggregation over states by a minimum or a maximum from aggregation by a weighted sum. Tropical independence does, which is why we formulate the aggregation over states by tropical independence.

Do the familiar arguments against independence and the sure-thing principle apply to tropical expected utility? The most prominent ones, the paradoxa of Allais (1953) and Ellsberg (1961), rest on reversals of strict preference when a common consequence is changed. Since the weak sure-thing principle holds, a common consequence can turn a strict preference into indifference but never into the opposite strict preference. Hence neither the common-consequence pattern of the Allais paradoxon nor the three-color Ellsberg pattern can be produced by tropical expected utility, under pessimism or under optimism.

Example 2 (Allais paradoxon)**.** To see this for the paradoxon of Allais (1953), consider a lottery with 100 tickets, and let the state be the ticket drawn. State s_1 is ticket 1, state s_2 is one of the tickets 2 to 11, and state s_3 is one of the tickets 12 to 100. While Allais (1953) considers objective probabilities, we let the decision maker disbelieve the states to degrees $\rho(s_1)$, $\rho(s_2)$, and $\rho(s_3)$. Consider the acts, with outcomes in millions of dollars,

	s_1	s_2	s_3
A	1	1	1
B	0	5	1
C	1	1	0
D	0	5	0

In experiments, typically a majority of decision makers choose A over B and D over C (Blavatskyy, Ortmann, and Panchenko, 2022). That is, the decision maker prefers the sure million to a gamble with a small chance of nothing, but when both options risk nothing anyway, she prefers the chance of five million. Note that the acts A and B agree on s_3, and so do C and D. With $E = \{s_1, s_2\}$, we have $A = A_E 1$, $B = B_E 1$, $C = A_E 0$, and $D = B_E 0$. By the weak sure-thing principle, $A \succ B$ implies $C \succsim D$, which conflicts with the typical choice pattern. Explicitly, under pessimism, normalize $u(0) = 0$, so that $0 < u(1) < u(5)$. Then $U_{+1,u}(A) = u(1)$ and

$$U_{+1,u}(B) = \min\{\rho(s_1), u(5) + \rho(s_2), u(1) + \rho(s_3)\}.$$

Hence $A \succ B$ requires $\rho(s_1) < u(1)$. That is, the decision maker does not disbelieve getting nothing firmly enough to accept the risk. Moreover,

$$U_{+1,u}(C) = \min\{u(1) + \rho(s_1), u(1) + \rho(s_2), \rho(s_3)\}, \qquad U_{+1,u}(D) = \min\{\rho(s_1), u(5) + \rho(s_2), \rho(s_3)\}.$$

If $D \succ C$, then $\rho(s_3) > U_{+1,u}(C)$, so $U_{+1,u}(C) = u(1) + \min\{\rho(s_1), \rho(s_2)\}$, and $\rho(s_1) > U_{+1,u}(C)$, so $U_{+1,u}(C) = u(1) + \rho(s_2)$. Hence $D \succ C$ requires $\rho(s_1) > u(1) + \rho(s_2) \geq u(1)$. The decision maker would have to disbelieve state s_1 both less and more than $u(1)$. The same computation with max in place of min shows that the same holds under optimism. Thus, like subjective expected utility, tropical expected utility cannot encompass the Allais paradoxon.

Example 3 (Ellsberg paradoxon)**.** For the Ellsberg example, consider an urn with 30 red balls and 60 balls that are blue or green in unknown proportion. One ball is drawn. While Ellsberg (1961) considers bets on draws from urns, we take the color of the drawn ball as the state, i.e.,

$\Omega = \{R, B, G\}$. Consider the acts

	R	B	G
X	100	0	0
Y	0	100	0
X'	100	0	100
Y'	0	100	100

In experiments, the typical finding is ambiguity aversion, i.e., $X \succ Y$ and $Y' \succ X'$ (Trautmann and van de Kuilen, 2015). That is, the decision maker prefers to bet on the color whose proportion is known. The acts X and Y agree on G, and so do X' and Y'. With $E = \{R, B\}$, we have $X = X_E 0$, $Y = Y_E 0$, $X' = X_E 100$, and $Y' = Y_E 100$. By the weak sure-thing principle, $X \succ Y$ implies $X' \succsim Y'$, which conflicts with the typical choice pattern. Explicitly, under pessimism, normalize $u(0) = 0$, so that $u(100) > 0$. Then

$$U_{+1,u}(X) = \min\{u(100) + \rho(R), \rho(B), \rho(G)\}, \qquad U_{+1,u}(Y) = \min\{\rho(R), u(100) + \rho(B), \rho(G)\}.$$

If $\rho(B) \leq \rho(R)$, then $U_{+1,u}(X) \leq \min\{\rho(B), \rho(G)\} \leq U_{+1,u}(Y)$. Hence $X \succ Y$ requires $\rho(R) < \rho(B)$. Analogously, $Y' \succ X'$ requires $\rho(B) < \rho(R)$. The decision maker would have to disbelieve blue more than red and red more than blue. The same computation with max in place of min shows that the same holds under optimism. Thus, like subjective expected utility, tropical expected utility cannot encompass the Ellsberg paradoxon. In particular, the orientation is not an ambiguity attitude. Ambiguity concerns differences between sources of uncertainty, such as events with known and with unknown probabilities (Abdellaoui, Baillon, Placido, and Wakker, 2011, p. 702). Since all events are judged by one ranking function, tropical expected utility has a single source. It is a qualitative analog of subjective expected utility rather than a model of ambiguity.

These arguments therefore apply to tropical expected utility as much as to subjective expected utility. Arguments that rely on objective randomization, such as the common-ratio effect or a preference for randomization, have no counterpart in our model, which has no objective randomization. A further argument applies to tropical expected utility with more force. By Lemma 7 in the Appendix, a pessimistic decision maker is indifferent between X and $X \wedge Y$ whenever $X \precsim Y$, although $X \wedge Y$ yields strictly less in some states. Tropical independence thus implies an insensitivity to outcomes in states that do not determine the evaluation, which is the price of non-compensatory evaluation.

4.2 Behavior Beyond Alternative Decision Theories

While tropical expected utility shares these limitations with subjective expected utility, it captures behavior that alternative decision theories cannot. We highlight three patterns that yield clean tests. For simplicity, we consider a pessimistic decision maker with linear utility $u(x) = x$ (see Section 5.1 on the characterization of the tropical expected utility representation with linear utility).

Example 4 (Locally irrelevant states)**.** Let $\Omega = \{\omega_1, \omega_2\}$ and $\rho(\omega_1) = \rho(\omega_2) = 0$. The decision maker is indifferent between $X = (0, 100)$ and $X' = (0, 1000)$, since both have value 0. Yet state ω_2 is non-null, since she strictly prefers X to $Y = (0, -50)$, which has value -50. Under subjective expected utility with a strictly increasing utility function, raising the outcome in a state of positive probability always yields a strictly better act. More generally, under pessimism with at least two non-null states, raising the outcome in a state that does not attain the minimum in $U_{+1,u}$ leaves the act's value unchanged. A state can thus matter for some acts and be irrelevant for others.

In the previous example, all ranks are 0, and the pessimistic criterion reduces to the maximin criterion of Wald (1950), who states it as minimizing the maximum loss, and to the Hurwicz criterion $\alpha \min_{\omega\in\Omega} X(\omega) + (1-\alpha)\max_{\omega\in\Omega} X(\omega)$ with $\alpha = 1$ (Hurwicz, 1951; Arrow and Hurwicz, 1972). Milnor (1954, Theorems 3 and 4) axiomatizes both criteria. With positive ranks, tropical expected utility departs from this criterion as well, as the following example demonstrates.

Example 5 (Ignored losses in non-null states)**.** Let $\Omega = \{\omega_1, \omega_2\}$, $\rho(\omega_1) = 0$, and $\rho(\omega_2) = 3$. Then

$$\begin{aligned} U_{+1,u}(0, 100) &= \min\{0, 103\} = 0, \\ U_{+1,u}(0, -2) &= \min\{0, 1\} = 0, \\ U_{+1,u}(0, -10) &= \min\{0, -7\} = -7. \end{aligned}$$

So the decision maker is indifferent between $(0, -2)$ and $(0, 100)$, although the former pays less in ω_2, and ω_2 is non-null, since she strictly prefers $(0, 100)$ to $(0, -10)$. Under Choquet expected utility (Schmeidler, 1989) with any capacity ν and a strictly increasing utility function, indifference between $(0, -2)$ and $(0, 100)$ requires the capacity ν to satisfy

$$\nu(\{\omega_2\})\big(u(100) - u(0)\big) = -\big(1 - \nu(\{\omega_1\})\big)\big(u(0) - u(-2)\big).$$

The left side is nonnegative and the right side nonpositive, so $\nu(\{\omega_2\}) = 0$ and $\nu(\{\omega_1\}) = 1$, and then $(0, -10) \sim (0, 100)$. This covers rank-dependent utility, whose capacity is a transformation of a probability (Quiggin, 1982), and disappointment aversion, which on binary acts is a special case of rank-dependent utility (Gul, 1991, p. 677). Under α-maxmin expected utility (Ghirardato, Maccheroni, and Marinacci, 2004) with any set of priors and any $\alpha \in [0, 1]$, indifference analogously requires that every prior assigns probability 0 to ω_2, and again $(0, -10) \sim (0, 100)$. In particular, the criteria of Wald (1950) and Hurwicz (1951), the special cases in which the set of priors contains all probability measures, strictly prefer $(0, 100)$ to $(0, -2)$. The Hurwicz value $\alpha u(0) + (1-\alpha)u(100)$ of $(0, 100)$ is weakly larger than $u(0)$, the value $\alpha u(-2) + (1-\alpha)u(0)$ of $(0, -2)$ is weakly smaller than $u(0)$, and the two are equal only if $\alpha = 0$ and $\alpha = 1$ at once. Wald's criterion is the case $\alpha = 1$. Under the pessimistic qualitative utility of Dubois, Prade, and Sabbadin (2001, p. 462), which evaluates an act X by $\min_{\omega\in\Omega} \max\{1 - \pi(\omega), u(X(\omega))\}$ on a finite ordinal scale, with a strictly increasing utility u and degrees of possibility π with $\pi(\omega_1) = 1$, indifference requires $1 - \pi(\omega_2) \geq u(0)$, and then also $(0, -10) \sim (0, 100)$. A single indifference together with a single strict preference thus refutes Choquet expected utility, including subjective expected utility, rank-dependent utility, disappointment aversion, and neo-additive capacities (Chateauneuf, Eichberger, and Grant, 2007), α-maxmin expected utility, including the criteria of Wald (1950) and Hurwicz (1951), and pessimistic qualitative utility.

Example 6 (Thresholds for ignoring losses)**.** Let $\Omega = \{\omega_1, \omega_2\}$, $\rho(\omega_1) = 0$, and $\rho(\omega_2) = 3$. Let X_ℓ be the act that pays 10 in ω_1 and $10 - \ell$ in ω_2, i.e., a loss of ℓ in the disbelieved state. Its value is $\min\{10, 13 - \ell\}$. For $\ell \leq 3$, the decision maker is indifferent between X_ℓ and the sure amount 10. That is, she ignores the loss altogether. For $\ell > 3$, the value is $13 - \ell$, so every additional unit of loss counts fully. Under subjective expected utility with $0 < \mu(\omega_2) < 1$ and linear utility, the certainty equivalent $10 - \mu(\omega_2)\ell$ decreases in ℓ from the first unit on, and by the fraction $\mu(\omega_2)$ of every unit. Under the pessimistic qualitative utility of Dubois, Prade, and Sabbadin (2001), with $\pi(\omega_1) = 1$, the value of X_ℓ is $\min\{u(10), \max\{1 - \pi(\omega_2), u(10 - \ell)\}\}$. If $1 - \pi(\omega_2) \geq u(10)$, no loss in ω_2 is ever taken into account. Otherwise, small losses count fully, and once $u(10 - \ell)$ falls below $1 - \pi(\omega_2)$, further losses no longer count. Catastrophic losses in implausible states are thus neglected, which is the reverse of the pattern of tropical expected utility. Tropical expected utility thus produces an all-or-nothing response to losses in disbelieved states. Losses up to the degree of disbelief are ignored, and larger losses dominate the evaluation.

These patterns reflect the non-compensatory character of the criterion. The first two examples are violations of strict monotonicity in non-null states, which subjective expected utility requires and which a single choice can refute. The second example also refutes Choquet expected utility, α-maxmin expected utility, and pessimistic qualitative utility. The third example describes people who neglect moderate losses in scenarios they consider unlikely but react strongly to catastrophic losses in the same scenarios, unlike pessimistic qualitative utility, which neglects the catastrophic ones.

5 Constant Disbelief Exchange Rate

We now focus on tropical expected utility with linear utility, $u(x) = |t|x$, and, as its most special but important case, the criteria with $t = \sigma$. Since utility and disbelief are measured in the same units, one step of disbelief is then worth $1/|t|$ units of money at every level of wealth. We call this the disbelief exchange rate. It is constant. For general u, one step of disbelief at wealth x is worth $1/(u(x) - u(x-1))$ units of money, so the disbelief exchange rate varies with wealth. Utility is linear here in two senses. The utility function is (standard) linear, $u(x) = |t|x$, i.e., the tropical power $x^{\odot|t|}$, and the evaluation preserves tropical sums of acts and is tropically homogeneous of degree $|t|$, so that $U_t/|t|$ is tropically linear in the act (see Remark 2). The first implies the second, and Theorem 2 characterizes both by strong tropical independence.

In the sense of Schipper (2026b, Proposition 2), the risk attitude is constant if and only if utility is affine, $u(x) = mx + u(0)$ with an integer $m \geq 1$. For $t \in \mathbb{Z} \setminus \{0\}$, let

$$U_t(X) := \operatorname{sgn}(t) \cdot \min_{\omega \in \Omega} \big\{ t \cdot X(\omega) + \rho(\omega) \big\},$$

i.e., $U_t = U_{\operatorname{sgn}(t),u}$ with linear utility $u(x) = |t|x$. Thus $U_t(X) = \min_{\omega\in\Omega}\{tX(\omega)+\rho(\omega)\}$ if $t > 0$ and $U_t(X) = \max_{\omega\in\Omega}\{|t|X(\omega) - \rho(\omega)\}$ if $t < 0$. In tropical notation, with $\oplus^{+1} = \min$ and $\odot = +$, we have $t \cdot X(\omega) = X(\omega)^{\odot t}$ and $U_t(X) = \big(\bigoplus\{\rho(\omega) \odot X(\omega)^{\odot t} : \omega \in \Omega\}\big)^{\odot \operatorname{sgn}(t)}$. The most special cases are $U_1(X) = \min_{\omega\in\Omega}\{X(\omega) + \rho(\omega)\}$ and $U_{-1}(X) = \max_{\omega\in\Omega}\{X(\omega) - \rho(\omega)\}$. If $u(x) = mx + u(0)$, then $U_{\sigma,u} = U_{\sigma m} + u(0)$, since the constant $\sigma u(0)$ can be taken out of the minimum. So linear utility means a representation by U_t, with $t = \sigma m$, whose sign is the orientation. In Schipper (2026b), $|t|$ is the intensity of the risk attitude. Schipper (2026b, Proposition 7 (iii)) describes the investment demand under pessimism with linear utility.

Linear utility is characterized by one further property.

P6 (Constant additivity) For all $X, Y \in \mathcal{X}$ and $k \in \mathbb{Z}$, $X \succsim Y$ if and only if $X + k \succsim Y + k$.

Adding the same sure amount to two acts does not change their ranking. That is, the preference does not depend on wealth, so the decision maker exhibits both decreasing and increasing aversion to risk in the sense of Schipper (2026b). In expected utility, this property characterizes constant absolute risk aversion. It is natural for stakes that are small relative to wealth, but less so for large stakes, where people often become more willing to take risks as they get richer. Theorem 1 does not require constant additivity.

Constant additivity concerns tropical scalar multiplication, $c \odot X = X + c$. Together with tropical independence, it covers both tropical operations, so that tropical linear combinations $(c \odot X) \oplus^\sigma Z$ take the place of mixtures in the form of the independence axiom. This is strong tropical independence.

P3* (Strong tropical independence) There is $\sigma \in \{-1, +1\}$ such that, for all $X, Y, Z \in \mathcal{X}$ and $c \in \mathbb{Z}$, if $X \succsim Y$, then $(c \odot X) \oplus^\sigma Z \succsim (c \odot Y) \oplus^\sigma Z$. We call the case $\sigma = +1$ the min-plus form and the case $\sigma = -1$ the max-plus form.

For $\sigma = +1$, P3* requires that, for all $X, Y, Z \in \mathcal{X}$ and $c \in \mathbb{Z}$, if $X \succsim Y$, then $(X+c)\wedge Z \succsim (Y+c)\wedge Z$. For $\sigma = -1$, it requires that, if $X \succsim Y$, then $(X+c)\vee Z \succsim (Y+c)\vee Z$. That is, adding the same sure amount to two acts and then capping both by the same act from above, or raising both to the same floor, does not reverse their ranking. The next lemma shows that strong tropical independence is exactly tropical independence together with constant additivity.

Lemma 2. *Let $\succsim$ satisfy P1 and P2. Then $\succsim$ satisfies P3* in the min-plus (max-plus) form if and only if it satisfies P3^{+} (P3^{-}) and P6.*

Remark 1 (Tropical mixture set)**.** Herstein and Milnor (1953, p. 292) call a set $\mathcal{S}$ a *mixture set* if any $a, b \in \mathcal{S}$ and $\mu \in [0,1]$ determine an element $\mu a + (1-\mu)b \in \mathcal{S}$ such that (1) $1a + (1-1)b = a$, (2) $\mu a + (1-\mu)b = (1-\mu)b + \mu a$, and (3) $\lambda[\mu a + (1-\mu)b] + (1-\lambda)b = (\lambda\mu)a + (1-\lambda\mu)b$. Our set of acts is a tropical mixture set. For weights $a, b \in \mathbb{N} \cup \{\infty\}$ with $a \oplus b = 0$, the tropical counterpart of $\mu + (1-\mu) = 1$, the *tropical mixture* of acts X and Y is $\big((a \odot X^{\odot\sigma}) \oplus (b \odot Y^{\odot\sigma})\big)^{\odot\sigma}$, i.e., state by state $\min\{X + a, Y + b\}$ if $\sigma = +1$ and $\max\{X - a, Y - b\}$ if $\sigma = -1$.

- The weights $(0, \infty)$ yield X, which is (1).
- The tropical sum is commutative, which is (2).
- Mixing the mixture with weights (a, b) again with Y, with outer weights (a', b'), yields the mixture of X and Y with weights $(a' \odot a, (a' \odot b) \oplus b')$, which again have tropical sum 0. This is (3), since $\lambda\mu$ and $1 - \lambda\mu = \lambda(1-\mu) + (1-\lambda)$ become $a' \odot a$ and $(a' \odot b) \oplus b'$.

Unlike Savage's acts, the acts thus form a mixture set without objective lotteries, which is why in our context it is appropriate to speak of independence. Equal weights $a = b = 0$, the counterpart of the 50–50 mixtures in Herstein and Milnor's Axiom 3 (p. 293), yield the tropical sum $X \oplus^{\sigma} Z$. Our P3 is therefore an analog of their Axiom 3, with weak preference in place of indifference.

Remark 2 (Tropical homogeneity and linearity)**.** With linear utility, the evaluation preserves tropical sums of acts and is tropically homogeneous of degree $|t|$. For $t > 0$, acts X, Y, and $c, d \in \mathbb{Z}$,

$$U_t\big((c \odot X) \oplus^{+1} (d \odot Y)\big) = \big((tc) \odot U_t(X)\big) \oplus^{+1} \big((td) \odot U_t(Y)\big),$$

so U_t preserves tropical sums and satisfies $U_t(c \odot X) = c^{\odot t} \odot U_t(X)$, i.e., it is tropically homogeneous of degree t. Dividing by t, the normalized evaluation U_t/t preserves tropical sums and commutes with tropical multiplication by scalars. It is therefore a linear functional in the sense of idempotent analysis (Kolokoltsov, 2001, Theorem 1), with ranks $\rho/|t|$, which lie on the grid $\frac{1}{|t|}\mathbb{Z}$. Analogously, for $t < 0$, U_t preserves tropical sums in the max-plus semiring and is tropically homogeneous of degree $|t|$, with $|t|c$ and $|t|d$ in place of tc and td, and $U_t/|t|$ is linear. To verify this for $t > 0$, note that $(c \odot X) \oplus^{+1} (d \odot Y)$ is the act $\min\{X + c, Y + d\}$. Since $t > 0$, multiplying by t preserves minima, so $t \min\{X(\omega) + c, Y(\omega) + d\} = \min\{tX(\omega) + tc, tY(\omega) + td\}$ in every state. Adding $\rho(\omega)$ and minimizing over ω, the two minima can be taken in either order, and the constants tc and td can be taken out of the minimum over states, so that

$$\begin{aligned} U_t\big(\min\{X + c, Y + d\}\big) &= \min_{\omega\in\Omega} \min\big\{tX(\omega) + tc + \rho(\omega),\ tY(\omega) + td + \rho(\omega)\big\} \\ &= \min\Big\{tc + \min_{\omega\in\Omega}\{tX(\omega) + \rho(\omega)\},\ td + \min_{\omega\in\Omega}\{tY(\omega) + \rho(\omega)\}\Big\} \\ &= \min\big\{tc + U_t(X),\ td + U_t(Y)\big\}. \end{aligned}$$

For $t < 0$, $(c \odot X) \oplus^{-1} (d \odot Y)$ is the act $\max\{X + c, Y + d\}$, and the same steps with $|t|$ in place of t, maxima in place of minima, and $-\rho(\omega)$ in place of $\rho(\omega)$ yield $U_t\big(\max\{X + c, Y + d\}\big) = \max\big\{|t|c + U_t(X),\ |t|d + U_t(Y)\big\}$.

This homogeneity, and the linearity of $U_t/|t|$, concern the act. It is the tropical counterpart of the linearity of the expectation in the random variable, $E[aX + bY] = aE[X] + bE[Y]$, which expected utility exhibits only under risk neutrality. Strong tropical independence is exactly what it requires, and Theorem 2 below is a Riesz-type representation for preferences (see the discussion after Theorem 2). The counterpart of the linearity of expected utility in mixtures, i.e., in probabilities, is different. It is tropical linearity in the ranking function, which holds for every utility function and both orientations. Write $U_{\sigma,u}(X;\rho)$ to make the ranking function explicit. If ρ and ρ' are ranking functions and $a, b \in \mathbb{N}$ with $\min\{a, b\} = 0$ (i.e., the tropical sum $a \oplus b$ is the tropical one), then $\rho'' := (a \odot \rho) \oplus^{+1} (b \odot \rho')$ is a ranking function, where $\min\{a, b\} = 0$ plays the role of $\alpha + (1-\alpha) = 1$. Then

$$U_{\sigma,u}(X;\rho'') = \Big(\big(a \odot U_{\sigma,u}(X;\rho)^{\odot\sigma}\big) \oplus \big(b \odot U_{\sigma,u}(X;\rho')^{\odot\sigma}\big) \Big)^{\odot\sigma},$$

i.e., $U_{+1,u}(X;\rho'') = \min\{U_{+1,u}(X;\rho) + a,\ U_{+1,u}(X;\rho') + b\}$ under pessimism and $U_{-1,u}(X;\rho'') = \max\{U_{-1,u}(X;\rho) - a,\ U_{-1,u}(X;\rho') - b\}$ under optimism. With general utility, only half of linearity in the act is lost. Since u is strictly increasing, it commutes with minima and maxima, so that

$$U_{\sigma,u}(X \oplus^{\sigma} Y) = U_{\sigma,u}(X) \oplus^{\sigma} U_{\sigma,u}(Y)$$

for all acts X and Y, whatever the utility function. The tropical sum is a lattice operation that uses only the order of outcomes, and every strictly increasing transformation of outcomes preserves it. This is why tropical independence can be imposed without restricting utility. In expected utility, by contrast, the expected utility of the state-wise minimum of two acts is not the minimum of their expected utilities. Tropical scalar multiplication, in contrast, uses the additive structure of outcomes. Adding c to an act changes the term of each state by $u(X(\omega) + c) - u(X(\omega))$, so $U_{\sigma,u}(c \odot X) = c' \odot U_{\sigma,u}(X)$ for a constant c' holds for all acts only if these increments do not depend on $X(\omega)$, i.e., only if u is affine (see also Theorem 2). Tropical expected utility with general utility is therefore tropically linear in beliefs and preserves tropical sums of acts, but it is not homogeneous in acts, just as expected utility is linear in probabilities but not in outcomes.

Without tropical scalar multiplication, the cardinal content of utility must come from elsewhere. In Theorem 1, P5 supplies it. Counting indifference classes between component acts measures utility differences in the same units as ranks, and requiring the same count in every non-null state pins down utility up to an additive constant, as tradeoff consistency does in subjective expected utility (Wakker, 1989). The orientation is thus an ordinal property, revealed by how the decision maker treats tropical sums of acts, while the intensity of the risk attitude is a cardinal property, revealed by counting (Schipper, 2026b).

Under strong tropical independence, the utility function is linear, and most of the properties of Theorem 1 become redundant. Of the Archimedean property, only bounded disbelief P4(ii) is needed. Its alternative is then determined by the orientation of P3* (see the proof of Lemma 3).

Lemma 3. *If $\succsim$ satisfies P1, P2, P3*, and P4(ii), then it satisfies P3, P4, and P5 for the same σ as P3*, and P6.*

By Lemma 3, Theorem 1 applies, and constant additivity P6 forces the utility function to be linear. This yields a characterization of tropical expected utility with linear utility.

Theorem 2. *A binary relation $\succsim$ on $\mathcal{X}$ satisfies P1, P2, P3*, and P4(ii) if and only if there exist $t \in \mathbb{Z} \setminus \{0\}$ and a ranking function ρ such that, for all acts $X, Y \in \mathcal{X}$,*

$$X \succsim Y \quad \text{if and only if} \quad \Big(\bigoplus_{\omega\in\Omega} X(\omega)^{\odot t} \odot \rho(\omega) \Big)^{\odot \operatorname{sgn}(t)} \geq \Big(\bigoplus_{\omega\in\Omega} Y(\omega)^{\odot t} \odot \rho(\omega) \Big)^{\odot \operatorname{sgn}(t)},$$

and U_t takes every integer value, i.e., $U_t(\mathcal{X}) = \mathbb{Z}$. Moreover, the following hold.

(i) *If $\succsim$ satisfies P3* for σ, then the representation holds with* $\operatorname{sgn}(t) = \sigma$. *If at least two states are non-null,* $\operatorname{sgn}(t)$ *is unique.*

(ii) *A state ω is null if and only if $\rho(\omega) = \infty$.*

(iii) *If $t > 0$, then $|t|$ is the number of indifference classes of acts X with $0 \precsim X \prec 1$. If $t < 0$, then $|t|$ is the number of indifference classes of acts X with $0 \prec X \precsim 1$. For given* $\operatorname{sgn}(t)$, *t and ρ are unique. Without the normalization $U_t(\mathcal{X}) = \mathbb{Z}$, $\succsim$ may also be represented by $U_{t'}$ with another ranking function, for instance with $t' = kt$ and ranks $k\rho$ for an integer $k \geq 2$, but in such a case $|t'| \geq |t|$.*

Written out with the min-plus notation for pessimism and max-plus for optimism, the representation reads as follows. For $t > 0$, the decision maker is pessimistic, and for all acts $X, Y \in \mathcal{X}$,

$$X \succsim Y \quad \text{if and only if} \quad \min_{\omega\in\Omega} \big\{tX(\omega) + \rho(\omega)\big\} \geq \min_{\omega\in\Omega} \big\{tY(\omega) + \rho(\omega)\big\}.$$

For $t < 0$, the decision maker is optimistic, and for all acts $X, Y \in \mathcal{X}$,

$$X \succsim Y \quad \text{if and only if} \quad \max_{\omega\in\Omega} \big\{|t|X(\omega) - \rho(\omega)\big\} \geq \max_{\omega\in\Omega} \big\{|t|Y(\omega) - \rho(\omega)\big\}.$$

The sign of t is the orientation, while its size is a matter of scale. By part (iii), $|t|$ is one plus the number of indifference classes strictly between two consecutive sure amounts, and by constant additivity this number is the same for every pair $k, k+1$. Preferences over acts alone do not separate $|t|$ from the scale of ranks. Indeed, $U_t/|t|$ depends only on $\rho/|t|$, so a preference represented by U_t is also represented by $U_{\operatorname{sgn}(t)}$ with ranks $\rho/|t|$ on the finer grid $\frac{1}{|t|}\mathbb{Z}$. The normalization $U_t(\mathcal{X}) = \mathbb{Z}$ fixes the scale. It makes consecutive indifference classes differ by one unit, so that ranks are integers measured in preference steps. Absent the normalization, the disbelief exchange rate $1/|t|$ is not separated from the scale of disbelief.

Theorem 2 is the preference analog of a representation theorem in idempotent analysis. In ordinary analysis, linear functionals on spaces of functions are integrals with respect to measures. The idempotent counterpart states that linear functionals on idempotent semimodules of functions, i.e., maps that preserve $\oplus^{+1}$ and commute with tropical multiplication by scalars, are idempotent integrals $h \mapsto \bigoplus\{f(x) \odot h(x)\}$, the tropical sum over all x, for a unique function f (Kolokoltsov and Maslov, 1987, Theorem 1, for the max-plus semiring; Kolokoltsov and Maslov, 1997, Theorem 1.9, and Kolokoltsov, 2001, Theorem 1, for the min-plus semiring). Strong tropical independence P3* is the preference counterpart of this linearity. Property P3$^+$ makes the evaluation preserve state-wise minima, i.e., $\oplus^{+1}$, and Property P6 makes it tropically homogeneous of degree $|t|$, $U_t(c \odot X) = c^{\odot|t|} \odot U_t(X)$, so that $U_t/|t|$ commutes with tropical multiplication by constants. Theorem 2 is thus a Riesz-type representation for preferences in idempotent analysis. Compared with the result for functionals, our setting has four additional features. First, the properties are stated for a preference rather than for a real-valued functional, so that the evaluation must be constructed. Second, outcomes are integers, and without solvability there may be indifference classes strictly between two consecutive sure amounts. To give them integer values, the evaluation stretches the money scale by $|t|$, valuing the sure amount k at $|t|k$. This is where the coefficient t arises. Third, Theorem 1 drops the commutation with tropical scalars altogether and identifies a potentially nonlinear utility function u jointly with a ranking function ρ. Fourth, the min-plus and the max-plus case share all properties except the form of P3*, with the sign of t selecting the semiring.

5.1 Ranks Measured in Money

When one step of disbelief is worth one unit of money, ranks are measured in money. This case is characterized by one further axiom.

P7 (Solvability) For every $X \in \mathcal{X}$, there is $k \in \mathbb{Z}$ with $X \sim k$.

Every act is indifferent to some sure amount of money. Behaviorally, the decision maker can always name a certainty equivalent on the given grid of outcomes. In expected utility, certainty equivalents exist by continuity, but they need not lie on a grid. With integer outcomes, P7 is a strong requirement. It says that the decision maker's evaluations are no finer than the outcome grid, so that degrees of disbelief are measured in units of money. We use it only to single out the disbelief exchange rate 1, i.e., $|t| = 1$.

Solvability singles out this exchange rate under the properties of both Theorem 1 and Theorem 2. Under those of Theorem 1, it forces linear utility with slope 1, so that P7 implies P6.

Proposition 1. *Let $\succsim$ satisfy P1 to P5, and let $U_{\sigma,u}$ be the representation of Theorem 1. Then $\succsim$ satisfies P7 if and only if $u(x) = x + u(0)$ for all x. In this case, $\succsim$ is represented by U_σ, i.e., by U_1 or U_{-1}, and satisfies P6.*

With linear utility, solvability makes the Archimedean axiom redundant.

Lemma 4. *If $\succsim$ satisfies P1, P2, P3*, and P7, then it satisfies P4(ii).*

By Lemma 4, Theorem 2 applies, and solvability forces the disbelief exchange rate 1, i.e., $|t| = 1$. This yields a characterization of tropical expected utility with ranks measured in money.

Corollary 1 (Ranks measured in money)**.** *A binary relation $\succsim$ on $\mathcal{X}$ satisfies P1, P2, P3*, and P7 if and only if there exist $\sigma \in \{-1, +1\}$ and a ranking function ρ such that, for all acts $X, Y \in \mathcal{X}$,*

$$X \succsim Y \quad \text{if and only if} \quad \Big(\bigoplus_{\omega\in\Omega} X(\omega)^{\odot\sigma} \odot \rho(\omega)\Big)^{\odot\sigma} \geq \Big(\bigoplus_{\omega\in\Omega} Y(\omega)^{\odot\sigma} \odot \rho(\omega)\Big)^{\odot\sigma}.$$

Moreover, the following hold.

(i) If $\succsim$ satisfies P3 for σ, then the representation holds with this σ. If at least two states are non-null, σ is unique.*

(ii) A state ω is null if and only if $\rho(\omega) = \infty$.

(iii) For given σ, ρ is unique. The representation is the certainty equivalent, i.e., $X \sim U_\sigma(X)$ for every act X.

Written out, for $\sigma = +1$, the decision maker is pessimistic, and for all acts $X, Y \in \mathcal{X}$,

$$X \succsim Y \quad \text{if and only if} \quad \min_{\omega\in\Omega} \big\{X(\omega) + \rho(\omega)\big\} \geq \min_{\omega\in\Omega} \big\{Y(\omega) + \rho(\omega)\big\}.$$

For $\sigma = -1$, the decision maker is optimistic, and for all acts $X, Y \in \mathcal{X}$,

$$X \succsim Y \quad \text{if and only if} \quad \max_{\omega\in\Omega} \big\{X(\omega) - \rho(\omega)\big\} \geq \max_{\omega\in\Omega} \big\{Y(\omega) - \rho(\omega)\big\}.$$

Remark 3 (Ranks revealed in money)**.** With $t = \sigma$, one step of disbelief is worth one unit of money, so the rank of a state is directly revealed in money. For $\omega \in \Omega_+$, $0_\omega y \sim \rho(\omega)$ for large y under P3$^+$, and $0_\omega y \sim -\rho(\omega)$ for small y under P3$^-$. That is, under P3$^+$, $\rho(\omega)$ is the sure amount the decision maker considers equivalent to an act that yields nothing in state ω and a large outcome otherwise. Under P3$^-$, $-\rho(\omega)$ is the sure amount equivalent to an act that yields nothing in state ω and a large loss otherwise. To verify this under P3$^+$, let $\succsim$ be represented by U_1. For $y \geq \rho(\omega)$,

$$U_1(0_\omega y) = \min\left\{\rho(\omega), \min_{\omega' \neq \omega}\{y + \rho(\omega')\}\right\} = \rho(\omega),$$

since $y + \rho(\omega') \geq y \geq \rho(\omega)$, and $U_1(k) = k$ for every sure amount k, since the minimum rank is 0. So $0_\omega y \sim \rho(\omega)$. Under P3$^-$, the duality lemma (Lemma 6 (D5) in the appendix) reduces the claim to the case P3$^+$.

For $t > 1$, the same computation yields $U_t(0_\omega y) = \rho(\omega)$ for $ty \geq \rho(\omega)$, while the sure amount k has value tk. So $0_\omega y$ is worth $\rho(\omega)/t$ units of money and lies weakly between the sure amounts $\lfloor \rho(\omega)/t \rfloor$ and $\lceil \rho(\omega)/t \rceil$. It is indifferent to a sure amount only if t divides $\rho(\omega)$. Sure amounts then reveal ranks only up to the money grid. The exact rank is the number of indifference classes of acts X with $0 \precsim X \prec 0_\omega y$.

As utility is scaled up, disbelief becomes worth nothing relative to utility, and tropical expected utility approaches Wald's (1950) maximin criterion on the states of finite rank under pessimism, and the corresponding maximax criterion under optimism (Schipper, 2026b).

6 Discussion

Summary. We characterized expected utility for subjective ranking functions using a tropical expected utility functional. That is, starting from preferences over acts, we derived jointly a subjective ranking function, representing the decision maker's personal disbeliefs, and a utility function such that the expected utility functional is the idempotent integral of utility with respect to the ranking function. The functional features either a pessimistic or optimistic disposition toward risk, its orientation. The utility features the intensity of the risk attitude so that familiar notions of risk aversion carry over in tropical form as studied in a companion paper (see Schipper, 2026b). The representations, differing by how the decision maker trades off disbelief for utility, are nested as shown in Table 2.

Axioms	Representation	Orientation	Disbelief exchange rate
P1–P5	$U_{\sigma,u}$, u strictly increasing	$\sigma \in \{-1, +1\}$	$1/(u(x) - u(x-1))$ at wealth x
P1, P2, P3*, P4(ii)	U_t, $t \in \mathbb{Z} \setminus \{0\}$	$\sigma = \operatorname{sgn}(t)$	constant, $1/\lvert t \rvert$
P1, P2, P3*, P7	U_σ	$\sigma = t$	1

Table 2: Nested representations.

Related behavioral characterizations. We are not the first to characterize a decision criterion based on ranking functions. Giang and Shenoy (2000) translate the von Neumann–Morgenstern axioms to Spohn's ranking functions. Their objects of choice are ranking lotteries, i.e., prizes with exogenously given (and thus objective) ranking functions. Compound ranking lotteries are reduced by the laws of ranking functions, i.e., the rank of a prize is the minimum, over the second-stage ranking lotteries, of the rank of the lottery plus the rank of the prize in it. Analogous to von Neumann and Morgenstern (1947), the utility of a prize is elicited by a ranking lottery to which the prize is indifferent and that puts finite ranks only on the best and the worst prize. With

probabilities, the probability of the worst prize is one minus the probability of the best. With ranks, the degree of disbelief κ_b in the best prize and the degree of disbelief κ_w in the worst prize only satisfy $\min\{\kappa_b, \kappa_w\} = 0$. Hence utilities are pairs (κ_b, κ_w) or, equivalently, what Spohn (2012) calls two-sided ranks $v = \kappa_w - \kappa_b \in \mathbb{Z} \cup \{-\infty, \infty\}$. A prize with $v > 0$ is indifferent to the ranking lottery that disbelieves the worst prize with rank v and the best prize not at all. Similarly, a prize with $v < 0$ is indifferent to the ranking lottery that disbelieves the best prize with rank $-v$ and the worst prize not at all. The prize with $v = 0$ is indifferent to the ranking lottery that disbelieves neither the best nor the worst prize. It separates gains, $v > 0$, from losses, $v < 0$. The utility of a lottery is obtained by applying the idempotent min-plus integral to each component of the pair (Giang and Shenoy, 2000, Theorem 1). In terms of v, a ranking lottery that assigns the degree of disbelief δ_i to prize i has utility $\min_i\{\delta_i + \max\{v_i, 0\}\}$ if some prize i of rank 0 has v_i weakly larger than zero, and $\max_i\{\min\{v_i, 0\} - \delta_i\}$ otherwise. Thus the decision maker is pessimistic over gains and optimistic over losses relative to the zero payoff, and prizes on the other side of zero are evaluated as if they were zero. On lotteries over prizes with v weakly larger than zero, the criterion is of our form $U_{+1,u}$, with v in the role of $u \circ X$, and on lotteries over prizes with v weakly smaller than zero, it is of our form $U_{-1,u}$. The approach has several limitations. First, in their theory the existence of ranks is assumed as a primitive rather than derived from behavior like in our approach. Since these ranks attached to prizes are exogenously given, they must be objective rather than subjective. However, we do not know of any compelling "randomization" device generating objective ranks and thus objective ranking lotteries that would be required for testing their theory with behavioral experiments. Second, the disposition to risk, the orientation, is tied exogenously to the sign of the utility rather than being derived from the choices of the decision maker. The zero prize acts as a reference point, with pessimism over gains and optimism over losses. In this respect, their approach is closer to prospect theory (Kahneman and Tversky, 1979), with its risk aversion over gains and risk seeking over losses, than to von Neumann–Morgenstern expected utility, in which the risk attitude is a property of the derived utility function over all prizes. Their criterion, like those of Pearl (1993) and Wilson (1995), which switch orientation the other way round (Table 1), therefore violates P3 and is not covered by our results. Third, the criterion is insensitive to how good and how bad prizes are. A ranking lottery that assigns rank 0 to both a very good and a very bad prize is equivalent to the zero prize, however good and however bad the two prizes are. In particular, improving the good prize or worsening the bad one leaves the ranking lottery's utility unchanged, so the criterion is only weakly monotone in prizes.

Spohn (2025) modifies the axioms of Giang and Shenoy (2000). He keeps their objects of choice, ranking lotteries, and their reduction of compound ranking lotteries. Instead of comparing each prize with a ranking lottery over the best and the worst prize, he compares it only with ranking lotteries over the best prize and the prize itself (Spohn, 2025, Axiom 9). He also requires that, in a ranking lottery that disbelieves neither of two prizes, only the better prize counts (Axiom 10). His representation theorem yields a unique disutility function ℓ from prizes to the natural numbers, which measures the distance from the best prize, and the decision maker minimizes $\min_i\{\delta_i + \ell(o_i)\}$ over ranking lotteries that assign the degree of disbelief δ_i to prize o_i (Spohn, 2025, Theorem 3.1). With $u \circ X = -\ell$, this is our optimistic criterion $U_{-1,u}$, and with $\ell(x) = -x$ it is U_{-1}. As in Giang and Shenoy (2000), ranks are exogenously given and objective, and the orientation is built into the axioms. Spohn (2025, p. 268) notes that, since there are no objective ranks, ranks and disutilities should be measured simultaneously from preferences, as in Savage (1954), and he leaves this for future work. Our results provide such a joint measurement, for his criterion as well as for its pessimistic counterpart, and they both are embedded in a family with graded risk attitudes.

Dubois, Prade, and Sabbadin (2001) characterize subjective expected utility for qualitative possibility measures introduced by Zadeh (1978) and Dubois and Prade (1998). A possibility

measure assigns to each state a degree of possibility on a linearly ordered scale from 0 (impossible) to 1 (fully possible), with some state fully possible, and the possibility of an event is the highest possibility of its states. Unlike a probability measure, a possibility measure is maxitive rather than additive. It is like a ranking function turned upside down (see Wilson, 1995, Section 6.2), but different from ranking functions, differences in degrees of possibility do not carry any meaning. Only the order of degrees matters. In the style of Savage (1954), Dubois, Prade, and Sabbadin (2001, Theorems 5 and 6) derive from a complete preorder over acts a possibility measure, with degree $\pi(\omega)$ for state ω, and a utility function u on outcomes, both with values in the same finite ordinal scale, which we label by numbers in $[0, 1]$. Their pessimistic and optimistic criteria are characterized by separate axioms, pessimism and optimism, just as our orientation is revealed by our properties P3$^+$ or P3$^-$. Their pessimistic criterion is $\min_\omega \max\{1-\pi(\omega), u(X(\omega))\}$, compared with our $\min_\omega\{\rho(\omega) + u(X(\omega))\}$. Both are idempotent integrals, theirs in the semiring with min and max, i.e., a Sugeno integral (Dubois, Prade, and Sabbadin, 2001, p. 462), and ours in the min-plus semiring. Replacing the tropical product by the maximum has two consequences. First, utility and disbelief are compared but not traded off, so there is no analog of the intensity $|t|$. Second, a pessimist in their sense evaluates a state ω only up to $1-\pi(\omega)$, so any outcome with utility below $1-\pi(\omega)$ counts as $1-\pi(\omega)$. She therefore neglects catastrophic rather than moderate losses in implausible states (see Example 6), while in our criterion every worsening of an outcome in a state of finite rank lowers its contribution. Dubois and Prade (1995) characterize a counterpart for possibilistic lotteries in the style of von Neumann and Morgenstern (1947).

How does tropical expected utility relate to Shackle's potential surprise? Shackle (1949) proposes a decision theory based on potential surprise. The decision maker assigns to every outcome x of a course of action a degree of potential surprise $y(x)$, and she evaluates the pair $(x, y(x))$ by the intensity $\varphi(x, y)$ of enjoyment or distress by anticipation, which increases with the desirability of x and decreases with y (pp. 18–20). Relative to a neutral outcome, she focuses on the gain of maximal enjoyment and on the loss of maximal distress, and she replaces each by the equally stimulating outcome that carries no potential surprise, her focus-gain and her focus-loss (pp. 19 and 25). Courses of action are then compared by their pairs of focus-gain and focus-loss on a gambler indifference-map (pp. 30–31). Shackle does not specify φ. With states in place of outcomes, ranks in place of degrees of potential surprise, and enjoyment $u(x) - y$ and distress $-(u(x) + y)$, the focus-gain in units of utility is $\max_\omega\{u(X(\omega)) - \rho(\omega)\}$ over the states with gains, which is our optimistic criterion, and the focus-loss is $\min_\omega\{u(X(\omega)) + \rho(\omega)\}$ over the states with losses, which is our pessimistic criterion. Shackle's decision maker thus evaluates gains optimistically and losses pessimistically, like Pearl (1993) and Wilson (1995), and combines the two by a further preference. Our decision maker instead evaluates all outcomes with a single orientation, and no neutral outcome separates gains from losses. Like Giang and Shenoy (2000) and Spohn (2025), Shackle takes degrees of potential surprise as primitive. They are assessed by introspection, by asking "what intensity of shock or surprise he would feel" (p. 11). Our results provide a behavioral measurement of potential surprise in units of utility.

Variational preferences and risk measures. Two neighboring literatures in economics and finance deal with functionals of similar form. Since a linear function on the set of probability measures attains its minimum at a point mass, for every act X,

$$\min_\mu \Big\{ \sum_\omega \mu(\omega) u(X(\omega)) + \sum_\omega \mu(\omega)\rho(\omega) \Big\} = \min_\omega \big\{ u(X(\omega)) + \rho(\omega) \big\} = U_{+1,u}(X),$$

where μ ranges over the probability measures on Ω. The left side is an instance of the variational representation of Maccheroni, Marinacci, and Rustichini (2006, Theorem 3), with the linear and grounded cost $\sum_\omega \mu(\omega)\rho(\omega)$. With $\rho \in \{0, \infty\}$, it reduces to evaluation by the worst case on

the states of rank 0. The right side is our pessimistic tropical expected utility representation. The identity explains why tropical expected utility does not need probabilities. The variational functional is concave in the profile of utilities, and a concave functional is the lower envelope of the ordinary linear functionals above it, i.e., of expectations with respect to probabilities. The probabilities on the left side are the coefficients of these linear functionals. Our criterion, on the right side, is linear in the tropical sense. As a function of the profile of utilities, it is an idempotent integral whose density is the ranking function (Section 5). Its coefficients are ranks, not probabilities, and the minimum on the left side is attained at point masses, i.e., at states. The probabilities in the variational form are thus an artifact of representing a tropically linear functional with ordinary linear functionals.

Beyond this identity, the similarity is formal. Their setting is that of Anscombe and Aumann (1963), with lotteries as outcomes and an affine utility function. Their axioms combine acts by mixtures, notably weak certainty independence and uncertainty aversion (Maccheroni, Marinacci, and Rustichini, 2006, Axioms A.2 and A.5), and the cost function is interpreted as an index of ambiguity aversion over probabilistic models. In our setting, outcomes are integers and the decision maker holds no probabilities. Our properties combine acts by state-wise minima or maxima, and ρ is a degree of disbelief in states. Moreover, only the pessimistic criterion takes the variational form, since variational preferences are ambiguity averse (Maccheroni, Marinacci, and Rustichini, 2006, Proposition 7).

In the theory of risk measures, the negative of the pessimistic criterion with $t = 1$ is the penalized maximum loss of the loss $-X$, i.e., $-U_1(X) = \max_\omega\{-X(\omega) - \rho(\omega)\}$. Zapata (2025, Theorem 4.1) characterizes the penalized maximum loss as a monetary risk measure that is maxitive and continuous from below. Maxitive monetary risk measures are convex (Zapata, 2025, Remark 3.2), and a real-valued convex function on a finite-dimensional space is continuous. So on a finite state space, continuity from below follows from maxitivity. Our Properties P2, P6, and P3$^+$ are the preference counterparts of monotonicity, cash additivity, and maxitivity.

Why are there only two orientations of the disposition? That the orientation takes only two values has precedents. Besides the behavioral characterization of qualitative possibility measures by Dubois, Prade, and Sabbadin (2001), also under complete ignorance, which corresponds to $\rho \equiv 0$, Maskin (1979, Theorem 10) obtains lexicographic maximin or maximax with ordinal utility, and Congar and Maniquet (2010, Theorem 1) show that no compromise between the two is possible even if indifference may be intransitive. On finite state spaces, α-maxmin expected utility satisfies the axioms by which Ghirardato, Maccheroni, and Marinacci (2004) separate perceived ambiguity from the reaction to it only if $\alpha \in \{0, 1\}$ (Eichberger, Grant, Kelsey, and Koshevoy, 2011, Theorem 2). With neo-additive capacities, the degree of optimism is separated from beliefs only through the uniqueness of the representation (Chateauneuf, Eichberger, and Grant, 2007, Lemma A.1), and it is undefined if the decision maker perceives no ambiguity (p. 556).

Why do we need so little richness? Our representation theorems require less richness than subjective expected utility theory. Savage (1954) requires an infinite state space that can be partitioned finely, Anscombe and Aumann (1963) take lotteries as outcomes, and Wakker (1989) takes a connected topological space of consequences, such as a Euclidean space. In each case, richness together with continuity serves to measure probabilities that are real-valued and additive, and to derive a utility function that is unique up to positive affine transformations. We need neither rich events nor mixtures, since tropical expected utility aggregates states by a minimum or a maximum rather than by an average. An act is indifferent to one of its component acts, so ranks never have to be compared with fractions of events, and the rank of each state is measured separately against the outcome scale. Hence finitely many states suffice. Continuity is replaced by counting. Since ranks and utilities are integers, they are measured in a common discrete unit by

counting indifference classes, which is the role of Properties P4 and P5. Wakker (1991, Theorem 5) replaces continuity by restricted solvability and an Archimedean axiom, which requires that every bounded standard sequence is finite. Our Property P4(i) implies this axiom, since the elements of a standard sequence lie in distinct indifference classes. Unlike Wakker's (1991) Archimedean axiom, which he calls merely technical, P4(i) restricts preferences, as discussed in Section 3. Some richness remains, but it is countable. Outcomes range over all integers, and the representation takes every integer value, a discrete analog of solvability. This is also why utility is unique up to an additive constant rather than up to a positive affine transformation. Multiplying utility by a positive constant other than one would either change the decision maker's disbelief exchange rate or, if the ranks are multiplied by the same constant, violate that the representation takes every integer value. Hence, of the positive affine transformations, only the addition of a constant remains, i.e., tropical scalar multiplication.

How does P5 relate to state independence and tradeoff consistency? Property P5 is the counterpart of state independence in subjective expected utility, such as Savage's (1954) P3 or the state independence of Anscombe and Aumann (1963), which requires that the ranking of outcomes does not depend on the state. Property P5 requires more, namely that utility differences do not depend on the state. That is, it expresses state independence of utility differences rather than of utilities. In this respect, P5 resembles tradeoff consistency, called cardinal coordinate independence in Wakker (1989), which together with continuity characterizes subjective expected utility with finitely many states. There, the tradeoff between two outcomes in a state is revealed by preferences between acts that differ in this state, while the outcomes in the other states serve as a gauge. Tradeoff consistency requires that tradeoffs revealed in this way are consistent with each other, within a state and across states. Wakker (1991, Sections 3 and 4) shows that it implies the sure-thing principle, and hence additivity across states. Like tradeoff consistency, P5 requires that tradeoffs are consistent across states, but it does not imply the sure-thing principle, which tropical expected utility satisfies only in its weak form (see Section 4.1). Additivity across states, in its tropical form, comes instead from tropical independence, P3.

Property P5 also measures tradeoffs differently. In subjective expected utility, raising the outcome in state ω from x to x' changes the value of an act by

$$\mu(\omega)\big(u(x') - u(x)\big),$$

so the utility difference is scaled by the probability of the state and is revealed by comparison with a utility difference in another state, the gauge. In tropical expected utility, the component acts of x and x' in state ω have the values $u(x) + \rho(\omega)$ and $u(x') + \rho(\omega)$, and

$$\big(u(x') + \rho(\omega)\big) - \big(u(x) + \rho(\omega)\big) = u(x') - u(x)$$

does not depend on the state. Since every integer value is attained, there are $u(x') - u(x) + 1$ indifference classes between the two component acts in every state, so the tradeoff can be measured directly by counting. Counting does not require particular indifferences to exist, which would require richness of outcomes, so P5 works with integer outcomes.

Under tropical expected utility, tradeoffs cannot be revealed through a gauge. Following Köbberling and Wakker (2003, Section 3), consider two states and write an act as the pair of its outcomes in the first and in the second state. From the indifference relation, they derive a relation between tradeoffs. We say that receiving x instead of x' in the first state is as good a tradeoff as receiving y instead of y' if $(x, z) \sim (x', z')$ and $(y, z) \sim (y', z')$ for some outcomes z and z' in the second state, i.e., if both are exactly offset by receiving z instead of z' in the second state. Tradeoff consistency requires that this derived relation breaks if one of the four outcomes

x, x', y, y' is improved (Köbberling and Wakker, 2003, Definition 2). Now let both states have rank 0, let u be the identity, and consider pessimism. The acts $(0, 10)$, $(5, 0)$, and $(7, 0)$ all have value 0. With $z = 10$ and $z' = 0$, the indifferences $(0, 10) \sim (5, 0)$ and $(0, 10) \sim (7, 0)$ reveal that receiving 0 instead of 5 is as good a tradeoff as receiving 0 instead of 7. Improving 5 to 7 thus does not break the relation, which violates tradeoff consistency. The reason is that the evaluation is determined by the lowest outcome. In $(0, 10)$ the outcome in the second state does not count, and in $(5, 0)$ and $(7, 0)$ the outcome in the first state does not count. Outcomes in the gauge state thus either determine the evaluation or do not count, and they never offset a difference in the first state. Köbberling and Wakker (2003, Section 4.2) show that for Choquet expected utility it suffices to restrict tradeoff consistency to comonotonic sets of acts, on which the evaluation is additive. The tropical analog would restrict it to sets of acts on which the same state attains the minimum. On such a set, the evaluation depends only on the outcome in this state, so no other state can serve as a gauge. Property P5 therefore measures tradeoffs within a single state against the preference scale itself, by counting indifference classes.

A Proofs

Translations of the Integers

Lemma 5. *A function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ is a strictly increasing bijection of $\mathbb{Z}$ onto itself if and only if $f(n) = n + f(0)$ for all $n \in \mathbb{Z}$.*

The "if" direction is immediate. For the "only if" direction, see Hrbacek and Jech (1999, Exercise 4.10).

Duality

Most proofs treat pessimism and obtain optimism from the following duality. Define the *dual* relation $\succsim^*$ by $X \succsim^* Y$ if and only if $-Y \succsim -X$, and for a utility function u, let $\tilde{u}(x) := -u(-x)$, which is again strictly increasing and integer-valued. The dual of $\succsim^*$ is $\succsim$, and $\tilde{\tilde{u}} = u$.

Let $m_P(\succsim)$ denote the number of indifference classes of acts X with $0 \precsim X \prec 1$, and let $m_O(\succsim)$ denote the number of indifference classes of acts X with $0 \prec X \precsim 1$, whenever these numbers are finite, as under P4(i). Under a representation by U_t, the two numbers coincide, since the set of attained values is invariant under adding $|t|$, so a half-open interval of length $|t|$ contains the same number of attained values whether it is open on the left or on the right. We use the one that matches the orientation.

Lemma 6. *The following hold, and each holds with the roles of $\succsim$ and $\succsim^*$ exchanged.*

(D1) $\succsim$ satisfies P1 (P2, P6, P7) if and only if $\succsim^*$ does.

(D2) $\succsim$ satisfies P3 (P3) for σ if and only if $\succsim^*$ satisfies it for $-\sigma$.*

(D3) $\succsim$ and $\succsim^*$ have the same null states.

(D4) $\succsim$ satisfies P4(i) (P4(ii), P5) if and only if $\succsim^*$ does.

(D5) $\succsim$ is represented by $U_{-1,u}$ if and only if $\succsim^*$ is represented by $U_{+1,\tilde{u}}$, with the same ranking function, and $U_{-1,u}$ takes every integer value if and only if $U_{+1,\tilde{u}}$ does. In particular, for $t > 0$, $\succsim$ is represented by U_{-t} if and only if $\succsim^*$ is represented by U_t. If $\succsim$ satisfies P6, then $m_O(\succsim) = m_P(\succsim^*)$.

PROOF. Negation of acts satisfies

$$-(X \wedge Y) = (-X) \vee (-Y), \tag{3}$$
$$-\big((c \odot X) \wedge Z\big) = \big((-c) \odot (-X)\big) \vee (-Z), \tag{4}$$
$$-\big(x_\omega y\big) = (-x)_\omega(-y), \tag{5}$$
$$X \precsim^* Z \precsim^* Y \iff -Y \precsim -Z \precsim -X. \tag{6}$$

Since the dual of $\succsim^*$ is $\succsim$, each statement for the pair $(\succsim, \succsim^*)$ yields the same statement for $(\succsim^*, \succsim)$. Hence it suffices to show (D1), (D3), and (D4) for P4 and P5 in one direction. For (D2), it suffices to show the equivalence for $\sigma = -1$, since the equivalence for $\sigma = +1$ then follows by exchanging the roles of $\succsim$ and $\succsim^*$.

(D1) to (D3) By definition, $\succsim^*$ satisfies P1 if and only if $\succsim$ does. For P2, let $\succsim$ satisfy P2. If $X \geq Y$, then $-Y \geq -X$, so $-Y \succsim -X$ by P2 for $\succsim$, i.e., $X \succsim^* Y$. Moreover, for $k \in \mathbb{Z}$,

$$\begin{aligned} k+1 \succ^* k &\iff -k \succ -(k+1) \\ &\iff (-k-1)+1 \succ -k-1. \end{aligned}$$

The first equivalence is the definition of $\succ^*$. The second holds since $-k = (-k-1)+1$ and $-(k+1) = -k-1$. The last statement is the second part of P2 for $\succsim$ at $-k-1$. For P7, $\succsim^*$ satisfies it if and only if $\succsim$ does, since $-X \sim k$ if and only if $X \sim^* -k$. For P6, let $\succsim$ satisfy P6, and let $X, Y \in \mathcal{X}$ and $k \in \mathbb{Z}$. Then

$$\begin{aligned} X \succsim^* Y &\iff -Y \succsim -X \\ &\iff -Y-k \succsim -X-k \\ &\iff -(Y+k) \succsim -(X+k) \\ &\iff X+k \succsim^* Y+k. \end{aligned}$$

The first and the last equivalence are the definition of $\succsim^*$. The second is P6 for $\succsim$ with $-k$. The third holds since $-Y-k = -(Y+k)$ and $-X-k = -(X+k)$. For P3, $\succsim^*$ satisfies P3$^+$ if and only if, for all X, Y, Z, $-Y \succsim -X$ implies $-(Y \wedge Z) \succsim -(X \wedge Z)$, i.e., by (3), $(-Y) \vee (-Z) \succsim (-X) \vee (-Z)$. Since $X \mapsto -X$ is a bijection of $\mathcal{X}$, this is P3$^-$ for $\succsim$. The same argument with (4) applies to P3*, with $-c$ in place of c. Null states coincide, since X and Y agree off ω if and only if $-X$ and $-Y$ do, and since $X \sim^* Y$ if and only if $-X \sim -Y$.

(D4) By (5), negation maps the act $x_\omega y$ to $(-x)_\omega(-y)$. By the definition of $\succsim^*$, $x_\omega y \sim^* x_\omega \bar{y}$ for all $y \geq \bar{y}$ if and only if $(-x)_\omega z \sim (-x)_\omega(-\bar{y})$ for all $z \leq -\bar{y}$. Likewise, $x_\omega y \sim^* x_\omega \bar{y}$ for all $y \leq \bar{y}$ if and only if $(-x)_\omega z \sim (-x)_\omega(-\bar{y})$ for all $z \geq -\bar{y}$. Hence, for a non-null state ω and an outcome x, the first alternative of P4(ii) for $\succsim^*$ at (x, ω) is the second alternative for $\succsim$ at $(-x, \omega)$, with $-\bar{y}$ in place of $\bar{y}$, and vice versa. Since $-x$ ranges over $\mathbb{Z}$ as x does, and since the null states coincide by (D3), $\succsim^*$ satisfies P4(ii) if and only if $\succsim$ does. Moreover, an act is a component act of outcome x in state ω for $\succsim^*$ if and only if its negation is a component act of outcome $-x$ in state ω for $\succsim$, i.e., $c^*(x, \omega) = -c(-x, \omega)$. By (6), negation maps the indifference classes of $\succsim^*$ between two acts onto the indifference classes of $\succsim$ between their negations of those two acts, in reverse order. In particular, it maps the classes between $c^*(x, \omega)$ and $c^*(x', \omega)$ onto the classes between $c(-x', \omega)$ and $c(-x, \omega)$. Since $x < x'$ if and only if $-x' < -x$, $\succsim^*$ satisfies P4(i) and P5 if and only if $\succsim$ does.

(D5) For every act X,

$$\begin{aligned} U_{+1,\tilde{u}}(-X) &= \min_{\omega\in\Omega}\big\{\tilde{u}(-X(\omega)) + \rho(\omega)\big\} \\ &= \min_{\omega\in\Omega}\big\{ -u(X(\omega)) + \rho(\omega)\big\} \\ &= -\max_{\omega\in\Omega}\big\{u(X(\omega)) - \rho(\omega)\big\} \\ &= -U_{-1,u}(X). \end{aligned} \tag{7}$$

The first equality is the definition of $U_{+1,\tilde{u}}$. The second equality follows from the definition of $\tilde{u}$. The third equality holds since $\min_{\omega\in\Omega} a_\omega = -\max_{\omega\in\Omega}(-a_\omega)$ for $a_\omega := -u(X(\omega)) + \rho(\omega)$, also when $a_\omega = \infty$, in which case $-a_\omega = -\infty$. The last equality is the definition of $U_{-1,u}$. Hence

$$\begin{aligned} U_{-1,u} \text{ represents } \succsim &\iff \big[\, -Y \succsim -X \iff U_{-1,u}(-Y) \geq U_{-1,u}(-X)\,\big] \text{ for all acts } X, Y \\ &\iff \big[\, X \succsim^* Y \iff U_{+1,\tilde{u}}(X) \geq U_{+1,\tilde{u}}(Y)\,\big] \text{ for all acts } X, Y \\ &\iff U_{+1,\tilde{u}} \text{ represents } \succsim^* . \end{aligned}$$

The first equivalence holds since $X \mapsto -X$ is a bijection of $\mathcal{X}$. The second follows from the definition of $\succsim^*$, from (7) applied to $-X$ and to $-Y$, which yields $U_{-1,u}(-X) = -U_{+1,\tilde{u}}(X)$ and $U_{-1,u}(-Y) = -U_{+1,\tilde{u}}(Y)$, and from the fact that multiplying an inequality by -1 reverses it. The third is the definition of a representation. By (7), moreover, $U_{-1,u}$ takes every integer value if and only if $U_{+1,\tilde{u}}$ does. For $u(x) = tx$ with $t > 0$, we have $\tilde{u} = u$, $U_{-1,u} = U_{-t}$, and $U_{+1,u} = U_t$. Finally, let $\succsim$ satisfy P6. An act X satisfies $0 \precsim^* X \prec^* 1$ if and only if $-1 \prec -X \precsim 0$, by the definition of $\succsim^*$, i.e., by P6 with $k = 1$, if and only if $0 \prec 1 - X \precsim 1$. By the definition of $\succsim^*$ and P6, $X \sim^* X'$ if and only if $1 - X \sim 1 - X'$. So $X \mapsto 1 - X$ maps the indifference classes of $\succsim^*$ in $\{X : 0 \precsim^* X \prec^* 1\}$ one-to-one onto those of $\succsim$ in $\{Z : 0 \prec Z \precsim 1\}$. Hence the two sets have the same number of indifference classes, and $m_P(\succsim^*) = m_O(\succsim)$ whenever these numbers are finite. □

Proof of Theorem 1

The idea of the proof is as follows. We first prove sufficiency, necessity, and uniqueness for $\sigma = +1$. Sufficiency proceeds in five steps. Step 1 constructs a utility function V on acts, using P1, P2, and P4(i). Step 2 shows that V is tropically additive across states, $V(X) = \bigoplus_{\omega\in\Omega} v(X(\omega), \omega)$, using P3. Step 3 shows that the state functions $v(\cdot, \omega)$ are finite on non-null states, so that component acts exist, using P4(ii) and P5. Step 4 shows that they differ across states only by a constant, $v(x, \omega) = \rho(\omega) \odot u(x)$, using P5. Step 5 combines Steps 2 and 4, and P2 yields that u is strictly increasing. We then show that σ is unique and obtain the case $\sigma = -1$ by duality (Lemma 6).

Sufficiency. Let $\succsim$ satisfy P1, P2, P3$^+$, P4, and P5.

Step 1: Utility by counting. Let $\mathcal{C} := \{[X] : X \in \mathcal{X}\}$, linearly ordered by $\succsim$ (P1). For $D \in \mathcal{C}$, the set $\{D' \in \mathcal{C} : [0] \precsim D' \prec D\}$ if $D \succsim [0]$, and the set $\{D' \in \mathcal{C} : D \precsim D' \prec [0]\}$ if $D \prec [0]$, are finite by P4(i), since they consist of classes between $[0]$ and D. Define $V : \mathcal{C} \longrightarrow \mathbb{Z}$ by

$$V(D) := \begin{cases} \big|\{D' \in \mathcal{C} : [0] \precsim D' \prec D\}\big| & \text{if } D \succsim [0], \\ -\big|\{D' \in \mathcal{C} : D \precsim D' \prec [0]\}\big| & \text{if } D \prec [0]. \end{cases} \tag{8}$$

That is, $V(D)$ counts the classes from the class of the sure amount 0 up to D, with a negative sign below $[0]$. In particular, since no class D' satisfies $[0] \precsim D' \prec [0]$,

$$V(0) = 0. \tag{9}$$

We show that V is an order-isomorphism of $\mathcal{C}$ onto $\mathbb{Z}$. That such an isomorphism exists is an instance of Hrbacek and Jech (1999, Exercise 4.10, p. 85), since $\mathcal{C}$ is linearly ordered without endpoints by P1 and P2, and every closed interval of classes is finite by P4(i). The counting in (8) yields the isomorphism explicitly, anchored at $[0]$.

First, V is strictly increasing. Let $D \prec E$. If $[0] \precsim D$, then $\{D' : [0] \precsim D' \prec D\} \cup \{D\} \subseteq \{D' : [0] \precsim D' \prec E\}$, so $V(D) + 1 \leq V(E)$ by (8). If $E \prec [0]$, then $\{D' : E \precsim D' \prec [0]\} \cup \{D\} \subseteq \{D' : D \precsim D' \prec [0]\}$, so $V(D) + 1 \leq V(E)$ by (8). If $D \prec [0] \precsim E$, then $V(D) \leq -1 < 0 \leq V(E)$ by (8).

Second, V is onto $\mathbb{Z}$. Let $n \geq 0$. By P4(i), the classes D with $[0] \precsim D \precsim [n]$ are finitely many. We denote them by $D_0 \prec D_1 \prec \cdots \prec D_m$ with $D_0 = [0]$ and $D_m = [n]$. By (1), $[0] \prec [1] \prec \cdots \prec [n]$ are among them, so $m \geq n$. For $i \leq m$, the classes D' with $[0] \precsim D' \prec D_i$ are $D_0, \ldots, D_{i-1}$, so $V(D_i) = i$ by (8). Hence $V(D_n) = n$. Let $n < 0$. By P4(i), the classes D with $[n] \precsim D \precsim [0]$ are finitely many. We denote them by $D_{-m} \prec \cdots \prec D_{-1} \prec D_0$ with $D_{-m} = [n]$ and $D_0 = [0]$. By (1), $[n] \prec [n+1] \prec \cdots \prec [0]$ are among them, so $m \geq -n$. For $1 \leq i \leq m$, the classes D' with $D_{-i} \precsim D' \prec [0]$ are $D_{-i}, \ldots, D_{-1}$, so $V(D_{-i}) = -i$ by (8). Hence $V(D_n) = n$.

For an act X, we write $V(X) := V([X])$. Since V is strictly increasing and $\succsim$ is complete (P1), for all acts X and Y,

$$X \succsim Y \quad \text{if and only if} \quad V(X) \geq V(Y). \tag{10}$$

Claim 1. For all integers $k \geq 0$, $V(k) \geq k$.

Proof of Claim 1. We proceed by induction on k. For the base case, $k = 0$, $V(0) = 0$ by (9). Assume as induction hypothesis $V(k) \geq k$. Then $V(k+1) > V(k) \geq k$, where the first inequality follows from (1) and (10) and the second is the induction hypothesis. Since V is integer-valued, $V(k+1) \geq k+1$.

Step 2: State-dependent tropically additive utility. For $\omega \in \Omega$ and $a \in \mathbb{Z}$, $V(a_\omega y)$ is nondecreasing in y by P2 and (10), and we let

$$v(a, \omega) := \lim_{y \to \infty} V(a_\omega y) \in \mathbb{Z} \cup \{\infty\}. \tag{11}$$

For all $a \leq a'$ and y, $a_\omega y \leq a'_\omega y$, so $a_\omega y \precsim a'_\omega y$ by P2, and $V(a_\omega y) \leq V(a'_\omega y)$ by (10). Letting $y \to \infty$ yields $v(a, \omega) \leq v(a', \omega)$. Hence $v(\cdot, \omega)$ is nondecreasing.

We show that V is the minimum over states of v. Let X be an act, and let $M \geq \max_{\omega \in \Omega} X(\omega)$. For every state ω',

$$\Big(\bigwedge_{\omega \in \Omega} X_\omega M \Big)(\omega') = \min_{\omega \in \Omega} \Big(\big(X_\omega M\big)(\omega') \Big) = \min\{X(\omega'), M\} = X(\omega').$$

The first equality follows from the definition of $\bigwedge$. The second holds since the act $X_{\omega'} M$ pays $X(\omega')$ at ω', and every other act $X_\omega M$ pays M at ω'. The third holds since $M \geq X(\omega')$. Hence

$$X = \bigwedge_{\omega \in \Omega} X_\omega M. \tag{12}$$

By P1, since Ω is finite, there is $\omega' \in \Omega$ with $X_{\omega'} M \precsim X_\omega M$ for all $\omega \in \Omega$. By (12) and Lemma 7, $X \sim X_{\omega'} M$. Hence, for all $M \geq \max_{\omega \in \Omega} X(\omega)$,

$$\begin{aligned} V(X) &= V\big(X_{\omega'} M\big) \\ &= \min_{\omega \in \Omega} V\big(X_\omega M\big). \end{aligned}$$

The first equality follows from (10). The second holds since $V(X_{\omega'}M) \leq V(X_\omega M)$ for all $\omega \in \Omega$, by (10). Therefore

$$\begin{aligned} V(X) &= \lim_{M\to\infty} \min_{\omega\in\Omega} V\big(X_\omega M\big) \\ &= \min_{\omega\in\Omega} \lim_{M\to\infty} V\big(X_\omega M\big) \\ &= \min_{\omega\in\Omega} v(X(\omega),\omega). \end{aligned}$$

The first equality holds since the left-hand side does not depend on M. The second holds since each $V(X_\omega M)$ is nondecreasing in M, and the limit of a minimum of finitely many nondecreasing sequences is the minimum of their limits. The third follows from (11), with $a = X(\omega)$ and $y = M$. Hence, for every act X,

$$V(X) = \min_{\omega\in\Omega} v(X(\omega),\omega). \tag{13}$$

Claim 2. For every state ω,

$$\omega \text{ is null} \quad \text{if and only if} \quad v(a,\omega) = \infty \text{ for all } a \in \mathbb{Z}. \tag{14}$$

Proof of Claim 2. Let ω be null. For all a and y, the acts $a_\omega y$ and y differ only at ω, so $a_\omega y \sim y$. Hence, for all a,

$$v(a,\omega) = \lim_{y\to\infty} V(a_\omega y) = \lim_{y\to\infty} V(y) = \infty.$$

The first equality follows from (11), the second from $a_\omega y \sim y$ and (10), and the third from Claim 1.

Conversely, let $v(a,\omega) = \infty$ for all a, and let X be an act. Since $V(X)$ is finite, the minimum in (13) is attained at a state other than ω, so

$$V(X) = \min_{\omega'\neq\omega} v(X(\omega'),\omega').$$

The right-hand side does not depend on $X(\omega)$. Hence $V(X) = V(Y)$ for all acts X and Y that differ only at ω, so $X \sim Y$ by (10), and ω is null.

Step 3: Component acts. We first record two facts. First, by (13), applied to the act $a_\omega y$, for all $\omega \in \Omega$ and $a, y \in \mathbb{Z}$,

$$V(a_\omega y) = \min_{\omega'\in\Omega} v\big((a_\omega y)(\omega'),\omega'\big) = \min\Big\{v(a,\omega), \min_{\omega'\neq\omega} v(y,\omega')\Big\}. \tag{15}$$

The second equality holds since $(a_\omega y)(\omega) = a$ and $(a_\omega y)(\omega') = y$ for $\omega' \neq \omega$. Second, for all $\omega \in \Omega_+$ and $x \in \mathbb{Z}$,

$$\text{if } v(x,\omega) < \infty, \text{ then there is a component act } c(x,\omega) \text{ with } V(c(x,\omega)) = v(x,\omega). \tag{16}$$

Indeed, $V(x_\omega y)$ is an integer, nondecreasing in y, with finite limit $v(x,\omega)$ by (11). So there is $\bar{y}$ with $V(x_\omega y) = v(x,\omega)$ for all $y \geq \bar{y}$. Then $x_\omega y \sim x_\omega \bar{y}$ for all $y \geq \bar{y}$ by (10), so $x_\omega \bar{y}$ is a component act $c(x,\omega)$ with $V(c(x,\omega)) = v(x,\omega)$.

The goal of this step is to show that

$$v(a,\omega) < \infty \quad \text{for all } \omega \in \Omega_+ \text{ and all } a \in \mathbb{Z}. \tag{17}$$

Suppose by contradiction that $v(a,\omega_0) = \infty$ for some $\omega_0 \in \Omega_+$ and $a \in \mathbb{Z}$. Since $v(\cdot,\omega_0)$ is nondecreasing, $v(x,\omega_0) = \infty$ for all $x \geq a$. We show that the component acts of a and of some $x > a$ are indifferent in ω_0 but not in some other non-null state ω_1, which contradicts P5.

First, for all $x \geq a$ and all component acts $c(a, \omega_0)$ and $c(x, \omega_0)$,

$$c(a, \omega_0) \sim c(x, \omega_0). \tag{18}$$

Indeed, by (15) and $v(a, \omega_0) = v(x, \omega_0) = \infty$, for all y,

$$V(a_{\omega_0} y) = \min_{\omega' \neq \omega_0} v(y, \omega') = V(x_{\omega_0} y),$$

so $a_{\omega_0} y \sim x_{\omega_0} y$ by (10). Moreover, $V(x_{\omega_0} y) \to v(x, \omega_0) = \infty$ as $y \to \infty$ by (11). This contradicts the first alternative of P4(ii) for x and ω_0. To see this, note that the first alternative of P4(ii) requires the existence of $\bar{y}$ with $x_{\omega_0} y \sim x_{\omega_0} \bar{y}$ for all $y \geq \bar{y}$. By (10), it implies $V(x_{\omega_0} y) = V(x_{\omega_0} \bar{y}) \in \mathbb{Z}$ for all $y \geq \bar{y}$, so the limit would be finite, a contradiction. Hence every component act $c(x, \omega_0) = x_{\omega_0} \bar{y}$ comes from the second alternative, i.e., $x_{\omega_0} y \sim c(x, \omega_0)$ for all $y \leq \bar{y}$. The same holds for a, with some $\bar{y}'$ in place of $\bar{y}$. For $y \leq \min\{\bar{y}, \bar{y}'\}$,

$$c(a, \omega_0) \sim a_{\omega_0} y \sim x_{\omega_0} y \sim c(x, \omega_0).$$

Second, we argue that there are $\omega_1 \in \Omega_+$ with $\omega_1 \neq \omega_0$ and $x > a$ such that

$$v(a, \omega_1) < v(x, \omega_1) < \infty. \tag{19}$$

For every $k \geq a$, (13) yields $V(k) = \min_{\omega \in \Omega} v(k, \omega)$. Since $V(k) \in \mathbb{Z}$ and $k \geq a$ implies $v(k, \omega_0) = \infty$, every minimizer is a state other than ω_0. By the pigeonhole principle, since there are infinitely many $k \geq a$ and Ω is finite, some $\omega_1 \neq \omega_0$ is a minimizer for infinitely many $k \geq a$, i.e., $v(k, \omega_1) = V(k)$ for these k. Since $v(\cdot, \omega_1)$ is nondecreasing, $v(a, \omega_1) \leq V(k) < \infty$, so $\omega_1 \in \Omega_+$ by Claim 2. Let x be one of these k with $x > \max\{0, a, v(a, \omega_1)\}$. By Claim 1, $v(x, \omega_1) = V(x) \geq x > v(a, \omega_1)$.

Third, by (16) and (19), there are component acts with $V(c(a, \omega_1)) = v(a, \omega_1) < v(x, \omega_1) = V(c(x, \omega_1))$, so $c(a, \omega_1) \prec c(x, \omega_1)$ by (10). By (18), $c(a, \omega_0) \sim c(x, \omega_0)$. So between the component acts of a and x, there is exactly one indifference class in ω_0 but at least two in ω_1, contradicting P5. This proves (17).

By (17) and (16), for all $x \in \mathbb{Z}$ and $\omega \in \Omega_+$, there is a component act $c(x, \omega)$ with

$$V(c(x, \omega)) = v(x, \omega). \tag{20}$$

From now on, $c(x, \omega)$ denotes such a component act.

Step 4: Additive decomposition. Let $\omega, \omega' \in \Omega_+$ and $x < x'$. By (20) from Step 3, $V(c(x, \omega)) = v(x, \omega) \leq v(x', \omega) = V(c(x', \omega))$, where the inequality follows from $v(\cdot, \omega)$ being nondecreasing. Since by Step 1 V is an order-isomorphism of $\mathcal{C}$ onto $\mathbb{Z}$, the number of indifference classes of acts Z with $c(x, \omega) \precsim Z \precsim c(x', \omega)$ is $v(x', \omega) - v(x, \omega) + 1$. The same holds for ω'. By P5, these numbers are equal, so $v(x', \omega) - v(x, \omega) = v(x', \omega') - v(x, \omega')$. Hence, for all $\omega, \omega' \in \Omega_+$,

$$v(x, \omega) - v(x, \omega') \text{ does not depend on } x \in \mathbb{Z}. \tag{21}$$

Define

$$u(x) := V(x) \quad \text{for } x \in \mathbb{Z}, \qquad \rho(\omega) := v(0, \omega) \quad \text{for } \omega \in \Omega.$$

Recall that $V(0) = 0$ by (9). So, by (13), $\min_{\omega \in \Omega} \rho(\omega) = \min_{\omega \in \Omega} v(0, \omega) = V(0) = 0$. By (17) and Claim 2, $\rho(\omega) \in \mathbb{Z}$ if $\omega \in \Omega_+$ and $\rho(\omega) = \infty$ otherwise. So ρ is a ranking function à la Spohn (2012)! For all $x \in \mathbb{Z}$ and $\omega \in \Omega_+$,

$$\begin{aligned} u(x) &= \min_{\omega' \in \Omega_+} v(x, \omega') \\ &= \min_{\omega' \in \Omega_+} \big(v(x, \omega) - v(0, \omega) + v(0, \omega')\big) \\ &= v(x, \omega) - \rho(\omega). \end{aligned}$$

The first equality follows from the definition of u, (13), and Claim 2. The second follows from (21). The third holds since $\min_{\omega' \in \Omega_+} v(0, \omega') = \min_{\omega' \in \Omega} \rho(\omega') = 0$. With Claim 2 for $\omega \notin \Omega_+$,

$$v(x, \omega) = u(x) + \rho(\omega) \quad \text{for all } x \in \mathbb{Z} \text{ and } \omega \in \Omega. \tag{22}$$

Step 5: Representation. By (13) and (22), for every act X,

$$V(X) = \min_{\omega \in \Omega} \big\{ u(X(\omega)) + \rho(\omega) \big\} = U_{+1,u}(X), \tag{23}$$

where the second equality follows from the definition of $U_{+1,u}$. By (10) from Step 1, $U_{+1,u}$ represents $\succsim$. Since V is onto $\mathbb{Z}$ by Step 1, $U_{+1,u}(\mathcal{X}) = \mathbb{Z}$. Since $u(x) = V(x)$, u is strictly increasing by (1) and (10). This proves sufficiency.

Properties of $U_{+1,u}$. Let $U_{+1,u}$, defined by

$$U_{+1,u}(X) = \min_{\omega \in \Omega} \big(u(X(\omega)) + \rho(\omega) \big)$$

for a strictly increasing $u : \mathbb{Z} \to \mathbb{Z}$ and a ranking function ρ, represent the preference relation $\succsim$. Observe that each term is either an integer or ∞. Since $\min_\omega \rho(\omega) = 0$, some state ω has $\rho(\omega) = 0$, and its term $u(X(\omega))$ is an integer. A minimum over finitely many elements of $\mathbb{Z} \cup \{\infty\}$, at least one of which is an integer, is an integer. Thus, $U_{+1,u}(X) \in \mathbb{Z}$. Next observe that for every constant act k, $U_{+1,u}(k) = \min_{\omega \in \Omega} \big(u(k) + \rho(\omega) \big) = u(k) + \min_{\omega \in \Omega} \rho(\omega) = u(k)$ since $\min_\omega \rho(\omega) = 0$.

Claim 3. For all $x, y \in \mathbb{Z}$ with $x \leq y$,

$$u(y) - u(x) \geq y - x. \tag{24}$$

Proof of Claim 3. We proceed by induction on $y - x$. For the base case, $y = x$, equality holds. Assume as induction hypothesis $u(y) - u(x) \geq y - x$. Then $u(y+1) - u(x) > u(y) - u(x) \geq y - x$, where the first inequality follows from u being strictly increasing and the second is the induction hypothesis. Since u is integer-valued, $u(y+1) - u(x) \geq y + 1 - x$.

Claim 4. For every state ω with $\rho(\omega) < \infty$, all $x \in \mathbb{Z}$, and all $y \geq x + \rho(\omega)$,

$$U_{+1,u}(x_\omega y) = u(x) + \rho(\omega). \tag{25}$$

Proof of Claim 4. By the definition of $U_{+1,u}$,

$$U_{+1,u}(x_\omega y) = \min \Big\{ u(x) + \rho(\omega), \min_{\omega' \neq \omega} \big(u(y) + \rho(\omega') \big) \Big\}.$$

For every $\omega' \neq \omega$,

$$u(y) + \rho(\omega') \geq u(y) \geq u(x) + (y - x) \geq u(x) + \rho(\omega).$$

The first inequality follows from $\rho(\omega') \geq 0$. The second follows from Claim 3, which applies since $y - x \geq \rho(\omega) \geq 0$. The third follows from $y - x \geq \rho(\omega)$. Hence the minimum is attained by the term $u(x) + \rho(\omega)$.

Part (ii) of Theorem 1. For every state ω,

$$\omega \text{ is null} \quad \text{if and only if} \quad \rho(\omega) = \infty.$$

Let $\rho(\omega) = \infty$, and let X and Y be acts with $X(\omega') = Y(\omega')$ for all $\omega' \neq \omega$. Since the term for ω is ∞,

$$U_{+1,u}(X) = \min_{\omega' \neq \omega} \big(u(X(\omega')) + \rho(\omega') \big) = U_{+1,u}(Y).$$

Since $U_{+1,u}$ represents $\succsim$, $X \sim Y$. By the definition of null states, ω is null.

For the converse, we show the contrapositive and let $\rho(\omega) < \infty$ and $y \geq \rho(\omega)$. Claim 4 applies with $x = 0$ and with $x = -1$, since $y \geq \rho(\omega)$, and yields

$$U_{+1,u}((-1)_\omega y) = u(-1) + \rho(\omega) < u(0) + \rho(\omega) = U_{+1,u}(0_\omega y),$$

where the inequality follows from u being strictly increasing. Since $U_{+1,u}$ represents $\succsim$, $(-1)_\omega y \prec 0_\omega y$. These two acts agree everywhere except at ω, so ω is not null.

Necessity. Let $U_{+1,u}$ represent $\succsim$ as above. We verify the properties. Only the proof of P5 uses that $U_{+1,u}$ takes every integer value.

P1 holds since $U_{+1,u}$ represents $\succsim$ and $\geq$ on $\mathbb{Z}$ is complete and transitive.

P2: If $X \geq Y$, then $U_{+1,u}(X) \geq U_{+1,u}(Y)$, since u is increasing and the minimum is nondecreasing in each argument, so $X \succsim Y$. Moreover, by *Properties of* $U_{+1,u}$, $U_{+1,u}(k+1) = u(k+1) > u(k) = U_{+1,u}(k)$, so $k+1 \succ k$.

P3$^+$: By (2), $U_{+1,u}(X \wedge Z) = \min\{U_{+1,u}(X), U_{+1,u}(Z)\}$ for all acts X and Z. So, for all acts X, Y, and Z,

$$\begin{aligned} X \succsim Y &\implies U_{+1,u}(X) \geq U_{+1,u}(Y) \\ &\implies \min\{U_{+1,u}(X), U_{+1,u}(Z)\} \geq \min\{U_{+1,u}(Y), U_{+1,u}(Z)\} \\ &\implies U_{+1,u}(X \wedge Z) \geq U_{+1,u}(Y \wedge Z) \\ &\implies X \wedge Z \succsim Y \wedge Z. \end{aligned}$$

The first and the last implication hold since $U_{+1,u}$ represents $\succsim$. The second holds since the minimum is nondecreasing in each argument. The third follows from (2), applied to X and Z and to Y and Z.

P4(i): Let $X \precsim Y$. Every act Z with $X \precsim Z \precsim Y$ has a value in $\{U_{+1,u}(X), \ldots, U_{+1,u}(Y)\}$, and acts with the same value are indifferent. So there are at most $U_{+1,u}(Y) - U_{+1,u}(X) + 1$ classes of such acts.

P4(ii): Let $\omega \in \Omega_+$ and $x \in \mathbb{Z}$. By part (ii) of Theorem 1, $\rho(\omega) < \infty$, and, by Claim 4, $U_{+1,u}(x_\omega y) = u(x) + \rho(\omega)$ for all $y \geq x + \rho(\omega)$. So the first alternative of P4(ii) holds with $\bar{y} := x + \rho(\omega)$.

Claim 5. For all $x \in \mathbb{Z}$, $\omega \in \Omega_+$, and all component acts $c(x, \omega)$,

$$U_{+1,u}(c(x,\omega)) = u(x) + \rho(\omega). \tag{26}$$

Proof of Claim 5. Let $c(x,\omega) = x_\omega \bar{y}$. By part (ii) of Theorem 1, $\rho(\omega) < \infty$. The component act comes from one of the two alternatives of P4(ii).

If it comes from the first, then $x_\omega y \sim x_\omega \bar{y}$ for all $y \geq \bar{y}$. Taking $y \geq \max\{\bar{y}, x + \rho(\omega)\}$, Claim 4 yields $U_{+1,u}(c(x,\omega)) = U_{+1,u}(x_\omega y) = u(x) + \rho(\omega)$.

If it comes from the second, there are two cases. If some $\omega' \neq \omega$ is non-null, then $\rho(\omega') < \infty$ by part (ii) of Theorem 1, and

$$U_{+1,u}(x_\omega y) \leq u(y) + \rho(\omega')$$

for all y, since the minimum is weakly smaller than its term for ω'. Since $u(y) \to -\infty$ as $y \to -\infty$ by Claim 3, and $\rho(\omega')$ is finite, $U_{+1,u}(x_\omega y) \to -\infty$. The second alternative then fails, so there is no such component act. If ω is the only non-null state, then $\rho(\omega') = \infty$ for all $\omega' \neq \omega$ by part (ii) of Theorem 1. So $U_{+1,u}(x_\omega y) = u(x) + \rho(\omega)$ for all y, and in particular $U_{+1,u}(c(x,\omega)) = u(x) + \rho(\omega)$.

P5: Here we use the normalization $U_{+1,u}(\mathcal{X}) = \mathbb{Z}$. Let $\omega \in \Omega_+$ and $x < x'$. By Claim 5, an act Z satisfies $c(x,\omega) \precsim Z \precsim c(x',\omega)$ if and only if $u(x) + \rho(\omega) \leq U_{+1,u}(Z) \leq u(x') + \rho(\omega)$. Since $U_{+1,u}$

takes every integer value, and since $Z \sim Z'$ if and only if $U_{+1,u}(Z) = U_{+1,u}(Z')$, because $U_{+1,u}$ represents $\succsim$, there are $u(x') - u(x) + 1$ indifference classes of acts Z with $c(x,\omega) \precsim Z \precsim c(x',\omega)$. This number does not depend on $\omega \in \Omega_+$, which is P5.

Uniqueness of u and ρ. Let $U_{+1,u}$ with ranking function ρ and $U_{+1,u'}$ with ranking function ρ' both represent $\succsim$ and take every integer value. We show that $u' = u + b$ for some $b \in \mathbb{Z}$ and $\rho' = \rho$. Since both represent $\succsim$, for all acts X and Y,

$$U_{+1,u}(X) \geq U_{+1,u}(Y) \quad \text{if and only if} \quad X \succsim Y \quad \text{if and only if} \quad U_{+1,u'}(X) \geq U_{+1,u'}(Y).$$

So $f(U_{+1,u}(X)) := U_{+1,u'}(X)$ is well defined and strictly increasing. Since both take every integer value, f is a bijection of $\mathbb{Z}$ onto itself.

By Lemma 5, $f(n) = n + f(0)$ for all $n \in \mathbb{Z}$, so $U_{+1,u'} = U_{+1,u} + b$ with $b := f(0)$. For every constant act x, $U_{+1,u}(x) = u(x)$ and $U_{+1,u'}(x) = u'(x)$ by *Properties of* $U_{+1,u}$, so $u' = u + b$. For $\omega \in \Omega_+$ and $x \in \mathbb{Z}$, Claim 5 for both representations yields

$$u'(x) + \rho'(\omega) = U_{+1,u'}(c(x,\omega)) = U_{+1,u}(c(x,\omega)) + b = u(x) + \rho(\omega) + b.$$

By part (ii) of Theorem 1, $\rho(\omega)$ and $\rho'(\omega)$ are finite, so $\rho'(\omega) = \rho(\omega)$ by $u' = u + b$. For $\omega \notin \Omega_+$, $\rho'(\omega) = \rho(\omega) = \infty$ by part (ii) of Theorem 1.

The case σ = −1. By (D1), (D2), and (D4) of Lemma 6, $\succsim$ satisfies P1, P2, P3$^-$, P4, and P5 if and only if $\succsim^*$ satisfies P1, P2, P3$^+$, P4, and P5. By (D5), $\succsim$ is represented by $U_{-1,u}$, taking every integer value, if and only if $\succsim^*$ is represented by $U_{+1,\tilde{u}}$, taking every integer value, with the same ρ. By (D3), the two relations have the same null states. Hence sufficiency, necessity, and parts (ii) and (iii) of Theorem 1 for $\sigma = -1$ follow from the case $\sigma = +1$ applied to $\succsim^*$. Here, if $\succsim^*$ is represented by U_{+1,u^*}, then, by (D5) with the roles of $\succsim$ and $\succsim^*$ exchanged, $\succsim$ is represented by $U_{-1,u}$ with $u := \tilde{u}^*$, i.e., $u(x) = -u^*(-x)$.

Part (i) of Theorem 1. Since sufficiency was shown under P3$^+$ for $\sigma = +1$ and under P3$^-$ for $\sigma = -1$, the first statement of (i) holds. If at least two states are non-null, σ is unique by Lemma 1. If only one state ω_0 is non-null, then part (ii) of Theorem 1 and $\min_{\omega\in\Omega} \rho(\omega) = 0$ yield $\rho(\omega_0) = 0$, so that

$$U_{\sigma,u}(X) = u(X(\omega_0)) \quad \text{for all acts } X \text{ and both signs of } \sigma.$$

Since u is strictly increasing, $u(\min\{a,b\}) = \min\{u(a), u(b)\}$ and $u(\max\{a,b\}) = \max\{u(a), u(b)\}$. Hence $U_{\sigma,u}(X \wedge Z) = \min\{U_{\sigma,u}(X), U_{\sigma,u}(Z)\}$ and $U_{\sigma,u}(X \vee Z) = \max\{U_{\sigma,u}(X), U_{\sigma,u}(Z)\}$ for all acts X and Z. So P3$^+$ and P3$^-$ both hold, as in the proof of necessity, and σ is not unique. This completes the proof of Theorem 1. □

Remark 4. In the proof of sufficiency in Theorem 1 for $\sigma = +1$, Steps 1 to 5 use P4(ii) only in Step 3, and P5 only in Steps 3 and 4. If, for every $\omega \in \Omega_+$ and $x \in \mathbb{Z}$, $x_\omega y$ is eventually constant in indifference class as $y \to \infty$, then (17) holds without the argument of Step 3, and Steps 1 to 5 use P5 only for the component acts $c(x,\omega)$ of (20).

Proof of Lemma 1

Suppose $\succsim$ satisfies P3$^+$ and P3$^-$, and let $\omega_1 \neq \omega_2$ be non-null. By the sufficiency part of Theorem 1, P1, P2, P3$^+$, P4, and P5 yield a representation $U_{+1,u}$ with ranking function ρ. By part (ii) of Theorem 1, $\rho(\omega_1)$ and $\rho(\omega_2)$ are finite. Let $y > \max\{\rho(\omega_1), \rho(\omega_2)\}$. Since $\omega_1 \neq \omega_2$ and $y > 0$, the act $0_{\omega_1} y \vee 0_{\omega_2} y$ pays y in every state, so it is the constant act y. For $i = 1, 2$,

$$U_{+1,u}(y) = u(y) \geq u(0) + y > u(0) + \rho(\omega_i) = U_{+1,u}(0_{\omega_i} y).$$

The first equality follows from *Properties of* $U_{+1,u}$ in the proof of Theorem 1. The first inequality follows from Claim 3 in the proof of Theorem 1 with $x = 0$, which applies since $y > 0$. The second inequality holds by the choice of y. The last equality follows from Claim 4 in the proof of Theorem 1 with $x = 0$, which applies since $y \geq \rho(\omega_i)$. Since $U_{+1,u}$ represents $\succsim$, $y \succ 0_{\omega_i} y$ for $i = 1, 2$. By P1, $0_{\omega_i} y \precsim 0_{\omega_{3-i}} y$ for $i = 1$ or $i = 2$. By P3$^-$ and Lemma 7 with $n = 2$, $X_1 = 0_{\omega_i} y$, $X_2 = 0_{\omega_{3-i}} y$, and $j = 2$, $y = 0_{\omega_i} y \vee 0_{\omega_{3-i}} y \sim 0_{\omega_{3-i}} y$, which contradicts $y \succ 0_{\omega_{3-i}} y$. □

Tropical Independence for Finitely Many Acts

The following lemma states the equivalent form of tropical independence discussed in Section 3 for finitely many acts.

Lemma 7. *Let* $\succsim$ *satisfy P1 and P2. Then* $\succsim$ *satisfies P3*$^+$ *if and only if, for all* $n \geq 1$*, all acts* $X_1, \ldots, X_n$*, and all* $j \leq n$*,*

$$X_j \precsim X_i \text{ for all } i \leq n \quad \text{implies} \quad X_1 \wedge \cdots \wedge X_n \sim X_j. \tag{27}$$

It satisfies P3$^-$ *if and only if, for all* $n \geq 1$*, all acts* $X_1, \ldots, X_n$*, and all* $j \leq n$*,*

$$X_j \succsim X_i \text{ for all } i \leq n \quad \text{implies} \quad X_1 \vee \cdots \vee X_n \sim X_j. \tag{28}$$

Proof. We prove the statement for P3$^+$ and obtain the one for P3$^-$ by duality.

First, we show that P3$^+$ implies (27). Suppose P3$^+$ holds, and let $X_j \precsim X_i$ for all $i \leq n$. Define

$$Z_0 := X_j \quad \text{and} \quad Z_k := X_k \wedge Z_{k-1} \quad \text{for } k = 1, \ldots, n.$$

Since X_j is among $X_1, \ldots, X_n$ and $\wedge$ is idempotent, $Z_n = X_1 \wedge \cdots \wedge X_n$. So it suffices to show that, for all $k = 0, \ldots, n$,

$$Z_k \sim X_j.$$

We proceed by induction on k. For the base case, $k = 0$, $Z_0 \sim X_j$ by reflexivity, which is implied by P1, since $Z_0 = X_j$. Let $k \geq 1$, and assume as induction hypothesis $Z_{k-1} \sim X_j$. Since $X_k \succsim X_j$ by the hypothesis with $i = k$, P3$^+$ with the common act Z_{k-1} yields

$$Z_k = X_k \wedge Z_{k-1} \succsim X_j \wedge Z_{k-1} = Z_{k-1}.$$

The first equality follows from the definition of Z_k, and the last from $X_j \geq Z_{k-1}$, which follows from the definition of Z_{k-1}. Moreover, $Z_{k-1} \geq Z_k$ by the definition of Z_k, so P2 yields $Z_{k-1} \succsim Z_k$. Hence $Z_k \sim Z_{k-1}$, and $Z_k \sim X_j$ by the induction hypothesis and transitivity (P1). This completes the induction.

Second, conversely, we show that (27) implies P3$^+$. Suppose that (27) holds. For $n = 2$, it says that the state-wise minimum of two acts is indifferent to the worse of the two. Let $X \succsim Y$, and let Z be an act. By completeness (P1), $Y \succsim Z$ or $Z \succsim Y$. If $Y \succsim Z$, then $X \succsim Z$ by transitivity, and (27) yields $X \wedge Z \sim Z$ and $Y \wedge Z \sim Z$, so $X \wedge Z \sim Y \wedge Z$. If $Z \succsim Y$, then (27) yields $Y \wedge Z \sim Y$. By completeness, $X \succsim Z$ or $Z \succsim X$, and (27) yields $X \wedge Z \sim Z$ in the first case and $X \wedge Z \sim X$ in the second. Since $Z \succsim Y$ by the case assumption and $X \succsim Y$ by the starting assumption, transitivity yields $X \wedge Z \succsim Y$ in both cases. Hence $X \wedge Z \succsim Y \sim Y \wedge Z$. Thus, whenever $X \succsim Y$, $X \wedge Z \succsim Y \wedge Z$, which is P3$^+$.

Finally, the statement for P3$^-$ follows from the one for P3$^+$, applied to the dual relation $\succsim^*$, which satisfies P1 and P2 by (D1) of Lemma 6. By (D2), $\succsim$ satisfies P3$^-$ if and only if $\succsim^*$ satisfies P3$^+$, i.e., if and only if, for all acts $X_1, \ldots, X_n$ and all $j \leq n$, $X_j \precsim^* X_i$ for all $i \leq n$ implies $X_1 \wedge \cdots \wedge X_n \sim^* X_j$. By the definition of $\succsim^*$ and (3), this says that $-X_j \succsim -X_i$ for all $i \leq n$ implies $(-X_1) \vee \cdots \vee (-X_n) \sim -X_j$. Since $-X_1, \ldots, -X_n$ range over all acts, this is (28). □

Proof of Lemma 2

We prove the statement for the min-plus form and obtain the one for the max-plus form by duality.

First, suppose P3* holds in the min-plus form. For $c = 0$, the definitions of $\odot$ and $\oplus^{+1}$ yield $(0 \odot X) \oplus^{+1} Z = X \wedge Z$, so P3* with $c = 0$ is P3$^+$. For P6, let $c \in \mathbb{Z}$. We show that $X \succsim Y$ implies $X + c \succsim Y + c$. The reverse implication then follows by applying this to $X + c$, $Y + c$, and $-c$, since $(X + c) + (-c) = X$. Let $X \succsim Y$, and let M be the constant act $\max_{\omega\in\Omega} \max\{X(\omega), Y(\omega)\} + c$. Since $M \geq X + c$ and $M \geq Y + c$,

$$(c \odot X) \oplus^{+1} M = (X + c) \wedge M = X + c \quad \text{and} \quad (c \odot Y) \oplus^{+1} M = (Y + c) \wedge M = Y + c.$$

So P3* with the common act M yields $X + c \succsim Y + c$.

Second, conversely, suppose P3$^+$ and P6 hold. Let $X \succsim Y$, $c \in \mathbb{Z}$, and let Z be an act. By P6, $c \odot X = X + c \succsim Y + c = c \odot Y$. By P3$^+$ with the common act Z, $(c \odot X) \oplus^{+1} Z \succsim (c \odot Y) \oplus^{+1} Z$. This is P3* in the min-plus form.

Finally, by (D2) of Lemma 6, $\succsim$ satisfies P3* in the max-plus form if and only if $\succsim^*$ satisfies it in the min-plus form. Since $\succsim^*$ satisfies P1 and P2 by (D1), the min-plus case applies to $\succsim^*$, so this holds if and only if $\succsim^*$ satisfies P3$^+$ and P6. By (D1) and (D2), this holds if and only if $\succsim$ satisfies P3$^-$ and P6. □

Proof of Lemma 3

By Lemma 2, P3* yields P3$^+$ and P6 in the min-plus form and P3$^-$ and P6 in the max-plus form. Since P4(ii) is assumed, it remains to show P4(i) and P5. We prove them under P3$^+$ and obtain the case for P3$^-$ by duality. We first show that P4(ii) then holds in its first alternative.

Suppose P3$^+$ holds. We use the following consequence of the definition of null states:

$$\text{if } X(\omega) = Y(\omega) \text{ for all } \omega \in \Omega_+, \text{ then } X \sim Y. \tag{29}$$

Indeed, let $\omega_1, \ldots, \omega_n$ be the null states, and for $i = 0, \ldots, n$, let X_i be the act that agrees with Y at $\omega_1, \ldots, \omega_i$ and with X at all other states. Then $X_0 = X$, and $X_n = Y$, since X and Y agree on Ω_+. For $i = 1, \ldots, n$, the acts X_{i-1} and X_i agree at every state other than ω_i, and ω_i is null, so $X_{i-1} \sim X_i$. By P1, $X \sim Y$.

Claim. For every $\omega \in \Omega_+$, there is $\bar{y}^\omega \in \mathbb{Z}$ such that $0_\omega y \sim 0_\omega \bar{y}^\omega$ for all $y \geq \bar{y}^\omega$.

Proof of the Claim. If $\Omega_+ = \{\omega\}$, then for every y, the acts $0_\omega y$ and $0_\omega 0$ agree at ω, so $0_\omega y \sim 0_\omega 0$ by (29). So the Claim holds with $\bar{y}^\omega = 0$.

Suppose now that $|\Omega_+| \geq 2$, and suppose, to the contrary, that there is $\omega \in \Omega_+$ such that for no $\bar{y} \in \mathbb{Z}$ do we have $0_\omega y \sim 0_\omega \bar{y}$ for all $y \geq \bar{y}$. By P4(ii) with $x = 0$, there is $\bar{y} \in \mathbb{Z}$ such that $0_\omega y \sim 0_\omega \bar{y}$ for all $y \leq \bar{y}$. Let $\hat{y} := \min\{\bar{y}, 0\}$. For all $y \leq \hat{y}$, we have $0_\omega y \sim 0_\omega \bar{y}$ and $0_\omega \hat{y} \sim 0_\omega \bar{y}$, so P1 yields

$$0_\omega y \sim 0_\omega \hat{y} \quad \text{for all } y \leq \hat{y}. \tag{30}$$

Step 1. For all $x \in \mathbb{Z}$ and $w \leq x$, $x_\omega w \sim x$. Let $x \in \mathbb{Z}$ and $w \leq x$, and let $y := \hat{y} + w - x$. Then $y \leq \hat{y}$, so $0_\omega y \sim 0_\omega \hat{y}$ by (30). P3$^+$ with the sure act $\hat{y}$, applied to $0_\omega y \succsim 0_\omega \hat{y}$ and to $0_\omega \hat{y} \succsim 0_\omega y$, yields $(0_\omega y) \wedge \hat{y} \sim (0_\omega \hat{y}) \wedge \hat{y}$. Since $y \leq \hat{y} \leq 0$, the act $(0_\omega y) \wedge \hat{y}$ pays $\min\{0, \hat{y}\} = \hat{y}$ at ω and $\min\{y, \hat{y}\} = y$ elsewhere, so it equals $\hat{y}_\omega y$. The act $(0_\omega \hat{y}) \wedge \hat{y}$ is the sure act $\hat{y}$. Hence $\hat{y}_\omega y \sim \hat{y}$. By P6 with $k = x - \hat{y}$, $\hat{y}_\omega y + (x - \hat{y}) \sim x$. Since $\hat{y}_\omega y + (x - \hat{y}) = x_\omega(y + x - \hat{y}) = x_\omega w$, we get $x_\omega w \sim x$.

Step 2. For every act X and every $x \in \mathbb{Z}$ with $x \geq X(\omega')$ for all $\omega' \in \Omega$, $X \sim X_\omega x$. Let $m := \min_{\omega'\in\Omega} X(\omega')$. All three acts $x_\omega m$, $x_\omega X$, and x pay x at ω, and $m \leq X(\omega') \leq x$ at every $\omega' \neq \omega$. Hence $x_\omega m \leq x_\omega X \leq x$, and P2 yields $x_\omega m \precsim x_\omega X \precsim x$. By Step 1 with $w = m \leq x$,

$x_\omega m \sim x$, so P1 yields $x_\omega X \sim x$. Let $Z := X_\omega x$. P3$^+$ with Z, applied to $x_\omega X \succsim x$ and to $x \succsim x_\omega X$, yields $(x_\omega X) \wedge Z \sim x \wedge Z$. The act $(x_\omega X) \wedge Z$ pays $\min\{x, X(\omega)\} = X(\omega)$ at ω and $\min\{X(\omega'), x\} = X(\omega')$ at every $\omega' \neq \omega$, so it equals X. Since $Z \leq x$, $x \wedge Z = Z$. Hence $X \sim X_\omega x$.

Step 3. Every $\omega' \neq \omega$ is null. Let $\omega' \neq \omega$, and let X, Y be acts with $X(\omega'') = Y(\omega'')$ for all $\omega'' \neq \omega'$. In particular, $X(\omega) = Y(\omega) =: L$. Let x be the largest outcome of X and Y. By Step 2, $X \sim L_\omega x$ and $Y \sim L_\omega x$, so P1 yields $X \sim Y$. Hence ω' is null.

By Step 3, $\Omega_+ \subseteq \{\omega\}$, which contradicts $|\Omega_+| \geq 2$. This proves the Claim.

Component acts. By the Claim, and since every larger threshold also satisfies the first alternative, we may take $\bar{y}^\omega \geq 0$ for all $\omega \in \Omega_+$. Let $\omega \in \Omega_+$ and $x \in \mathbb{Z}$. For every y, the acts $x_\omega y$ and $0_\omega(y - x) + x$ both pay x at ω and y elsewhere, so $x_\omega y = 0_\omega(y - x) + x$. Let $y \geq x + \bar{y}^\omega$. Then $0_\omega(y - x) \sim 0_\omega \bar{y}^\omega$ by the choice of $\bar{y}^\omega$, and P6, applied in both directions, yields $0_\omega(y - x) + x \sim 0_\omega \bar{y}^\omega + x$, i.e., $x_\omega y \sim x_\omega(x + \bar{y}^\omega)$. So P4(ii) holds in the first alternative with $\bar{y} := x + \bar{y}^\omega$. In the rest of the proof, we use the component acts $c(x, \omega) := x_\omega(x + \bar{y}^\omega)$, which satisfy

$$c(x, \omega) = c(0, \omega) + x \quad \text{for all } x \in \mathbb{Z} \text{ and } \omega \in \Omega_+. \tag{31}$$

P4(i). We first record three facts. First, $\Omega_+ \neq \emptyset$. Otherwise, (29) yields $1 \sim 0$, which contradicts P2.

Second, every act X satisfies

$$X \sim c(X(\omega), \omega) \quad \text{for some } \omega \in \Omega_+. \tag{32}$$

Indeed, let M be an integer with $M > \max_{\omega' \in \Omega} X(\omega')$ and $M \geq \max_{\omega' \in \Omega} X(\omega') + \max_{\omega' \in \Omega_+} \bar{y}^{\omega'}$. For every $\omega' \in \Omega$, the act $X_{\omega'} M$ pays $X(\omega')$ at ω' and $M > X(\omega'')$ at every $\omega'' \neq \omega'$. Hence $X = \bigwedge_{\omega' \in \Omega} X_{\omega'} M$. By P1, there is $\omega \in \Omega$ with $X_\omega M \precsim X_{\omega'} M$ for all $\omega' \in \Omega$, and Lemma 7 yields $X \sim X_\omega M$. The state ω is non-null. Otherwise, $X_\omega M \sim M$ by (29), while $M \succ \max_{\omega' \in \Omega} X(\omega')$ by (1) and $\max_{\omega' \in \Omega} X(\omega') \succsim X$ by P2. Then $X \sim X_\omega M \sim M \succ X$, a contradiction. Since $M \geq X(\omega) + \bar{y}^\omega$, the paragraph on component acts with $x = X(\omega)$ and $y = M$ yields $X_\omega M \sim c(X(\omega), \omega)$.

Third, for every $\omega \in \Omega_+$, there is $n^\omega \in \mathbb{Z}$ with

$$x + n^\omega \precsim c(x, \omega) \prec x + n^\omega + 1 \qquad \text{for all } x \in \mathbb{Z}. \tag{33}$$

To see this, recall that the act $c(0, \omega) = 0_\omega \bar{y}^\omega$ pays 0 at ω and $\bar{y}^\omega \geq 0$ elsewhere, so $\bar{y}^\omega \geq c(0, \omega) \geq 0$, and P2 yields $\bar{y}^\omega \succsim c(0, \omega) \succsim 0$. Let $N := \{k \in \mathbb{Z} : c(0, \omega) \succsim k\}$. Then $0 \in N$. If $k > \bar{y}^\omega$, then $k \succ \bar{y}^\omega \succsim c(0, \omega)$ by (1) and P2, so $k \notin N$. So N is a nonempty set of integers bounded above by $\bar{y}^\omega$, and hence has a maximum n^ω. Since $n^\omega \in N$, $n^\omega \precsim c(0, \omega)$. Since $n^\omega + 1 \notin N$, P1 yields $c(0, \omega) \prec n^\omega + 1$. Let $x \in \mathbb{Z}$. P6 with $k = x$ yields $x + n^\omega \precsim c(0, \omega) + x \prec x + n^\omega + 1$, and $c(0, \omega) + x = c(x, \omega)$ by (31).

Now let $j \in \mathbb{Z}$. We show that

$$\text{every act } X \text{ with } j \precsim X \prec j + 1 \text{ is indifferent to } c(j - n^\omega, \omega) \text{ for some } \omega \in \Omega_+. \tag{34}$$

Let X be an act with $j \precsim X \prec j + 1$. By (32), there is $\omega \in \Omega_+$ with $X \sim c(x, \omega)$, where $x := X(\omega)$. By P1, $j \precsim c(x, \omega) \prec j + 1$. If $x + n^\omega \geq j + 1$, then $c(x, \omega) \succsim x + n^\omega \succsim j + 1$ by (33) and P2, which contradicts $c(x, \omega) \prec j + 1$. If $x + n^\omega \leq j - 1$, then $c(x, \omega) \prec x + n^\omega + 1 \precsim j$ by (33) and P2, which contradicts $j \precsim c(x, \omega)$. Hence $j \leq x + n^\omega \leq j$, i.e., $x = j - n^\omega$, which proves (34). So the acts X with $j \precsim X \prec j + 1$ form at most $|\Omega_+|$ indifference classes.

Finally, let $X \precsim Y$, $a := \min_{\omega \in \Omega} X(\omega)$, and $b := \max_{\omega \in \Omega} Y(\omega)$. By P2, $a \precsim X$ and $Y \precsim b$. Let Z be an act with $X \precsim Z \precsim Y$, so that $a \precsim Z \precsim b$ by P1. Let $J := \{j \in \mathbb{Z} : j \precsim Z\}$. Then $a \in J$. If $j > b$, then $j \succ b \succsim Z$ by (1), so $j \notin J$. So J is a nonempty set of integers bounded above by b,

and hence has a maximum j^Z, with $a \le j^Z \le b$. If $j^Z = b$, then $Z \sim b$. Otherwise, $a \le j^Z \le b-1$, and $j^Z \precsim Z \prec j^Z + 1$ by the maximality of j^Z and P1. So every act Z with $X \precsim Z \precsim Y$ is either indifferent to b or satisfies $j \precsim Z \prec j+1$ for one of the $b-a$ integers $j \in \{a, \dots, b-1\}$. By (34), each such j contributes at most $|\Omega_+|$ indifference classes. Hence there are at most $(b-a)|\Omega_+|+1$ indifference classes of acts Z with $X \precsim Z \precsim Y$. This is P4(i).

P5. Step 1 in the proof of Theorem 1 uses only P1, P2, and P4(i), so it yields an order-isomorphism V of the indifference classes onto $\mathbb{Z}$ with (10). Since V is onto $\mathbb{Z}$, we may define $f : \mathbb{Z} \to \mathbb{Z}$ by $f(V(X)) := V(X+1)$. This is well defined and strictly increasing, since by (10) and P6, $V(X) \ge V(Y)$ if and only if $X \succsim Y$, if and only if $X+1 \succsim Y+1$, if and only if $V(X+1) \ge V(Y+1)$. It is onto $\mathbb{Z}$, since for every $n \in \mathbb{Z}$ there is an act Y with $V(Y) = n$, and then $f(V(Y-1)) = V(Y) = n$. By Lemma 5, f is a translation, i.e., there is $m \in \mathbb{Z}$ with

$$V(X+1) = V(X) + m \quad \text{for all acts } X. \tag{35}$$

Since $1 \succ 0$ by P2, $m = V(1) - V(0) \ge 1$. We show that $V(X+k) = V(X) + mk$ for all acts X and integers $k \ge 0$. We proceed by induction on k. For the base case, $k = 0$, equality holds. Assume as induction hypothesis $V(X+k) = V(X)+mk$. Then $V(X+k+1) = V(X+k)+m = V(X)+m(k+1)$, where the first equality follows from (35) and the second is the induction hypothesis. Let $\omega \in \Omega_+$ and $x < x'$. By (31), $c(x', \omega) = c(x, \omega) + (x' - x)$, so the case $X = c(x, \omega)$ and $k = x' - x$ yields

$$V(c(x', \omega)) - V(c(x, \omega)) = m(x' - x).$$

Since V is an order-isomorphism onto $\mathbb{Z}$, the number of indifference classes of acts Z with $c(x, \omega) \precsim Z \precsim c(x', \omega)$ is $m(x'-x)+1$ for every $\omega \in \Omega_+$, so the counts in P5 coincide for the component acts $c(x, \omega)$. Since the indifference class of a component act does not depend on the choice of $\bar{y}$ within the first alternative of P4(ii) (see Section 3), the same count holds for all component acts from the first alternative, in particular for those of (20). By the paragraph on component acts, $x_\omega y$ is eventually constant in indifference class as $y \to \infty$ for all $\omega \in \Omega_+$ and $x \in \mathbb{Z}$. By Remark 4, Steps 1 to 5 of the proof of Theorem 1 then use P5 only for these component acts. So $\succsim$ is represented by some $U_{+1,u}$ that takes every integer value, and the necessity part of Theorem 1 yields P5 for all component acts.

The case P3$^-$. By (D1), (D2), and (D4) of Lemma 6, the dual relation $\succsim^*$ satisfies P1, P2, P3* for $\sigma = +1$, and P4(ii). By the case P3$^+$, it satisfies P3 for $\sigma = +1$, P4, P5, and P6. By (D1), (D2), and (D4), $\succsim$ satisfies P3 for $\sigma = -1$, P4, P5, and P6. □

Proof of Theorem 2

We prove sufficiency, part (ii), necessity, and part (iii) for the min-plus form and obtain them for the max-plus form by duality. Part (i) and the remaining case of part (iii) come last.

Sufficiency. By Lemma 3, $\succsim$ satisfies P1 to P6, with P3$^+$. By Theorem 1 and its part (i), $\succsim$ is represented by $U_{+1,u}$ for a strictly increasing u and a ranking function ρ, with $U_{+1,u}(\mathcal{X}) = \mathbb{Z}$. So $U_{+1,u}$ is an order-isomorphism of the indifference classes onto $\mathbb{Z}$, and the arguments for (35) and for $m \ge 1$ in the proof of Lemma 3 apply to $V = U_{+1,u}$. So there is an integer $m \ge 1$ with

$$U_{+1,u}(X+1) = U_{+1,u}(X) + m \quad \text{for all acts } X. \tag{36}$$

Since $\min_{\omega \in \Omega} \rho(\omega) = 0$, $U_{+1,u}(k) = u(k)$ for every sure amount k. Applying (36) to the sure act $X = x$ yields

$$u(x+1) = u(x) + m \quad \text{for all } x \in \mathbb{Z}.$$

We show that $u(x) = mx + u(0)$ for all $x \in \mathbb{Z}$. We proceed by induction on $|x|$. For the base case, $x = 0$, equality holds. Assume as induction hypothesis $u(x) = mx + u(0)$ and $u(-x) = -mx + u(0)$ for some $x \geq 0$. Then

$$u(x+1) = u(x) + m = m(x+1) + u(0) \quad \text{and} \quad u(-x-1) = u(-x) - m = -m(x+1) + u(0),$$

where, in each equation, the first equality follows from the recursion, at x and at $-x-1$, respectively, and the second from the induction hypothesis. Hence, by the definitions of $U_{+1,u}$ and U_m,

$$U_{+1,u}(X) = \min_{\omega \in \Omega} \{mX(\omega) + u(0) + \rho(\omega)\} = U_m(X) + u(0).$$

Since $U_m = U_{+1,u} - u(0)$ and $U_{+1,u}$ represents $\succsim$, so does U_m. Since $U_{+1,u}(\mathcal{X}) = \mathbb{Z}$ and $u(0)$ is an integer, $U_m(\mathcal{X}) = \mathbb{Z}$.

Part (ii). Let $\succsim$ be represented by U_t with $t > 0$. Then $U_t = U_{+1,u}$ with $u(x) = tx$ strictly increasing. The proof of part (ii) in the proof of Theorem 1 does not use $U_{+1,u}(\mathcal{X}) = \mathbb{Z}$, so it applies to U_t.

Necessity. Let $\succsim$ be represented by U_t with $t > 0$. Then $U_t = U_{+1,u}$ with $u(x) = tx$, which is strictly increasing from $\mathbb{Z}$ to $\mathbb{Z}$. In the necessity part of the proof of Theorem 1, only the proof of P5 uses that $U_{+1,u}$ takes every integer value, and the proof of P4(ii) uses part (ii) of Theorem 1, which holds for U_t by part (ii). So the proofs of P1, P2, P3$^+$, and P4(ii) there apply to U_t. For every act X and $k \in \mathbb{Z}$,

$$U_t(X + k) = \min_{\omega \in \Omega} \{tX(\omega) + tk + \rho(\omega)\} = U_t(X) + tk. \tag{37}$$

P6: For all acts X, Y and $k \in \mathbb{Z}$,

$$\begin{aligned} X \succsim Y &\iff U_t(X) \geq U_t(Y) \\ &\iff U_t(X) + tk \geq U_t(Y) + tk \\ &\iff U_t(X + k) \geq U_t(Y + k) \\ &\iff X + k \succsim Y + k. \end{aligned}$$

The first and the last equivalence hold since U_t represents $\succsim$. The second adds tk to both sides. The third follows from (37), applied to X and to Y.

P3*: By Lemma 2, P3$^+$ and P6 yield P3* in the min-plus form.

Part (iii). Let $\succsim$ be represented by U_t with $t > 0$ and some ranking function ρ. By the paragraph *Properties of* $U_{+1,u}$ in the proof of Theorem 1, U_t is integer-valued, and $U_t(k) = tk$ for every sure amount k. Since U_t represents $\succsim$ and is integer-valued, an act X satisfies $0 \precsim X \prec 1$ if and only if $0 \leq U_t(X) \leq t - 1$. Since U_t represents $\succsim$, acts X and Y are indifferent if and only if $U_t(X) = U_t(Y)$. So the indifference classes of such acts take distinct values of U_t in $\{0, \ldots, t-1\}$, and

$$m_P(\succsim) \leq t.$$

Suppose, moreover, that $U_t(\mathcal{X}) = \mathbb{Z}$. Then every value in $\{0, \ldots, t-1\}$ is attained by some act, which lies in one of these classes. So the classes take all t values, and $t = m_P(\succsim)$. Since $m_P(\succsim)$ depends only on $\succsim$, t is unique. Applied to any representation by $U_{t'}$ with $t' > 0$, the first part yields $t' \geq m_P(\succsim) = t$. Finally, $U_t = U_{+1,u}$ with $u(x) = tx$ strictly increasing and $U_t(\mathcal{X}) = \mathbb{Z}$ is a representation as in Theorem 1, so ρ is unique by part (iii) of Theorem 1. This covers representations $U_{t'}$ with $\operatorname{sgn}(t') = \operatorname{sgn}(t)$. The case of opposite signs involves the max-plus form and uses part (i), so we complete it at the end of the proof.

The max-plus form. By (D1), (D2), and (D4) of Lemma 6, $\succsim$ satisfies P1, P2, P3* for $\sigma = -1$, and P4(ii) if and only if $\succsim^*$ satisfies P1, P2, P3* for $\sigma = +1$, and P4(ii). By (D5), $\succsim$ is represented by U_{-t} if and only if $\succsim^*$ is represented by U_t, with the same ρ. If $\succsim$ satisfies P1, P2, P3* for $\sigma = -1$, and P4(ii), then it satisfies P6 by Lemma 3. If $\succsim$ is represented by U_{-t}, then $\succsim^*$ satisfies P6 by necessity, and so does $\succsim$ by (D1). In both cases, $m_O(\succsim) = m_P(\succsim^*)$ by (D5). By (D5), moreover, U_{-t} takes every integer value if and only if U_t does, and by (D3), the two relations have the same null states. Applying the min-plus case to $\succsim^*$ yields sufficiency, necessity, and parts (ii) and (iii) for the max-plus form.

Part (i). The first statement follows from sufficiency in the min-plus and max-plus forms. Suppose that $\succsim$ is represented by U_t and by $U_{t'}$ with $t > 0 > t'$. By necessity in the min-plus and the max-plus form, $\succsim$ satisfies P1, P2, P4(ii), and P3* both for $\sigma = +1$ and for $\sigma = -1$. By Lemma 3, applied with $\sigma = +1$ and with $\sigma = -1$, it satisfies P3$^+$, P3$^-$, P4, and P5. So, by Lemma 1, at most one state is non-null. Hence, if at least two states are non-null, the sign of t is unique.

Completing part (iii). Let $\succsim$ also be represented by $U_{t'}$ with $\operatorname{sgn}(t') \neq \operatorname{sgn}(t)$. We show $|t'| \geq |t|$. As in part (i), at most one state is non-null. Since $\min_{\omega\in\Omega} \rho(\omega) = 0$, some state ω_0 has $\rho(\omega_0) = 0$. By part (ii), ω_0 is non-null. Since at most one state is non-null, every other state is null, and so has rank ∞ by part (ii). Hence $U_t(X) = |t|X(\omega_0)$ for every act X, and $U_t(\mathcal{X}) = |t|\mathbb{Z}$. Since $U_t(\mathcal{X}) = \mathbb{Z}$, $|t| = 1 \leq |t'|$. □

Proof of Proposition 1

We prove the statement for $\sigma = +1$ and indicate the changes for $\sigma = -1$ at the end. By the paragraph *Properties of* $U_{+1,u}$ in the proof of Theorem 1, $U_{+1,u}(k) = u(k)$ for every sure amount k. Since $U_{+1,u}$ represents $\succsim$, an act X satisfies $X \sim k$ if and only if $U_{+1,u}(X) = u(k)$. Hence P7 holds if and only if every value of $U_{+1,u}$ is a value of u. Since $U_{+1,u}$ takes every integer value by Theorem 1, and u takes integer values, this holds if and only if u is onto $\mathbb{Z}$. Since u is strictly increasing by Theorem 1, u is onto $\mathbb{Z}$ if and only if it is a bijection of $\mathbb{Z}$ onto itself, which, by Lemma 5, holds if and only if $u(x) = x + u(0)$ for all $x \in \mathbb{Z}$.

In this case, by the definition of $U_{+1,u}$, for every act X,

$$U_{+1,u}(X) = \min_{\omega\in\Omega} \big\{X(\omega) + u(0) + \rho(\omega)\big\} = U_1(X) + u(0).$$

Since $U_1 = U_{+1,u} - u(0)$ and $U_{+1,u}$ represents $\succsim$, so does U_1. Since $U_{+1,u}$ takes every integer value by Theorem 1 and $u(0)$ is an integer, so does U_1. By Theorem 2 with $t = 1$, $\succsim$ satisfies P3*, and by Lemma 2, it satisfies P6.

For $\sigma = -1$, the same steps apply to $U_{-1,u}$, for which $U_{-1,u}(k) = u(k)$, since $\min_{\omega\in\Omega} \rho(\omega) = 0$, and $U_{-1,u} = U_{-1} + u(0)$ if $u(x) = x + u(0)$. Theorem 2 and Lemma 2 apply with $t = -1$. □

Proof of Lemma 4

We prove the statement for $\sigma = +1$ and obtain the one for $\sigma = -1$ by duality. By Lemma 2, P3$^+$ and P6 hold. Let $\omega \in \Omega_+$. By P7, for every $y \in \mathbb{Z}$, there is $k_y \in \mathbb{Z}$ with

$$0_\omega y \sim k_y,$$

and by (1), it is unique. Since $0_\omega(y + 1) \geq 0_\omega y$, P2 yields $k_{y+1} \sim 0_\omega(y + 1) \succsim 0_\omega y \sim k_y$, and (1) yields $k_{y+1} \geq k_y$. So k_y is nondecreasing in y.

We show that k_y is bounded above. Suppose, to the contrary, that it is not. We show that

$$X \sim L_\omega X \quad \text{for every act } X \text{ and every integer } L \geq \max_{\omega' \in \Omega} X(\omega'). \tag{38}$$

Let X be an act, and let L be an integer with $L \geq \max_{\omega' \in \Omega} X(\omega')$. Since k_y is unbounded above, there is y_0 with $k_{y_0} > L - X(\omega)$. Let $y \geq \max\{L, y_0 + X(\omega)\}$. Since, as shown above, k_y is nondecreasing, and $y - X(\omega) \geq y_0$, $k_{y-X(\omega)} + X(\omega) \geq k_{y_0} + X(\omega) > L$. At ω, the act $\big(X_\omega y\big) \wedge L_\omega X$ pays $\min\{X(\omega), L\} = X(\omega)$, and at $\omega' \neq \omega$, it pays $\min\{y, X(\omega')\} = X(\omega')$, since $y \geq L \geq X(\omega')$. Hence

$$X = \big(X_\omega y\big) \wedge L_\omega X. \tag{39}$$

Since $X_\omega y = 0_\omega(y - X(\omega)) + X(\omega)$, P6 yields $X_\omega y \sim k_{y-X(\omega)} + X(\omega)$. Since $L \geq L_\omega X$ and (as shown above) $k_{y-X(\omega)} + X(\omega) \geq k_{y_0} + X(\omega) > L$, (1) and P2 yield

$$X_\omega y \sim k_{y-X(\omega)} + X(\omega) \succ L \succsim L_\omega X.$$

So $L_\omega X$ is the worse of the two acts in (39), and Lemma 7 yields (38). Applied to two acts X and Y that differ only at ω, with L weakly larger than all their outcomes, (38) yields $X \sim L_\omega X = L_\omega Y \sim Y$. So ω is null, which contradicts $\omega \in \Omega_+$.

Hence k_y, as a bounded nondecreasing sequence of integers, must be eventually constant, i.e., there is $\bar{y}^\omega$ with $k_y = k_{\bar{y}^\omega}$ for all $y \geq \bar{y}^\omega$. So $0_\omega y \sim 0_\omega \bar{y}^\omega$ for all $y \geq \bar{y}^\omega$. As in the paragraph on component acts in the proof of Lemma 3, P6 yields $x_\omega y \sim x_\omega(x + \bar{y}^\omega)$ for all $x \in \mathbb{Z}$ and $y \geq x + \bar{y}^\omega$, which is P4(ii).

Finally, by (D1) and (D2) of Lemma 6, if $\succsim$ satisfies P1, P2, P7, and P3* for $\sigma = -1$, then $\succsim^*$ satisfies P1, P2, P7, and P3* for $\sigma = +1$. By the case $\sigma = +1$, $\succsim^*$ satisfies P4(ii). By (D4), $\succsim$ satisfies P4(ii). □

Proof of Corollary 1

Let $\sigma \in \{-1, +1\}$. By the paragraph *Properties of* $U_{+1,u}$ in the proof of Theorem 1, with $u(x) = x$, and likewise for $\sigma = -1$, $U_\sigma(X)$ is an integer for every act X, and $U_\sigma(k) = k$ for every constant act k. Applied to the constant act $k = U_\sigma(X)$, this yields $U_\sigma(U_\sigma(X)) = U_\sigma(X)$. Hence, if U_σ represents $\succsim$, then

$$X \sim U_\sigma(X) \quad \text{for all acts } X. \tag{40}$$

Sufficiency. By Lemma 4, P4(ii) holds, so Theorem 2 applies. Suppose P3* holds for $\sigma = +1$, and let X be an act with $0 \precsim X \prec 1$. By P7, $X \sim k$ for some $k \in \mathbb{Z}$, so $0 \precsim k \prec 1$ by P1. By (1), $k \geq 1$ would imply $k \succsim 1$, and $k \leq -1$ would imply $k \prec 0$, so $k = 0$. So all such acts X are indifferent to 0, and $m_P(\succsim) = 1$. Suppose P3* holds for $\sigma = -1$, and let X be an act with $0 \prec X \precsim 1$. By analogous arguments, $X \sim 1$, and $m_O(\succsim) = 1$. By parts (i) and (iii) of Theorem 2, $\succsim$ is represented by U_σ if P3* holds for σ, σ is unique if at least two states are non-null, and ρ is unique. Part (ii) follows from part (ii) of Theorem 2. By (40), $X \sim U_\sigma(X)$.

Necessity. Let $\succsim$ be represented by U_σ. Since $U_\sigma(k) = k$ for every constant act k, U_σ takes every integer value. So, by Theorem 2 with $t = \sigma$, P1, P2, and P3* hold. By (40), P7 holds. □

References

[1] Abdellaoui, M., Baillon, A., Placido, L., Wakker, P.P. (2011). The rich domain of uncertainty: Source functions and their experimental implementation, American Economic Review 101, 695–723.

[2] Akian, M. (1999). Densities of idempotent measures and large deviations, Transactions of the American Mathematical Society 351, 4515–4543.

[3] Allais, M. (1953). Le comportement de l'homme rationnel devant le risque: Critique des postulats et axiomes de l'école américaine, Econometrica 21, 503–546.

[4] Anscombe, F.J., Aumann, R.J. (1963). A definition of subjective probability, Annals of Mathematical Statistics 34, 199–205.

[5] Arrow, K., Hurwicz, L. (1972). An optimality criterion for decision-making under ignorance, in: Carter, C.F., Ford, J.L. (eds.), Uncertainty and expectations in economics: Essays in honour of G.L.S. Shackle, Basil Blackwell, Oxford, 1–11.

[6] Baldwin, E., Klemperer, P. (2019). Understanding preferences: "Demand types", and the existence of equilibrium with indivisibilities, Econometrica 87, 867–932.

[7] Blavatskyy, P., Ortmann, A., Panchenko, V. (2022). On the experimental robustness of the Allais paradox, American Economic Journal: Microeconomics 14, 143–163.

[8] Chateauneuf, A., Eichberger, J., Grant, S. (2007). Choice under uncertainty with the best and worst in mind: Neo-additive capacities, Journal of Economic Theory 137, 538–567.

[9] Congar, R., Maniquet, F. (2010). A trichotomy of attitudes for decision-making under complete ignorance, Mathematical Social Sciences 59, 15–25.

[10] Del Moral, P., Doisy, M. (1999). Maslov idempotent probability calculus, I, Theory of Probability and Its Applications 43, 562–576.

[11] Develin, M., Sturmfels, B. (2004). Tropical convexity, Documenta Mathematica 9, 1–27.

[12] Dubois, D., Prade, H. (1995). Possibility theory as a basis for qualitative decision theory, in: Proceedings of the 14th International Joint Conference on Artificial Intelligence (IJCAI), Morgan Kaufmann, 1924–1930.

[13] Dubois, D., Prade, H. (1998). Possibility theory: Qualitative and quantitative aspects, in: D.M. Gabbay, P. Smets (eds.), Handbook of Defeasible Reasoning and Uncertainty Management Systems, Vol. 1, Kluwer, Dordrecht, 169–226.

[14] Dubois, D., Prade, H., Sabbadin, R. (2001). Decision-theoretic foundations of qualitative possibility theory, European Journal of Operational Research 128, 459–478.

[15] Ehrig, T., Schmidt, J. (2026). Logical reframing: A theory-based view of entrepreneurial persuasion, Strategic Entrepreneurship Journal, forthcoming.

[16] Eichberger, J., Grant, S., Kelsey, D., Koshevoy, G.A. (2011). The α-MEU model: A comment, Journal of Economic Theory 146, 1684–1698.

[17] Ellsberg, D. (1961). Risk, ambiguity, and the Savage axioms, Quarterly Journal of Economics 75, 643–669.

[18] Fargier, H., Sabbadin, R. (2005). Qualitative decision under uncertainty: Back to expected utility, Artificial Intelligence 164, 245–280.

[19] Ghirardato, P., Maccheroni, F., Marinacci, M. (2004). Differentiating ambiguity and ambiguity attitude, Journal of Economic Theory 118, 133–173.

[20] Giang, P.H., Shenoy, P.P. (2000). A qualitative linear utility theory for Spohn's theory of epistemic beliefs, in: Proceedings of the 16th Conference on Uncertainty in Artificial Intelligence, Morgan Kaufmann, 220–229.

[21] Goldszmidt, M., Pearl, J. (1992a). Reasoning with qualitative probabilities can be tractable, in: Dubois, D., Wellman, M.P., D'Ambrosio, B., Smets, P. (eds.), Proceedings of the 8th Conference on Uncertainty in Artificial Intelligence, Morgan Kaufmann, 112–120.

[22] Goldszmidt, M., Pearl, J. (1992b). Rank-based systems: A simple approach to belief revision, belief update, and reasoning about evidence and actions, in: Nebel, B., Rich, C., Swartout, W. (eds.), Proceedings of the Third International Conference on Principles of Knowledge Representation and Reasoning (KR'92), Morgan Kaufmann, 661–672.

[23] Goldszmidt, M., Pearl, J. (1996). Qualitative probabilities for default reasoning, belief revision, and causal modeling, Artificial Intelligence 84, 57–112.

[24] Gul, F. (1991). A theory of disappointment aversion, Econometrica 59, 667–686.

[25] Herstein, I.N., Milnor, J. (1953). An axiomatic approach to measurable utility, Econometrica 21, 291–297.

[26] Hrbacek, K., Jech, T. (1999). Introduction to set theory, 3rd edition, revised and expanded, Marcel Dekker, New York.

[27] Huber, F. (2019). Ranking theory, in: Pettigrew, R., Weisberg, J. (eds.), The open handbook of formal epistemology, PhilPapers, 397–436.

[28] Hurwicz, L. (1951). Optimality criteria for decision making under ignorance, Cowles Commission Discussion Paper, Statistics No. 370.

[29] Kahneman, D., Tversky, A. (1979). Prospect theory: An analysis of decision under risk, Econometrica 47, 263–291.

[30] Kern-Isberner, G., Skovgaard-Olsen, N., Spohn, W. (2021). Ranking theory, in: Knauff, M., Spohn, W. (eds.), The handbook of rationality, MIT Press, Cambridge, MA, 337–346.

[31] Köbberling, V., Wakker, P.P. (2003). Preference foundations for nonexpected utility: A generalized and simplified technique, Mathematics of Operations Research 28, 395–423.

[32] Kolokoltsov, V.N. (2001). Idempotent measures and their applications, IFAC Proceedings Volumes 34(13), 755–758.

[33] Kolokoltsov, V.N., Maslov, V.P. (1987). The general form of endomorphisms in the space of continuous functions with values in a numerical commutative semiring (with the operation $\oplus = \max$), Doklady Akademii Nauk SSSR 295(2), 283–287 (in Russian). English translation: Soviet Mathematics Doklady 36 (1988), 55–59.

[34] Kolokoltsov, V.N., Maslov, V.P. (1997). Idempotent analysis and its applications, Kluwer Academic Publishers, Dordrecht.

[35] Kyburg, H.E. (1961). Probability and the logic of rational belief, Wesleyan University Press, Middletown, CT.

[36] Maccheroni, F., Marinacci, M., Rustichini, A. (2006). Ambiguity aversion, robustness, and the variational representation of preferences, Econometrica 74, 1447–1498.

[37] Maskin, E. (1979). Decision-making under ignorance with implications for social choice, Theory and Decision 11, 319–337.

[38] Milnor, J. (1954). Games against nature, in: Thrall, R.M., Coombs, C.H., Davis, R.L. (eds.), Decision processes, Wiley, New York, 49–59.

[39] Ortoleva, P. (2012). Modeling the change of paradigm: Non-Bayesian reactions to unexpected news, American Economic Review 102, 2410–2436.

[40] Pearl, J. (1993). From conditional oughts to qualitative decision theory, in: Proceedings of the 9th Conference on Uncertainty in Artificial Intelligence, Morgan Kaufmann, 12–20.

[41] Pratt, J.W. (1964). Risk aversion in the small and in the large, Econometrica 32, 122–136.

[42] Quiggin, J. (1982). A theory of anticipated utility, Journal of Economic Behavior and Organization 3, 323–343.

[43] Savage, L.J. (1954). The foundations of statistics, Wiley.

[44] Schipper, B.C. (2026a). Aumann meets Spohn in the tropics: Agreement and speculation with ranking functions, mimeo, University of California, Davis.

[45] Schipper, B.C. (2026b). Risk attitudes in tropical expected utility with ranking functions, mimeo, University of California, Davis.

[46] Schipper, B.C. (2026c). Conditional ranking systems in tropical expected utility, mimeo, University of California, Davis.

[47] Schipper, B.C. (2026d). Eliciting ranking functions, mimeo, University of California, Davis.

[48] Schipper, B.C., Kern-Isberner, G. (2026). The theory of games with ranking functions, mimeo, University of California, Davis, and TU Dortmund University.

[49] Schmeidler, D. (1989). Subjective probability and expected utility without additivity, Econometrica 57, 571–587.

[50] Shackle, G.L.S. (1949). Expectation in economics, Cambridge University Press, Cambridge.

[51] Speyer, D., Sturmfels, B. (2009). Tropical mathematics, Mathematics Magazine 82, 163–173.

[52] Spohn, W. (1988). Ordinal conditional functions: A dynamic theory of epistemic states, in: Harper, W.L., Skyrms, B. (Eds.), Causation in decision, belief change, and statistics II, Dordrecht: Kluwer, 105–134.

[53] Spohn, W. (2012). The laws of belief: Ranking theory and its philosophical applications, Oxford University Press.

[54] Spohn, W. (2025). Rank-based decision theory and conditional deontic logic, Proceedings of Machine Learning Research 290, 263–272.

[55] Tran, N.M., Yu, J. (2019). Product-mix auctions and tropical geometry, Mathematics of Operations Research 44, 1396–1411.

[56] Trautmann, S.T., van de Kuilen, G. (2015). Ambiguity attitudes, in: Keren, G., Wu, G. (eds.), The Wiley Blackwell handbook of judgment and decision making, Wiley, Chichester, 89–116.

[57] von Neumann, J., Morgenstern, O. (1947). Theory of games and economic behavior, 2nd edition, Princeton University Press.

[58] Wakker, P.P. (1989). Additive representations of preferences: A new foundation of decision analysis, Kluwer Academic Publishers, Dordrecht.

[59] Wakker, P.P. (1991). Additive representations of preferences, a new foundation of decision analysis; the algebraic approach, in: Doignon, J.-P., Falmagne, J.-C. (Eds.), Mathematical psychology: Current developments, Springer, Berlin, 71–87.

[60] Wald, A. (1950). Statistical decision functions, Wiley, New York.

[61] Wilson, N. (1995). An order of magnitude calculus, in: Proceedings of the 11th Conference on Uncertainty in Artificial Intelligence, Morgan Kaufmann, 548–555.

[62] Zadeh, L.A. (1978). Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 1, 3–28.

[63] Zapata, J.M. (2025). Maxitive monetary risk measures: Worst-case risk assessment and sharp large deviations, Stochastics and Dynamics 25(2), 2550014.