Kuhn's Theorem for Games of the Extensive Form with Unawareness[*]

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Abstract

We extend Kuhn's Theorem to games of the extensive form with unawareness. This extension is not obvious: First, games of the extensive form with non-trivial unawareness involve a forest of partially ordered game trees rather than just one game tree. An information set at a node in one tree may consist of nodes in a less expressive tree. Consequently, perfect recall takes a more complicated form as players may also become aware of new actions during the play. Second, strategies can only be partially an object of ex ante choice in games with unawareness. Finally, nodes that a player may expect to reach with a strategy profile may not be the nodes that actually occur with this strategy profile, requiring us to define appropriate notions of equivalence of strategies. We show if a game of the extensive form with unawareness has perfect recall, then for each mixed strategy there is an equivalent behavior strategy but the converse does not hold under unawareness.

Keywords: Perfect recall, mixed strategy, behavior strategy, unawareness.

JEL-Classifications: C72, D83.

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1 Introduction

In games of the extensive form there are two notions of mixed strategies. First, a player can ex ante mix over her strategies, using a so called mixed strategy. Second, at each information set a player can mix interim over actions at that information set, using a so called behavior strategy. Kuhn's Theorem states that a game of the extensive form has perfect recall if and only if for each mixed strategy there is a behavior strategy that is equivalent in terms of the probabilities of reaching nodes in the game tree (Kuhn, 1953). The use of mixed strategies has been questioned (see a discussion in Osborne and Rubinstein, 1994, Chapter 3.2). However, even if players do not actually play mixed strategies, players may ex ante form probabilistic beliefs over strategies of opponents, so called conjectures. Such a conjecture may be viewed as a mixed strategy. Similarly, a player may form beliefs over opponents' actions that they may play at each of their information sets, and a profile of such beliefs, one for each of the opponent's information sets can be viewed as a behavior strategy. The Kuhn's Theorem implies that in games of perfect recall it is sufficient to work with behavior strategies. It highlights the role of the perfect recall assumption and facilitates the development of theory in games of the extensive form. Almost the entire literature invokes the perfect recall assumption and often invokes in some form or another Kuhn's Theorem. In that regard, it is fair to say that Kuhn's Theorem is one of the most fundamental theorems on games of the extensive form.

In games of the extensive form with unawareness, the notion of mixed strategy is even less compelling. A player may not be aware of all actions ex ante. Thus, she may not be able to ex ante conceive of all possible strategies and consequently may be unable to mix over them. Yet, given her awareness ex ante, she conceives of some partial strategies that she is able to mix over. It begs now the question whether for each (partial) mixed strategy there is an equivalent (partial) behavior strategy. Several subtleties prevent a quick straightforward answer: First, games of the extensive form with unawareness feature a forest of game trees rather than just one tree (see Halpern and Rêgo, 2014, Heifetz, Meier, and Schipper, 2013, Grant and Quiggin, 2013, Feinberg, 2021, Schipper, 2021; see Schipper 2014 for a review). The relevant information set at a node in one tree may actually reside in a less expressive tree, signifying the fact that the player to whom this information set belongs to is unaware of something. These obvious differences in the formalism to standard games of the extensive form make it non-trivial to define perfect recall in a meaningful way. Yet, the perfect recall assumption is instrumental for proving Kuhn's Theorem. In games of the extensive form with unawareness, perfect recall interacts with other properties on how awareness differs across trees. In particular, these properties imply that a player's awareness may not decrease during play. That is, players do not only not forget information but also do not forget awareness. Not surprisingly, it turns out that this property is also crucial for us proving the analogue of Kuhn's Theorem for games of the extensive form with unawareness.

There is another challenge that Kuhn's Theorem faces in games of the extensive form with unawareness. What is a meaningful notion of two strategies being equivalent to each other? Games of the extensive form with unawareness allow the game theorist to model differing players' subjective conceptions of the game during the play. Since (partial) strategies are objects of players' beliefs, the notion of a strategy *reaching* a node or an information set is also subjective. A player may believe that a strategy reaches a particular node or a particular information set even though this node or information set cannot be realized with this strategy. We formalize these two notions of a strategy being consistent with a node or information set in a game of the extensive form, show their relationship, and extend Kuhn's Theorem to both of these two notions in games of the extensive form with unawareness. We show that if the game of the extensive form with unawareness satisfies perfect recall, then for every mixed strategy there exists an equivalent behavior strategy. We show also with a counterexample that the converse of the theorem does not hold. Latter is in contrast to standard games of the extensive form without unawareness. While major textbooks[1] and even the seminal paper by Selten (1975) just state that perfect recall implies the equivalence of mixed and behavior strategies for standard games (without unawareness), Kuhn (1953) also proved the converse. This converse fails under unawareness. That is, we can find games in which for every mixed strategy there is an equivalent behavior strategy but that violates perfect recall (in a way that perfect recall cannot be violated in games without unawareness). In particular, we present an example where there is an action that the player herself is unaware of, but that leads to an information set of that player that can also be reached with an action that the player is aware of.

Kuhn's Theorem for games of the extensive form with unawareness helps to establish the foundation of games with unawareness. This paper is motivated by prior work on self-confirming equilibrium in which the result of this paper is invoked to define what we believe is a meaningful notion of equilibrium in games with unawareness, a notion that can be interpreted both as a steady state of conceptions and play (see Schipper, 2021). Various applications of games of the extensive form with unawareness have been developed in the literature. Verifiable disclosure and the failure of unraveling of information is analyzed by Heifetz, Meier, and Schipper (2021); see Li and Schipper (2024) for an experiment. Schipper and Woo (2019) study political campaigning. Filiz-Ozbay (2012) analyzes disclosure before insurance contracts, Francetich and Schipper (2024) analyze the incentives of an agent for raising the awareness of a principal before screening by the latter, and Pram and Schipper (2025) introduce dynamic elaboration VCG mechanisms for implementation of efficient outcomes at the pooled awareness level in conditional dominant strategies. We have now all game theoretic tools ready to revisit interesting problems in economics and explore how the presence of unawareness may change predictions;

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[1]Well known textbooks like Osborne and Rubinstein (1994) and Maschler, Solan, and Zamir (2013) state and prove one direction. Even Kuhn's (2002) own lecture notes just proves one direction. A notable exception is Ritzberger (2002) who presents the most comprehensive treatment in the literature and states and proves both directions.

see Schipper (2025) for a bibliography of the growing literature.

We should state upfront that this paper has nothing to contribute to decision making under absentmindedness (see Piccione and Rubinstein, 1997, and the special issue on imperfect recall in Games and Economic Behavior 1997) beyond the fact that perfect recall also implies non-absentmindedness in games of the extensive form with unawareness. Readers might be interested in more extreme notions of absentmindedess where players become unaware of events that they had been aware of previously, but such situations are outside current models of games of the extensive form with unawareness.

The paper is organized as follows: The next section spells out in detail games of the extensive form with unawareness. Section 3 introduces various notions of strategies. Kuhn's Theorem is extended to games of the extensive form with unawareness in Section 4.

2 Games of the Extensive form with Unawareness

In this section, we outline games of the extensive form with unawareness à la Heifetz, Meier, and Schipper (2013).[2] To define a game of the extensive form with unawareness Γ, consider first, as a building block, a finite game of perfect information with possibly simultaneous moves. That is, we allow for simultaneous moves as in Dubey and Kaneko, 1984, or in Osborne and Rubinstein, 1994, Chapter 6.3.2. The major purpose of this tree is to outline all physical moves. There is a finite set of players I and possibly a special player "nature" with index 0. We denote by I^0 the set of players including nature. Further, there is a nonempty finite set of "decision" nodes $\bar{D}$ and a player correspondence $P : \bar{D} \longrightarrow 2^{I^0} \setminus \{\emptyset\}$ that assigns to each node $n \in \bar{D}$, a nonempty set of "active" players $P(n) \subseteq I^0$. For every decision node $n \in \bar{D}$ and player $i \in P(n)$ who moves at that decision node, there is a nonempty finite set of actions $A_i(n)$. Moreover, there is a set of terminal nodes $\bar{Z}$. Each terminal node $z \in \bar{Z}$ is associated with a vector of payoffs $(u_i(z))_{i \in I}$, one for each player $i \in I$. We require that nodes in $\bar{N}$ are partially ordered by a precedence relation $\lessdot$ with which $(\bar{N}, \lessdot)$ forms an arborescence (that is, the predecessors of each node in $\bar{N}$ are totally ordered by $\lessdot$). This means that nodes in $\bar{N} := \bar{D} \cup \bar{Z}$ constitute

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[2]Although there are differences in the formalism between various approaches to games of the extensive form with unawareness (Halpern and Rêgo, 2014, Heifetz, Meier, and Schipper, 2013, Grant and Quiggin, 2013, Feinberg, 2021, Rêgo and Halpern, 2012, Ozbay, 2007; see Schipper, 2014, for a brief review), all approaches model unawareness that is consistent with the paradigm of "propositional awareness" as in Fagin and Halpern (1988) or Heifetz, Meier, and Schipper (2006). We use here the approach by Heifetz, Meier, and Schipper (2013) (also used by Schipper and Woo, 2019, Heifetz, Meier, and Schipper, 2021, Schipper, 2021, Meier and Schipper, 2024, Francetich and Schipper, 2024, Li and Schipper, 2024, and Pram and Schipper, 2025) because information sets in their approach can be interpreted as states of the mind of a player at a node. In their approach, information sets model both information and awareness rather than just "information if the player were aware of it". It also avoids having to define a separate awareness correspondence that for each node specifies which nodes the player is aware of.

a directed tree denoted by $\bar{T}$ and there is a unique node in $\bar{N}$ with no predecessors (i.e., the root of the tree). Finally, for each decision node $n \in \bar{D}$ there is a bijection ψ_n between the action profiles $\prod_{i \in P(n)} A_i(n)$ at n and n's immediate successors. Any terminal node in $\bar{Z}$ has no successors.

Note that so far we treat nature like any other player except that at terminal nodes we do not assign payoffs to nature.[3] We do not need to require that nature moves first or that nature moves according to a pre-specified probability distribution (although these assumptions can be imposed in our framework).

Consider now a finite join-semilattice $\mathbf{T}$ of subtrees of $\bar{T}$.[4] A subtree T is defined by a subset of nodes $N \subseteq \bar{N}$ with $N \cap \bar{D} \neq \emptyset$, along with $\prec$ the restriction of $\bar{\preceq}$ to N, where $(N, \prec)$ is also a tree. Two subtrees $T', T'' \in \mathbf{T}$ are ordered, written

$$T' \preceq T''$$

if the nodes of T' constitute a subset of the nodes of T''.

We require the following properties:

0. For any $n, n', n'' \in \bar{T}$ such that $n \bar{\preceq} n' \bar{\preceq} n''$, if there exists $n_T, n''_T \in T$ then there is $n'_T \in T$. That is, we do not allow decision nodes 'in-between' two nodes in the objective tree to be removed in a subtree unless one of the two nodes is also removed.

1. All the terminal nodes in each tree $T \in \mathbf{T}$ are in $\bar{Z}$. That is, we don't create "new" terminal nodes.

2. For every tree $T \in \mathbf{T}$, every node $n \in T$, and every active player $i \in P(n)$ there exists a nonempty subset of actions $A_i^T(n) \subseteq A_i(n)$ such that ψ_n maps the action profiles $A^T(n) = \prod_{i \in P(n)} A_i^T(n)$ bijectively onto n's successors in T.

3. For any tree $T \in \mathbf{T}$, if for two decision nodes $n, n' \in T$ with $i \in P(n) \cap P(n')$ it is the case that $A_i(n) \cap A_i(n') \neq \emptyset$, then $A_i(n) = A_i(n')$.

We illustrate Property 0 in Figure 1. Suppose that the tree modeling all physical moves is given by $\bar{T}$. Then T' is a subtree that does not satisfy Property 0. This is because nodes n, n', n'' are in $\bar{T}$ and $n \bar{\preceq} n' \bar{\preceq} n''$, however in T' there are copies of nodes n and n'' (that is, $n_{T'}$ and $n''_{T'}$) but there is no copy of n' in T'. That is, the decision node n' between n and n'' in $\bar{T}$ has been omitted in T' when $n_{T'}$ and $n''_{T'}$ exist and that is not allowed under Property 0.

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[3]Alternatively, we could assign at every terminal node the same payoff to nature.

[4]A join semi-lattice is a partially ordered set in which each pair of elements has a join, i.e., a least upper bound. Requiring the forest of trees to form a join-semilattice implies there is an objective set of rules governing the strategic interaction. Moreover, no matter which action a player may become aware of during the play, there is a representation that captures all actions that she is aware of.

Figure 1: Property 0
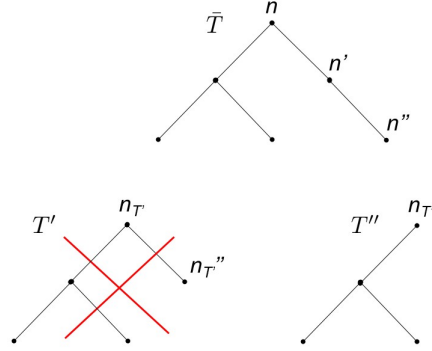
Subtree T'' on the other hand satisfies Property 0, because even though there is no copy of n' in T'', there is also no copy of n'' in T''.

Next, we illustrate Property 1 in Figure 2. Suppose that the tree the modeling all physical moves is given by $\bar{T}$. Then T' is a subtree satisfying Property 1. In contrast, tree T'' does not satisfy Property 1 because it contains a new terminal node. After cutting branches from tree $\bar{T}$ to create tree T'', there is now a node in T'' that was not a terminal node in the original tree $\bar{T}$.

Figure 2: Property 1
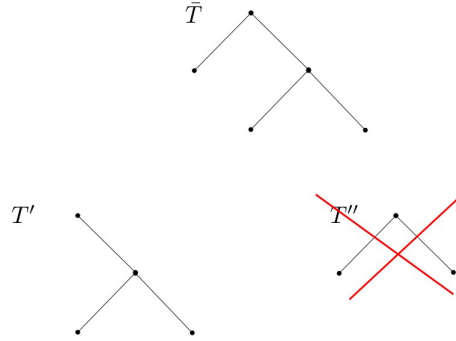
Within the family $\mathbf{T}$ of subtrees of $\bar{T}$, some nodes n appear in several trees $T \in \mathbf{T}$. In what follows, we will need to designate explicitly appearances of such nodes n in different trees as

distinct objects.[5] To this effect, in each tree $T \in \mathbf{T}$ label by n_T the copy in T of the node $n \in \bar{N}$ whenever the copy of n is part of the tree T, with the requirement that if the profile of actions $a_n \in A(n)$ leads from n to n', then a_{n_T} leads also from the copy n_T to the copy n'_T. For any $T, T', T'' \in \mathbf{T}$ with $T \preceq T' \preceq T''$ such that $n \in T''$, $n_{T'}$ is the copy of n in the tree T', n_T is the copy of n in the tree T, and $(n_{T'})_T$ is the copy of $n_{T'}$ in the tree T, we require that "nodes commute", $n_T = (n_{T'})_T$. For any $T \in \mathbf{T}$ and any $n \in T$, we let $n_T := n$ (i.e., the copy of $n \in T$ in T is n itself).

Denote by $\mathbf{D}$ the union of all decision nodes in all trees $T \in \mathbf{T}$, by $\mathbf{Z}$ the union of terminal nodes in all trees $T \in \mathbf{T}$, and by $\mathbf{N} = \mathbf{D} \cup \mathbf{Z}$. Copies n_T of a given node n in different subtrees T are now treated distinct from one another, so that $\mathbf{N}$ is a disjoint union of sets of nodes.

In what follows, when referring to a node in $\mathbf{N}$ we will typically avoid the subscript indicating the tree T for which $n \in T$ when no confusion arises. For a node $n \in \mathbf{N}$ we denote by T_n the tree containing n.[6]

Denote by N^T the set of nodes in the tree $T \in \mathbf{T}$. Similarly, denote by D_i^T the set of decision nodes in which player i is active in the tree $T \in \mathbf{T}$. Moreover, denote by Z^T the set of terminal nodes in the tree $T \in \mathbf{T}$. Finally, we let $\mathbf{D}_i$ denote the set of player i's decision nodes over all trees in $\mathbf{T}$. In the case of nature, $\mathbf{D}_0$ would be all nodes at which nature moves.

In games of the extensive form with unawareness, information sets model both information and awareness. At decision node n of player i in the tree $T_n \in \mathbf{T}$, the player may conceive the feasible paths to be described by a different (i.e., less expressive) tree $T' \in \mathbf{T}$. In such a case, her information set will be a subset of T' rather than of T_n and n will not be contained in the player's information set at n.

Formally, for each node $n \in \mathbf{N}$, define for each active player $i \in P(n) \setminus \{0\}$ a nonempty information set $h_i(n)$ with the following properties:[7]

U0 Confined awareness: If $n \in T$ and $i \in P(n)$, then there is $T' \in \mathbf{T}$, $T' \preceq T$ such that $h_i(n) \subseteq T'$.

U1 Generalized reflexivity: If $T' \preceq T$, $n \in T$, $h_i(n) \subseteq T'$ and T' contains a copy $n_{T'}$ of n, then $n_{T'} \in h_i(n)$.

I2 Introspection: If $n' \in h_i(n)$, then $h_i(n') = h_i(n)$.

I3 No divining of currently unimaginable paths, no expectation to forget currently conceiv-

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[5]It turns out that it is much easier to talk about copies of nodes in a less expressive tree than copies of histories in a less expressive tree. This is the main reason for why we make use of the "older" definition of games of extensive form based on nodes (Kuhn, 1953, Selten, 1975) rather than the more recent notion based on histories (Osborne and Rubinstein, 1994).

[6]Bold capital letters refer to sets of elements across trees.

[7]We keep the numbering consistent with Heifetz, Meier, and Schipper (2013).

able paths: If $n' \in h_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in P(n') \cap P(n'')$, then $h_i(n'') \subseteq T'$.

I4 No imaginary actions: If $n' \in h_i(n)$, then $A_i(n') \subseteq A_i(n)$.

I5 Distinct action names in disjoint information sets: For a subtree $T \in \mathbf{T}$, if there are decision nodes $n, n' \in T \cap \mathbf{D}$ with $A_i(n) = A_i(n')$, then $h_i(n') = h_i(n)$.

Properties (I2), (I4), and (I5) are standard for games of the extensive form, and properties (U0) and (U1) generalize standard properties of games of the extensive form to our generalized setting. At each information set of a player, property (I3) confines the player's *anticipation* of her future view of the game to the view she currently holds (even if, as a matter of fact, this view is about to be shattered as the game evolves). Figure 3 illustrates with an example and a counterexample each both U0 and U1.

Figure 3: Properties U0 and U1
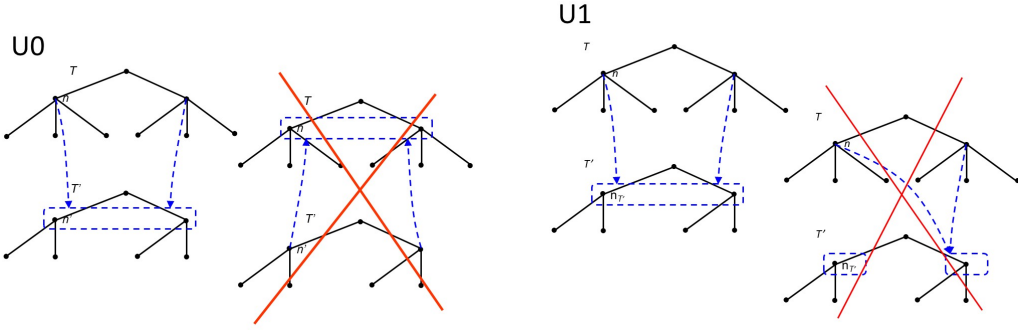
Central to our extension of Kuhn's Theorem is the condition of perfect recall (I6).

I6 Perfect recall: Suppose that player i is active in two distinct nodes n_1 and n_k, and there is a path $n_1, n_2, ..., n_k$ such that at n_1 player i takes the action a_i. If $n' \in h_i(n_k)$, then there exists a node $n'_1 \neq n'$ and a path $n'_1, n'_2, ..., n'_\ell = n'$ such that $h_i(n'_1) = h_i(n_1)$ and at n'_1 player i takes the action a_i.

This property is illustrated with an example in Figure 4 and two counterexamples in Figure 5. Suppose the tree modeling all physical moves is given by $\bar{T}$, and T', T'' are subtrees with $T' \preceq T'' \preceq \bar{T}$. To avoid clutter, we have included only the information sets of interest in the diagram which are drawn in blue. In each tree, player 1 is the only player active at the initial decision node and player 2 is the only player active at all other decision nodes, for example with respect to $\bar{T}$ player 1 is active at node n^* and player 2 is active at nodes n'_1, n''_1, n'

and n''. The other nodes are terminal nodes. We restrict attention to the path where player 1 plays L at node n^* and player 2 plays a_i at n_1, which is the action that has been colored red. The information set at n_k contains two nodes, n' and n'' both of which are in T'. Perfect recall requires 1) that both n' and n'' have a decision node of player 2 that precedes them in T' where a_i is played, a condition which is satisfied by n_1' and n_1'' respectively; and 2) that these nodes n_1' and n_1'' have the same information set as n_1, which is satisfied as all three nodes have the same information set in T''. Perfect recall requires these two conditions to be met for any two nodes of a player whenever one precedes another in a tree. The diagrams in Figure 5 are two examples in which the condition of perfect recall is not satisfied. In the left diagram, n' does not have a decision node of player 2 that precedes it in T' where a_i is played. In the right diagram, the node n_1' does not have the same information set as n_1.

It is known that in standard games of the extensive form without unawareness, perfect recall is necessary for the "playability" of strategies[8] and the existence of Nash equilibrium in behavior strategies; see for instance Wichardt (2008) (see also an earlier example by Luce and Raiffa, 1957, p. 160). Since standard games of the extensive form are special cases of games of the extensive form with unawareness, such arguments apply also to games of the extensive form with unawareness.

Figure 4: Perfect Recall Example
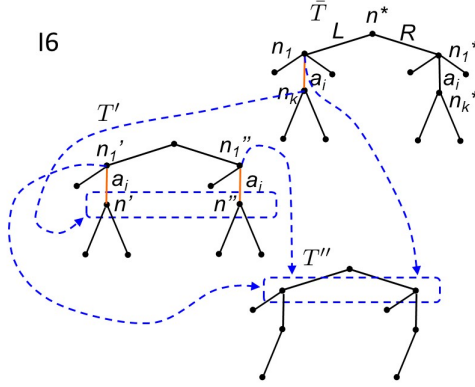
We provide two alternate characterizations of the perfect recall condition. First, perfect recall can be interpreted as players do not forget their experience throughout the game. This can be made explicit. For any player $i \in I$ and decision node of that player $n \in \mathbf{D}_i$, let $E_i(n)$ denote the record of player i's experience along the path to n (not including $h_i(n)$). I.e., $E_i(n)$ is the sequence of pairs (h_i, a_i) of player i's information sets and the action taken at these

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[8]A mixed strategy may require for instance to mix between strategy "play left after having played left" and "play right after having played right". Without perfect recall, such a strategy is not playable because when the player moves the second time, she has forgotten whether she played right or left before.

Figure 5: Perfect Recall Counterexamples
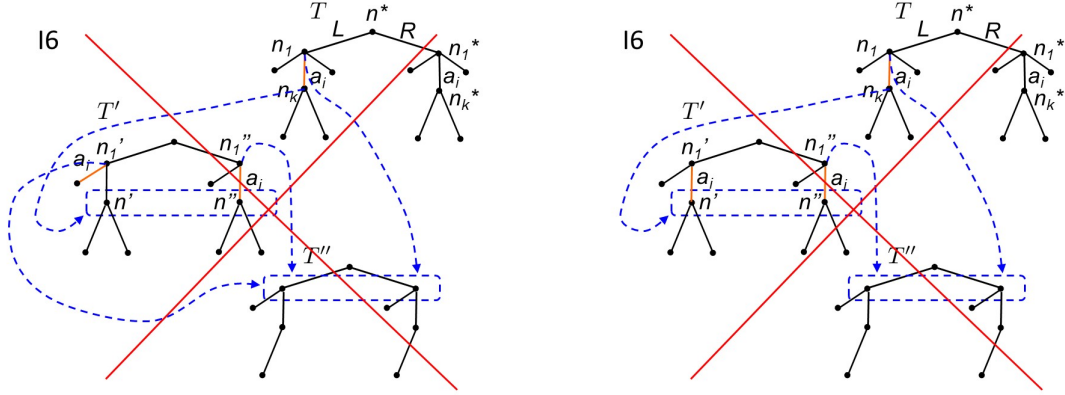
information sets in order of how they are encountered along the path to n. Perfect recall is now characterized as follows:

Remark 1 *A game of the extensive form with unawareness satisfies perfect recall (I6) if and only if for any player $i \in I$, $n \in \mathbf{D}_i$, $n' \in h_i(n)$ implies $E_i(n') = E_i(n)$.*

PROOF.   "$\Rightarrow$": Consider the non-trivial case $n' \neq n$. Suppose by contradiction that $E_i(n') \neq E_i(n)$. Then there exists a path $n_1, n_2, ..., n_k$ with $n = n_k$ for which there is no path $n'_1, n'_2, ..., n'_\ell$ with $n' = n'_\ell$ such that $h_i(n_1) = h_i(n'_1)$ and the action taken at n'_1 along the path n'_1, n'_2 is the action taken at n_1 along the path n_1, n_2. But this contradicts I6.

"$\Leftarrow$": Suppose not. Then there are two nodes n_1 and n_k with $n_1 \neq n_k$ with the path $n_1, n_2, ..., n_k$ where $n = n_k$ such that at n_1 player i takes action a_i along the path, but there is some $n' \in h_i(n)$ where there is no node $n'_1 \neq n'$ where there is a path $n'_1, n'_2, ..., n'_\ell = n'$, $h_i(n'_1) = h_i(n_1)$ and player i takes action a_i at n'_1 along the path. But this just means that player's records of experience are different in nodes n and n', i.e., $E_i(n') \neq E_i(n)$, a contradiction. $\square$

Another characterization would be something similar to the definition provided in Selten (1975). For two nodes n, n' where $T_n = T_{n'}$, define $n <_a n'$ to hold if action a is taken at n in order to get to n'.

Remark 2 *A game of the extensive form with unawareness satisfies perfect recall (I6) if and only if for any player $i \in I$, $n_1, n_k, n'_\ell \in \mathbf{D}_i$ with $h_i(n_k) = h_i(n'_\ell)$, $n_1 <_{a_i} n_k$ for some $a_i \in A_i(n_1)$ implies $n'_1 <_{a_i} n'_\ell$ for some n'_1 with $h_i(n'_1) = h_i(n_1)$.*

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PROOF. "⇒": Suppose by contradiction that $x <_a y$ for some $a \in A_i(x)$ but there does not exist x' such that $x' <_a y'$ with $h_i(x') = h_i(x)$. Then there exists a path $x, n_2, ..., n_k$ with $y = n_k$ for which there is no path $x', n_2', ..., n_\ell'$ with $y' = n_\ell'$ such that $h_i(x) = h_i(x')$ and the action taken at x' along the path x', n_2' is a. But this contradicts I6.

"⇐": Suppose not. Then there are two nodes n_1 and n_k with $n_1 \neq n_k$ with the path $n_1, n_2, ..., n_k$ such that at n_1 player i takes action a_i along the path, but there is some $n' \in h_i(n_k)$ where there is no node $n_1' \neq n'$ where there is a path $n_1', n_2', ..., n_\ell' = n'$, $h_i(n_1') = h_i(n_1)$ and player i takes action a_i at n_1' along the path. But this means that for player i, $h_i(n_k) = h_i(n_\ell')$ and $n_1 <_{a_i} n_k$ but there is no n_1' such that $n_1' <_{a_i} n_\ell'$ with $h_i(n_1') = h_i(n_1)$, a contradiction. □

In this form, the reader can more easily see that with respect to the perfect recall condition I6, the nodes n_1 and n_k must be in the same tree T, and the nodes n_1' and n_ℓ' must be in the same tree T', but T and T' may be different. We provide more elaboration on the perfect recall condition in Section 4.

We denote by H_i the set of player i's information sets in all trees. For an information set $h_i \in H_i$, we denote by T_{h_i} the tree containing h_i. For two information sets h_i, h_i' in a given tree T, we say that h_i precedes h_i' (or that h_i' succeeds h_i) if for every $n' \in h_i'$ there is a path $n, ..., n'$ in T such that $n \in h_i$. We denote it by $h_i \rightsquigarrow h_i'$.

The following property is implied by I2 and I4 (see Heifetz, Meier, and Schipper, 2013, Remark 1): If $n', n'' \in h_i$ where $h_i = h_i(n)$ is an information set for some $n \in \mathbf{D}_i$, then $A_i(n') = A_i(n'')$. Hence, if $n \in h_i$ we write also $A_i(h_i)$ for $A_i(n)$.

Properties U0, U1, I2, and I6 imply no absent-mindedness. This follows directly from Heifetz, Meier, and Schipper (2013, Remark 2).

> No Absent-mindedness: No information set h_i contains two distinct nodes n, n' on the same path in some tree.

The Perfect recall property I6 therefore guarantees that with the precedence relation $\rightsquigarrow$ player i's information sets H_i form an unrooted tree: For every information set $h_i' \in H_i$, the information sets preceding it $\{h_i \in H_i : h_i \rightsquigarrow h_i'\}$ are totally ordered by $\rightsquigarrow$.

Confined awareness (U0) and Perfect recall (I6) imply that a player cannot become unaware during the play (see Heifetz, Meier, and Schipper, 2013, Remark 6).

DA Awareness may only increase along a path: If there is a path $n, ..., n'$ in some subtree T'' such that player i is active in n and n', and $h_i(n) \subseteq T$ while $h_i(n') \subseteq T'$, then $T' \succeq T$.

To model unawareness proper, we impose as in Heifetz, Meier, and Schipper (2013) additional properties. They parallel properties of static unawareness structures in Heifetz, Meier, and Schipper (2006):

U4 Subtrees preserve ignorance: If $T \preceq T' \preceq T''$, $n \in T''$, $h_i(n) \subseteq T$ and T' contains the copy $n_{T'}$ of n, then $h_i(n_{T'}) = h_i(n)$.

U5 Subtrees preserve knowledge: If $T \preceq T' \preceq T''$, $n \in T''$, $h_i(n) \subseteq T'$ and T contains the copy n_T of n, then $h_i(n_T)$ consists of the copies that exist in T of the nodes of $h_i(n)$.

It is known that U5 implies U3, see Heifetz, Meier, and Schipper (2013, Remark 3):

U3 Subtrees preserve awareness: If $n \in T'$, $n \in h_i(n)$, $T \preceq T'$, and T contains a copy n_T of n, then $n_T \in h_i(n_T)$.

Properties U3 to U5 are illustrated in Figure 6 with an example and counterexample each.

Figure 6: Properties U3 to U5
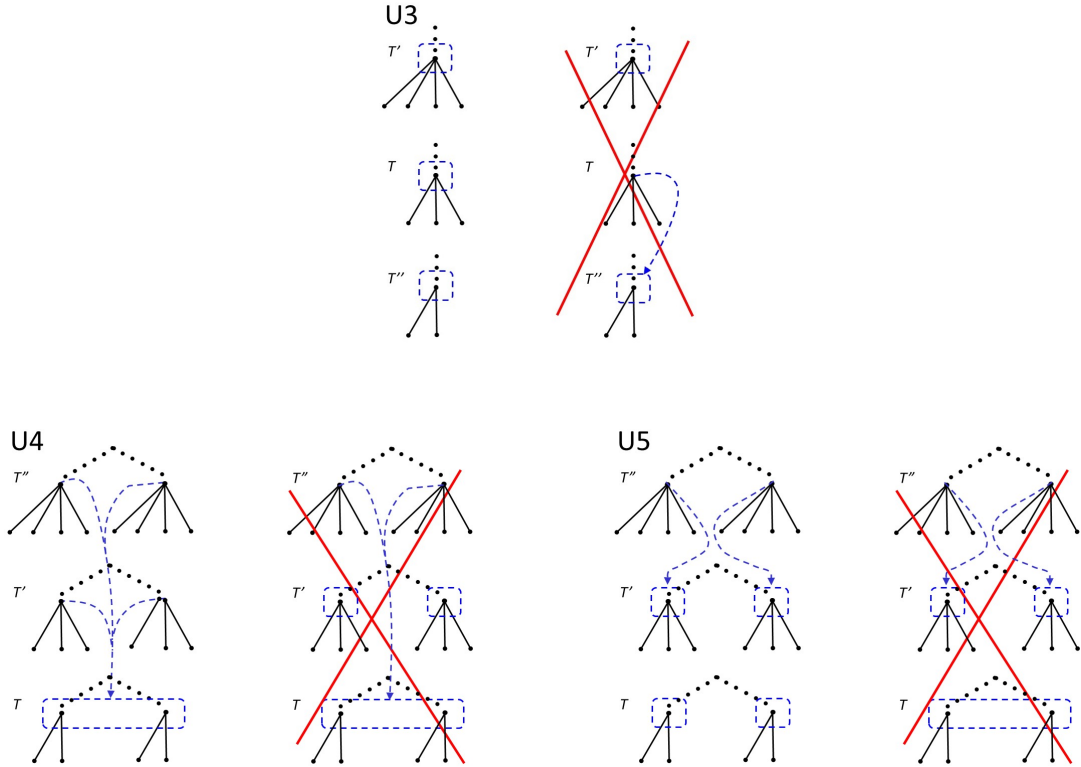
For trees $T, T' \in \mathbf{T}$ we denote by $T \rightarrowtail T'$ whenever for some node $n \in T$ and some player $i \in P(n)$ it is the case that $h_i(n) \subseteq T'$. Denote by $\hookrightarrow$ the transitive closure of $\rightarrowtail$. That is, $T \hookrightarrow T''$ if and only if there is a sequence of trees $T, T', \ldots, T'' \in \mathbf{T}$ satisfying $T \rightarrowtail T' \rightarrowtail \cdots \rightarrowtail T''$.

A *game of the extensive form with unawareness* Γ consists of a join-semilattice $\mathbf{T}$ of subtrees of a tree $\bar{T}$ satisfying properties 1–3 above, along with information sets $h_i(n)$ for every $n \in T$

with $T \in \mathbf{T}$ and $i \in P(n)$, and payoffs satisfying properties U0, U1, U4, U5, and I2-I5 above. We say that the game satisfies perfect recall if I6 also holds for all players $i \in I$.

For any game of the extensive form with unawareness Γ with set of trees $\mathbf{T}$, for any tree $T \in \mathbf{T}$, the T-*partial game* is the join-semisublattice of trees including T and also all trees T' in Γ satisfying $T \hookrightarrow T'$, with information sets as defined in Γ. A T-partial game is a game of the extensive form with unawareness, i.e., it satisfies all properties 1–3, U0, U1, U4, U5, and I2-I5 above, and we say that the game satisfies perfect recall if I6 also holds for all players $i \in I$.

We denote by H_i^T the set of player i's information sets in the T-partial game, $T \in \mathbf{T}$. This set contains not only i's information sets in the tree T but also in all trees $T' \in \mathbf{T}$ with $T \hookrightarrow T'$.

3 Strategies

For any collection of sets $(X_i)_{i \in I^0}$ we denote by

$$X := \prod_{i \in I^0} X_i, \quad X_{-i} := \prod_{j \in I^0 \backslash \{i\}} X_j$$

with typical elements x and x_{-i} respectively. For any collection of sets $(X_i)_{i \in I^0}$ and any tree $T \in \mathbf{T}$, we denote by X_i^T the set of objects in X_i restricted to the tree T and analogously for X^T and X_{-i}^T, where "restricted to the tree T" will become clear from the definitions below.

A *pure strategy* for player $i \in I$,

$$s_i \in S_i := \prod_{h_i \in H_i} A(h_i)$$

specifies an action of player i at each of her information sets $h_i \in H_i$. We let

$$s_0 \in S_0 := \prod_{n \in \mathbf{D}_0} A_0(n)$$

denote the "strategy" of nature, with $\mathbf{D}_0$ denoting the "decision" nodes of nature.

With the strategy s_i, at node $n \in D_i^{T_n}$ define player i's action at n to be $s_i(h_i(n))$, for $i \in I$. Thus, by U1 and I4 the strategy s_i specifies what player $i \in I$ does at each of her active nodes $n \in D_i^{T_n}$, both in the case that $n \in h_i(n)$ *and* in the case that $h_i(n)$ is a subset of nodes of a tree which is distinct from the tree T_n to which n belongs. In the first case, when $n \in h_i(n)$, we can interpret $s_i(h_i(n))$ as the action chosen by player i in node n. In the second case, when $n \notin h_i(n)$, $s_i(h_i(n))$ cannot be interpreted as the action chosen "consciously" by player i in n since she is not even aware of T_n. Instead, her state of mind at n is given by her information set $h_i(n)$ in a tree lower than T_n (denoted by $T_{h_i(n)}$). Thus, $s_i(h_i(n))$ is the physical move of player i in n in tree T_n induced by her "consciously" chosen action at her information set $h_i(n)$ in tree $T_{h_i(n)}$ (with $T_n \succ T_{h_i(n)}$). As an example, consider the game in Figure 7. The information

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set at node n in tree T_n lies in the lower tree $T_{h(n)}$ that misses the action "middle". This is indicated by the blue arrow and disk. When the player chooses "left" in $T_{h(n)}$ (as indicated by the red solid line beside the left edge), it induces also an action "left" at node n in tree T_n (as indicated by the red dashed line beside the left edge).

Figure 7: Action induced by a strategy
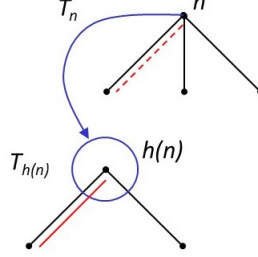
In a game of the extensive form with unawareness Γ the tree $\bar{T} \in \mathbf{T}$ represents the physical paths in the game; every tree in $\mathbf{T}$ that contains an information set represents the subjective view of the feasible paths in the mind of a player, or the view of the feasible paths that a player believes that another player may have in mind, etc. Moreover, as the actual play in $\bar{T}$ unfolds, a player may become aware of paths of which she was unaware earlier, and the way she views the game may be altered. Thus, in a game of the extensive form with unawareness, a strategy cannot be conceived as an ex ante plan of action. Formally, a strategy of player i is a list of answers to the questions "what would player $i \in I$ do if h_i were the set of nodes she considered as possible?", for $h_i \in H_i$ (and analogous for nature). A strategy of a player becomes meaningful as an object of beliefs of other players. How "much" of a player's strategy other players can conceive depends on their awareness given by the tree in which their information set is located. This leads to the notion of T-partial strategy. For a strategy $s_i \in S_i$ and a tree $T \in \mathbf{T}$, we denote by s_i^T the strategy in the T-partial game induced by s_i (i.e., $s_i^T(h_i) = s_i(h_i)$ for every information set $h_i \in H_i^T$ of player i in the T-partial game). (Recall that H_i^T not only contains information sets in the tree T but also in trees $T' \in \mathbf{T}$ with $T' \preceq T$.)

A *mixed strategy* of player $i \in I^0$, $\sigma_i \in \Delta(S_i)$, specifies a probability distribution over player i's set of pure strategies. With this notation, we let σ_0 be the probability distribution over "strategies" of nature. As mentioned already in the introduction, we don't consider mixed (or behavior) strategies necessarily as an object of choice of players but rather a conjecture over how a player would play.

A *behavior strategy* for player $i \in I$,

$$\beta_i \in B_i := \prod_{h_i \in H_i} \Delta(A_i(h_i))$$

is a collection of independent probability distributions, one for each of player i's information

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sets $h_i \in H_i$, where $\beta_i(h_i)$ specifies a mixed action in $\Delta(A_{h_i})$. With the behavior strategy β_i, at node $n \in D_i^{T_n}$ define player i's mixed action at n to be $\beta_i(h_i(n))$. Thus, the behavior strategy β_i specifies the mixed action of player $i \in I$ at each of her active decision nodes $n \in D_i^{T_n}$, both in the case that $n \in h_i(n)$ and in the case that $h_i(n)$ is a subset of nodes of a tree which is distinct from the tree T_n to which n belongs. It may be the case that $A_i(n) \supset A_i(h_i(n))$. Yet, we have automatically that β_i does not assign probabilities to actions in $A_n \setminus A_{h_i(n)}$. (I.e., at the decision node n of the richer tree T_n player i may have more actions than she is aware of at $h_i(n)$. In such a case, she is unable to use actions that she is unaware of.) With respect to nature, we let $\beta_0 \in B_0 = \prod_{n \in D_0} \Delta(A_0(n))$.

We say that a strategy profile $s = (s_j)_{j \in I} \in S$ *reaches a node* $n \in T$ if n is on the path of play in T given the players' actions and nature's moves $\left(s_j^T (h_j(n')) \right)_{j \in P(n')}$ in nodes $n' \in T$. Notice that by property (I4) ("no imaginary actions"), $s_j^T (h_j(n'))_{j \in I}$ is indeed well defined: even if $h_j(n') \nsubseteq T$ for some $n' \in T$, $\left(s_j^T (h_j(n')) \right)_{j \in P(n')}$ is a profile of actions which is actually available in T to the active players $j \in P(n')$ and possibly nature at n'. We say that a strategy profile $s \in S$ *reaches* the information set $h_i \in H_i$ if s reaches some node $n \in h_i$. We say that the strategy $s_i \in S_i$ *allows the node n to be reached* if there is a strategy profile $s_{-i} \in S_{-i}$ of the other players (and possibly nature) such that the strategy profile (s_i, s_{-i}) reaches n. Analogously, we say that the strategy profile $s_{-i} \in S_{-i}$ *allows the information set h_i to be reached* if there exists a strategy $s_i \in S_i$ such that the strategy profile (s_i, s_{-i}) reaches h_i. For each player $i \in I$, denote by $H_i(s)$ the set of information sets of i that are reached by the strategy profile s. This set may contain information sets in more than one tree.

We extend the definitions of information set reached to mixed and behavior strategies in the obvious way by considering nodes/information sets reached with strict positive probability.

4 Kuhn's Theorem

In games of the extensive form without unawareness but with perfect recall, Kuhn's Theorem asserts that for every mixed strategy profile there is an equivalent behavior strategy profile. Kuhn's Theorem can be extended to games of the extensive form with unawareness using a notion of equivalence based on the notion of reaching nodes. For any node n, any player $i \in I^0$, and any opponents' profile of strategies s_{-i} (including nature if any), let $\rho(n \mid \beta_i, s_{-i})$ and $\rho(n \mid \sigma_i, s_{-i})$ denote the probability that (β_i, s_{-i}) and (σ_i, s_{-i}) reach node n, respectively. For any player $i \in I^0$, a mixed strategy σ_i and a behavior strategy β_i are *equivalent* if for every profile of opponents' strategies $s_{-i} \in S_{-i}$ and every node $n \in \mathbf{N}$ of the game of the extensive form with unawareness $\rho(n \mid \sigma_i, s_{-i}) = \rho(n \mid \beta_i, s_{-i})$.

Let $S_i(n)$ be the set of all strategies of player i that allows n to be reached. That is, if $n \in T$ then $s_i \in S_i(n)$ if and only if there exist $s_{-i} \in S_{-i}$ such that the profile $(s_j^T(h_j(n')))_{j \in P(n')}$ in

$n' \in T$ reaches n.

In the following, we extend for better comparison a textbook proof of Kuhn's Theorem (e.g., Maschler, Solan, and Zamir, 2013, Chapter 6) to the more complicated set up of games of the extensive form with unawareness. It allows us to show which modifications are necessary as compared to standard games, and emphasizes that the basic idea of proof remains the same although the setting is considerably more complicated. We start with a lemma that is crucial for the proof of the main theorem.

Lemma 1 *Consider a game of the extensive form with unawareness Γ. If Γ satisfies perfect recall (i.e., I6), then for any player $i \in I$, $n \in N$ with $h_i(n) \in H_i$, and $n' \in h_i(n)$, $S_i(n) = S_i(n')$.*

Note the difference to standard games. First, games of the extensive form with unawareness involve forests of trees rather than just trees. Second, the perfect recall property applies now to information sets across trees. Third, the sets $S_i(n)$ are different from corresponding sets in standard games because n may be in a tree different from the tree "housing" the information set $h_i(n)$ of player i at n. This is significant because strategies are defined for each information set (rather than decision node) of player i.

PROOF OF LEMMA 1. Since Γ satisfies perfect recall, we have by Remark 1 that $n' \in h_i(n)$ implies $E_i(n') = E_i(n)$. Thus, the same information sets of player i that are reached along the path to n are reached along the path to n' (even though the information sets might appear in a subtree lower than the one containing the path to n). Moreover, at each of those information sets the same action is required to move along the path to n as to move along to the path to n'. Hence, any strategy that allows n to be reached also allows n' to be reached and vice versa. $\square$

We are now ready to state an extension of Kuhn's Theorem to games of the extensive form with unawareness.

Theorem 1 *In every game of the extensive form with unawareness, if player i has perfect recall, then for every mixed strategy of player i there exists an equivalent behavior strategy.*

PROOF. The theorem is proved in three steps. The first step defines the candidate of the behavior strategy. The second step shows that it is well-defined. The third step shows it to be equivalent to the mixed strategy.

Let σ_i be a mixed strategy of player i.

First, we define a candidate for the equivalent behavior strategy. Let $n \in \mathbf{D}_i$. Since $n \in \mathbf{D}_i$ we have $A_i(n) \neq \emptyset$. For any action $a_i \in A_i(n)$ of player i at n, define $S_i(n, a_i) := \{s_i \in$

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$S_i(n) : s_i(h_i(n)) = a_i\}$. That is, any strategy in $S_i(n, a_i)$ allows n to be reached and also prescribes action a_i at information set $h_i(n)$. This definition makes sense: First, recall that strategies ascribe actions to information sets (rather than nodes). Moreover, by definition of Γ there is an information set of player i at n that we denote by $h_i(n)$. There are two cases: First, $n \in h_i(n)$. In this case, for any $n' \in h_i(n)$ we have $A_i(n) = A_i(n')$ (Heifetz, Meier, and Schipper, 2013, Remark 1). Thus, we write $A_i(h_i(n))$ for actions available at any node in $h_i(n)$. Second, $n \notin h_i(n)$. (That's the case when n is in a tree more expressive than $h_i(n)$.) By I4 (No imaginary actions), for any $n' \in h_i(n)$ we have $A_i(n') \subseteq A_i(n)$. Note also that $A_i(n') \neq \emptyset$ since $n' \in \mathbf{D}_i$. If $a_i \in A_i(n) \setminus A_i(n')$, then $S_i(n, a_i) = \emptyset$ since no strategy of player i can ascribe an action to n that is not available at $h_i(n)$. Hence, in the following we consider sets $S_i(n, a_i)$ for $a_i \in A_i(h_i(n))$.

If player i's mixed strategy assigns strict positive probability to strategies reaching n, i.e., if $\sum_{s_i \in S_i(n)} \sigma_i(s_i) > 0$, then define for each action $a_i \in A_i(h_i(n))$,

$$\beta_i(h_i(n))(a_i) := \frac{\sum_{s_i \in S_i(n,a_i)} \sigma_i(s_i)}{\sum_{s_i \in S_i(n)} \sigma_i(s_i)}. \tag{1}$$

Otherwise, if $\sum_{s_i \in S_i(n)} \sigma_i(s_i) = 0$, define $\beta_i(h_i(n))$ in an arbitrary way provided that it constitutes a probability measure over actions available at $h_i(n)$. E.g., for all $a_i \in A_i(h_i(n))$,

$$\beta_i(h_i(n))(a_i) := \frac{1}{|A_i(h_i(n))|}. \tag{2}$$

Second, we want to show that β_i is well-defined. I.e., for each of player i's information sets $h_i \in H_i$, $\beta_i(h_i)$ is a probability measure on $A_i(h_i)$. Moreover, β_i is independent of player i's decision nodes. For the latter, it suffices to demonstrate it for the case $\sum_{s_i \in S_i(n)} \sigma_i(s_i) > 0$. Since Γ has perfect recall, i.e., Γ satisfies I6, we have by Lemma 1 that for any $n' \in h_i(n)$, $S_i(n') = S_i(n)$. Again, since $n' \in h_i(n)$, $S_i(n, a_i) = S_i(n', a_i)$ for all $a_i \in A_i(h_i)$. Observe that both the numerator and the denominator of the right-hand side of Equation (1) are independent of nodes in $h_i(n)$.

To show that for all $n \in \mathbf{D}_i$, $\beta_i(h_i(n))$ is a probability measure over $A_i(h_i(n))$, note first that, in the case of $\sum_{s_i \in S_i(n)} \sigma_i(s_i) = 0$, this follows directly from Equation (2).

If $\sum_{s_i \in S_i(n)} \sigma_i(s_i) > 0$, then Equation (1) defines a probability distribution over $A_i(h_i)$. To see this note that since $\sigma_i(s_i) \geq 0$ for all $s_i \in S_i$, both the numerator and the denominator are non-negative and hence $\beta_i(h_i)(a_i) \geq 0$ for all $a_i \in A_i(h_i)$. For any $n \in \mathbf{D}_i$ and $a_i, a_i' \in A_i(h_i(n))$ with $a_i \neq a_i'$, $S_i(n, a_i) \cap S_i(n, a_i') = \emptyset$. Moreover, $\bigcup_{a_i \in A_i(h_i(n))} S_i(n, a_i) = S_i(n)$. Thus, $\sum_{a_i \in A_i(h_i(n))} \sum_{s_i \in S_i(n,a_i)} \sigma_i(s_i) = \sum_{s_i \in S_i(n)} \sigma_i(s_i)$. It follows that $\sum_{a_i \in A_i(h_i)} \beta_i(h_i)(a_i) = 1$.

The third and last step is to show that the behavior strategy β_i is equivalent to the mixed strategy σ_i. The probability ρ_i that node n is allowed to be reached given mixed strategy σ_i of player i is: $\rho_i(n \mid \sigma_i) := \Sigma_{s_i \in S_i(n)} \sigma_i(s_i)$. When $S_i(n) = \emptyset$, $\rho_i(n \mid \sigma_i) := 0$. The probability

ρ_{-i} that node n is allowed to be reached given some profile of player i's opponents' (pure) strategies $s_{-i} \in S_{-i}$ is: $\rho_{-i}(n \mid s_{-i}) := \Sigma_{s_{-i} \in S_{-i}(n)} s_{-i}$ which just equals to 1 if $s_{-i} \in S_{-i}(n)$ and 0 if $s_{-i} \notin S_{-i}(n)$. Then, as the strategies between players are independent, we obtain that $\rho(n \mid \sigma_i, s_{-i}) = \rho_i(n \mid \sigma_i) \times \rho_{-i}(n \mid s_{-i}) = \Sigma_{s_i \in S_i(n)} \sigma_i(s_i)$ if $s_{-i} \in S_{-i}(n)$ and 0 otherwise. We obtain an analogous result for β_i due to the independence of strategies between players. For any $n \in \mathbf{N}$ we also have $\rho(n \mid \beta_i, s_{-i}) = \rho_i(n \mid \beta_i)$ if $s_{-i} \in S_{-i}(n)$ and 0 otherwise. As a result, in what follows we can focus without loss of generality on the case where $s_{-i} \in S_{-i}(n)$.

Fix a node $n \in \mathbf{D}_i \cup \mathbf{Z}$ and let $n_i^1, n_i^2, ..., n_i^L$ be a sequence of decision nodes of player i along the path from the root to n, not including n. By definition of Γ, there exists an information set of player i for each of the decision nodes $n_i^1, n_i^2, ..., n_i^L$. If $L = 0$, then player i has no information sets on the path from the root to n (not including n). In such a case, $S_i(n) = S_i$ and for any $s_{-i} \in S_{-i}$ we naturally define $\rho_i(n \mid \beta_i) = 1$. We therefore have $\rho(n \mid \beta_i, s_{-i}) = 1$ for $s_{-i} \in S_{-i}(n)$. Also, in this case $\rho(n \mid \sigma_i, s_{-i}) = \rho_i(n \mid \sigma_i) = \sum_{s_i \in S_i(n)} \sigma_i(s_i) = \Sigma_{s_i \in S_i} \sigma_i(s_i) = 1$ for $s_{-i} \in S_{-i}(n)$. Hence, β_i and σ_i are equivalent in this case.

Suppose now the case $L > 0$. Let $a_i^{n_i^\ell} \in A_i(h_i(n_i^\ell))$ denote the action of player i at node n_i^ℓ, $\ell = 1, ..., L$, that leads to $n_i^{\ell+1}$ in the case $\ell = 1, ..., L-1$ and to n in the case $\ell = L$. Again, assume without loss of generality that $s_{-i} \in S_{-i}(n)$.

We have

$$\rho(n \mid \beta_i, s_{-i}) = \rho_i(n \mid \beta_i) = \prod_{\ell=1}^{L} \beta_i(h_i(n_i^\ell))(a_i^{n_i^\ell}). \tag{3}$$

Assume without loss of generality that σ_i allows n to be reached (otherwise $\rho_i(n \mid \sigma_i) = \rho_i(n \mid \beta_i) = 0$). By definition of β_i,

$$\rho(n \mid \beta_i, s_{-i}) = \prod_{\ell=1}^{L} \frac{\sum_{s_i \in S_i(n_i^\ell, a_i^{n_i^\ell})} \sigma_i(s_i)}{\sum_{s_i \in S_i(n_i^\ell)} \sigma_i(s_i)}, \tag{4}$$

which is well-defined since σ_i is assumed to allow n to be reached and therefore also allows $n_i^1, ..., n_i^L$ to be reached.

Note that $S_i(n_i^{\ell+1}) = S_i(n_i^\ell, a_i^\ell)$. Thus

$$\sum_{s_i \in S_i(n_i^\ell, a_i^\ell)} \sigma_i(s_i) = \sum_{s_i \in S_i(n_i^{\ell+1})} \sigma_i(s_i)$$

and

$$\rho(n \mid \beta_i, s_{-i}) = \prod_{\ell=1}^{L} \frac{\sum_{s_i \in S_i(n_i^{\ell+1})} \sigma_i(s_i)}{\sum_{s_i \in S_i(n_i^\ell)} \sigma_i(s_i)}, \tag{5}$$

(where we take $n_{L+1} = n$) is the telescopic product for which the numerator of the ℓ-th term of the product equals to the denominator of the $\ell + 1$-th term of the product. Adjacent product

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terms cancel each other out. Thus,

$$\rho(n \mid \beta_i, s_{-i}) = \frac{\sum_{s_i \in S_i(n)} \sigma_i(s_i)}{\sum_{s_i \in S_i(n_i^1)} \sigma_i(s_i)}. \tag{6}$$

Since n_i^1 is in player i's first information set on the path towards n, we have $S_i(n_1) = S_i$. Hence,

$$\rho(n \mid \beta_i, s_{-i}) = \frac{\sum_{s_i \in S_i(n)} \sigma_i(s_i)}{\sum_{s_i \in S_i} \sigma_i(s_i)}. \tag{7}$$

Since trivially any strategy of player i allows her first information set to be reached, we have $\Sigma_{s_i \in S_i} \sigma_i(s_i) = 1$. Thus,

$$\rho(n \mid \beta_i, s_{-i}) = \frac{\sum_{s_i \in S_i(n)} \sigma_i(s_i)}{1} = \rho_i(n \mid \sigma_i, s_{-i}). \tag{8}$$

This completes the proof of the theorem. □

While most textbooks in game theory just focus on the arguably more relevant direction of Kuhn's Theorem for standard games without unawareness as Theorem 1 does for games with unawareness, Kuhn's (1953) original theorem is a characterization. That is, he proves that in every game of the extensive form without unawareness (in which all decision nodes have at least two actions), the game satisfies perfect recall *if and only if* for every mixed strategy there is an equivalent behavior strategy. However, under unawareness the converse of Theorem 1 does not hold. That is, there are some games of the extensive form with unawareness for which there exists an equivalent behavior strategy for every mixed strategy of player i, but player i does not have perfect recall. We present one such example in Figure 8. There are two trees, $\bar{T} \succ T$. There is a single player 1 moving at two information sets in this game, both of which are in T as indicated in blue. Since Player 1 does not have information sets in $\bar{T}$, she is never aware of the right action at node n. Consequently, we have an equivalence of mixed and behavior strategies. This is because Player 1 can achieve at T any distribution over terminal nodes with either notion of strategy, and because right is never playable at $h_1(n)$ (by either notion of strategy) the distribution induced at $\bar{T}$ must be identical to that at T. Perfect recall does not hold however, as right is played by Player 1 on the path from n to n'' in $\bar{T}$ but there is no path in T where right is played before n'''. That is, the game of Figure 8 is ruled out by the assumption of perfect recall in the theory of games of the extensive form with unawareness by Heifetz, Meier, and Schipper (2013).

Figure 8 raises several deeper questions. First, do we have the "right" notion of perfect recall for games with unawareness? According to Kuhn (1953, p. 213), perfect recall "is equivalent to the assertion that each player is allowed by the rules of the game to remember everything she knew at previous moves and all of her choices at those moves."[9] In games with unawareness,

[9] See also Ritzberger (1999) for a characterization of perfect recall and the discussion of perfect recall in Ritzberger (2002).

Figure 8: Counterexample for the converse of Theorem 1
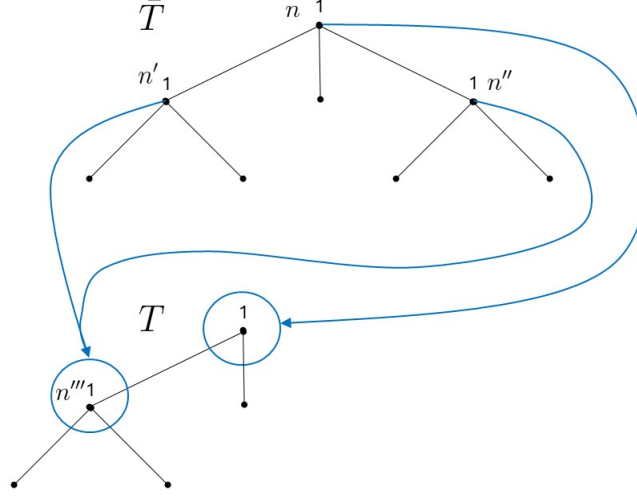
a player may be unaware of some of her own actions. If we were to allow a player to take an action that she is unaware of without her realizing that she took such an action, then she could not know and remember her choice at that previous move, hence violating perfect recall. This is exactly what happens in the example of Figure 8. The rules of physical moves of the game allow the player to go right initially even though she does not realize that she could go right. Moreover, even after going right, she does not realize that she went right leading her mistakenly to believe she went left. In contrast to standard games without unawareness, in games with unawareness the set of nodes where the player has the same state of mind can be larger than the set of nodes that she considers possible. This gives rise to situations where at her information set the player is unaware of some previous action that allows the node at which she has this information set. The perfect recall assumption rules this out. That is, at an information set the player cannot be unaware of her prior *own* action that allows a node at which she has this information set, ruling out scenarios such as in Figure 8. Note that this example only violates the memory condition for actions. It does satisfy the memory condition for information sets, i.e., at the latter information set in T, the player remembers everything she knew at the initial information set at T (all the while being completely unaware of the right action at n).

Another natural question that arises here is regarding the notion of strategy: Should the play of an action that the player is unaware of be allowed? In Figure 8, equivalence between strategies is only obtained due to right not being playable at $h_1(n)$. If within our notions of mixed and behavior strategies we allow the player to play right (despite being unaware of it) then equivalence would no longer hold as mixed strategies would be able to achieve a larger

set of outcomes than behavior strategies.[10] In the sense that the tree $\bar{T}$ models "actual" or "objective" play, one possible way to look at this question is whether we should allow the actual play and intended play (as perceived by the consciousness of the player) to diverge.[11] If we think of it in relation to real life situations, it certainly does not seem unreasonable to allow this. For instance, a fighter pilot during a stressful dogfight while intending to press one button in a complex cockpit may accidentally press another button he does not even consider at that moment and may entirely fail to realize that he has pressed the wrong button. In day to day life it is also not uncommon for us to make a mistake we cannot even conceive of at the time of acting, for example making typos in the essay we are writing due to pressing the wrong keys and unknowingly invoking shortcuts or commands in the software, all the while remaining completely unaware of the error. Capturing the above situations within our model with a generalized notion of strategy and a weaker notion of perfect recall is an interesting possibility and by no means do we dismiss it conceptually as it would allow us to study problems of *situation awareness* in military theory. However, to the extent that we aim to model rational play as the benchmark for real behavior, we want to rule out that players choose to play actions of which they are unaware of. After all, intentional choice can only be done among options that one can conceive of. And by restricting actions to only what is available to a player at an information set, we naturally rule out situations where the player plays actions she is not aware of in the "objective" tree $\bar{T}$.

4.1 Equivalence in Realization of Nodes

In games of the extensive form with unawareness there are two distinct notions of a strategy profile being consistent with a node. The first notion we introduced already at the end of Section 3 and called it a "strategy profile reaching a node". While a player may expect a strategy profile to reach a node, it can be the case in games with unawareness that a different node actually occurs. This is because the player is unaware of actions that a player with more awareness may take (since each player just considers the partial strategies consistent with her awareness level). This begs the question whether strategies that are equivalent with respect to nodes reached are also equivalent with respect to nodes that actually occur. Note that both notions of a node being consistent with a strategy are relevant. The notion of a strategy reaching a node is relevant for extensive-form rationalizability, more recently also called strong rationalizability due to Pearce (1984) and Battigalli (1997) and extended to games of the extensive form with unawareness by Heifetz, Meier, and Schipper (2013) whereas the notion of a node occurring

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[10] For example, consider mixed strategy $\frac{1}{2}(left, left) + \frac{1}{2}(right, right)$, where the first entry in the vector refers to the action chosen at the initial information set and the second entry refers to the action chosen at the latter information set. There is no equivalent behavior strategy that achieves the distribution over outcomes in $\bar{T}$.

[11] See Battigalli and De Vito (2021) for recent work distinguishing between intended play and actual play in standard games without unawareness.

with a strategy is crucial for self-confirming equilibrium (Battigalli and Guaitoli, 1997, and Battigalli and Bordoli, 2024) as extended to games of the extensive form with unawareness by Schipper (2021). As we will observe, the notions of node reached and node occurring coincide in games that feature only one tree (as in standard games without unawareness), but can diverge in games with unawareness that have more than one tree. Consequently, our model allows us to make explicit the differences between these two notions, compare them accordingly and illustrate some relationships between subjective and "objective" play.

We say that node $n \in \bar{T}$ in the upmost tree $\bar{T}$ *occurs* with strategy profile $s = (s_j)_{j \in I} \in S$ if n is on the path of play in $\bar{T}$ given the players' actions and nature's moves $(s_j (h_j(n')))_{j \in P(n')}$ in nodes $n' \in \bar{T}$. We extend the notion to any node in any tree by saying that node $n \in T$ occurs with strategy profile $s = (s_j)_{j \in I} \in S$ if there is $n' \in \bar{T}$ s.t. $n'_T = n$ and n' occurs with s. This is well-defined because $\mathbf{T}$ is a join semi-lattice and therefore there will always be an upmost tree. In particular, for any $T \in \mathbf{T}$ and $n \in T$ there is a node $n' \in \bar{T}$ such that $n'_T = n$. We say that the strategy $s_i \in S_i$ *allows the node n to occur* if there is a strategy profile $s_{-i} \in S_{-i}$ of the other players (and possibly nature) such that n occurs given the strategy profile (s_i, s_{-i}).

We say that information set $h_i \in H_i$ occurs with strategy profile $s \in S$ if some node $n \in \mathbf{D}_i$ with $h_i(n) = h_i$ occurs with s. Note that for this definition we do not require $n \in h_i$. Analogously, we say that the strategy $s_i \in S_i$ *allows the information set h_i to occur* if there is a strategy profile $s_{-i} \in S_{-i}$ of the other players (and possibly nature) such that h_i occurs given the strategy profile (s_i, s_{-i}).
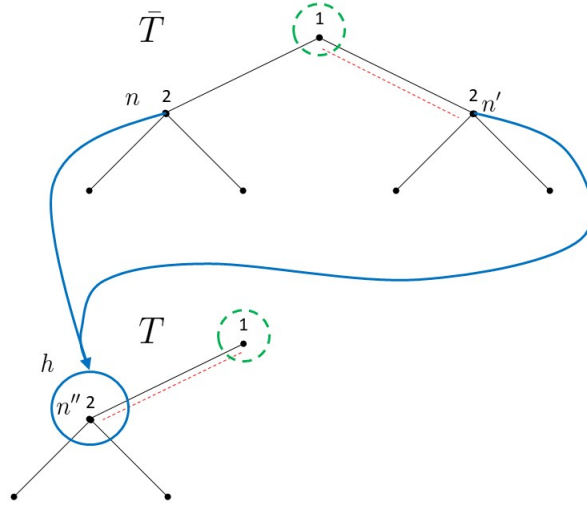
Figure 9: Illustration of Occur vs. Reached
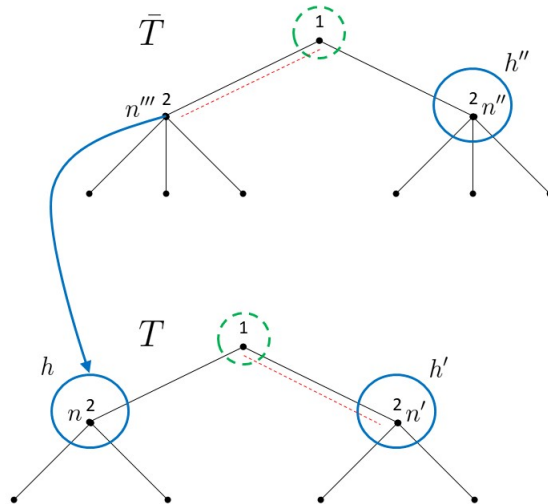
The following two examples will help to clarify the definition and its difference to the notion

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of a strategy reaching a node/information set. Consider first the example in Figure 9. There are two trees, $\bar{T} \succ T$, and two players, 1 and 2. Player 1 moves first. If she moves left in tree $\bar{T}$, then player 2 remains unaware of her middle action. This is shown in Figure 9 by the blue arrow and disk (i.e., information set h in T) upon player 1 moving left. Otherwise, if player 1 moves right in tree $\bar{T}$, player 2 becomes aware of middle (i.e., information set h''). (Player 1's initial information sets are indicated by disks with green dashed boundaries.) Consider the strategy of player 1 indicated by the red dashed edges. This strategy allows only nodes n''' and n' to be reached. As player 1 moves first, any strategy profile s of players which includes the above strategy for player 1 will also reach n''' and n'. Yet, the nodes that are allowed to occur with this strategy are n''' and n. Again, n''' and n will also occur given strategy profile s as player 1 moves first. Thus, this example shows that the nodes reached (resp., a strategy allows to be reached) may differ from nodes occurring (resp., a strategy allows to occur). Note though that they are not disjoint and that this intersection contains a node n''' in the upmost tree $\bar{T}$. In terms of information sets, the strategy profile s reaches only h' but the only information set occurring with this strategy profile is h. Thus, the example demonstrates that the information sets reached by a strategy profile may even be disjoint from the information sets occurring.

Figure 10: Another Illustration of Occur vs. Reached

The example in Figure 9 has the feature that if an information set is reached (resp., occurs) with a strategy then also a node in this information set is reached (resp., occurs). With respect to the notion of occurring, this may not be the case in general as the next example shows. In Figure 10 there are also two trees, $\bar{T}$ and T. Obviously, tree $\bar{T}$ is more expressive than T as it contains also the right action for player 1 (and all actions that follow). Player 2 is unaware of

player 1's right action and remains unaware of it even if player 1 takes the right action. This is indicated by the blue information set belonging to player 2 in tree T. That is, even if player 1 chooses right, player 2 thinks that left has been played by player 1. Consider the strategy of player 1 indicated by the red dashed edges and let s again denote a strategy profile of players that includes the above strategy of player 1. With respect to nodes, the strategy profile s reaches n' and n''. Yet, only n' occurs with this strategy profile. With respect to information set h, it is both reached and occurs given strategy profile s. Note though that h occurs despite the fact that its only element node n'' does not occur with the strategy profile s. This is not a defect of the notion of a node/information set occurring but simply reflects the fact that in a game with unawareness the node occurring may not be congruent with what the player believes to occur in her state of mind. Note though that a player is not deluded either as she just misses an important fact rather than "making things up".

We summarize the examples:

Remark 3 *In a game of the extensive form with unawareness, if s reaches n then it is not necessarily the case that n occurs with s. Similarly, if n occurs with s then it is not necessarily the case that s reaches n. Moreover, information set h may occur with a strategy profile s even though n with $h = \{n\}$ does not occur with s.*

For the upmost tree $\bar{T}$ the following observation follows directly from the definitions:

Remark 4 *Consider a game of the extensive form with unawareness with the upmost tree $\bar{T}$. A strategy profile s reaches $n \in \bar{T}$ if and only if n occurs with s.*

The observation means that the notions of node reached and node occurring with a strategy really depend on how these notions apply to less expressive trees $T \prec \bar{T}$. The notion of "reached" invokes the T-partial strategies to determine which node in T is reached and is thus a more subjective notion (from the point of view of a player who considers tree T and lower trees). The notion of "occur" invokes actions induced by the strategies in the highest possible tree. Thus it models the "actual" or "objective" play.

The following corollary follows now directly from the fact that any standard game of the extensive form (i.e., without unawareness) features just one tree.

Corollary 1 *In a standard game of the extensive form (i.e., without unawareness), a strategy profile s reaches n if and only if n occurs with s.*

We extend the definitions of information sets occurring to behavior and mixed strategies in the obvious way by considering nodes/information sets occurring with strict positive probability.

Let $N(s)$ denote the set of nodes in $\mathbf{N}$ that are reached with strategy profile s. Moreover, denote by $O(s)$ the set of nodes in $\mathbf{N}$ that occur with strategy profile s. We can now relate the notions of a node being reached and a node occurring with the following lemma:

Lemma 2 *Consider a game of the extensive form with unawareness. For any player $i \in I^0$ and strategies $s_i, s_i' \in S_i$, if $N(s_i, s_{-i}) = N(s_i', s_{-i})$ then $O(s_i, s_{-i}) = O(s_i', s_{-i})$ for any $s_{-i} \in S_{-i}$. The converse does not necessarily hold.*

PROOF.   For all $s_{-i} \in S_{-i}$, if $N(s_i, s_{-i}) = N(s_i', s_{-i})$ then $N(s_i, s_{-i}) \cap \bar{T} = N(s_i', s_{-i}) \cap \bar{T}$. By Remark 4 $O(s_i, s_{-i}) \cap \bar{T} = N(s_i, s_{-i}) \cap \bar{T}$ and $O(s_i', s_{-i}) \cap \bar{T} = N(s_i', s_{-i}) \cap \bar{T}$. Hence, $O(s_i, s_{-i}) \cap \bar{T} = O(s_i', s_{-i}) \cap \bar{T}$. By definition of node occurring, $O(s_i, s_{-i}) = O(s_i', s_{-i})$.

For the converse, we show a counterexample. Consider the extensive with unawareness in Figure 9. Further, let s_1 ascribe action "left" in tree $\bar{T}$ and "right" in tree T. Moreover, let s_1' ascribe action "left" both in tree $\bar{T}$ and T. Then information set h occurs both with s_1 and s_1'. In fact, $O(s_1, s_{-1}) = O(s_1', s_{-1})$ for any $s_1 \in S_1$. Yet, only strategy s_1' reaches h in T while s_1 reaches h' in T. $\qquad\square$

For any node n, any player $i \in I^0$, and any opponents' profile of strategies s_{-i} (including nature if any), let $o(n \mid \beta_i, s_{-i})$ and $o(n \mid \sigma_i, s_{-i})$ denote the probability that node n occurs with (β_i, s_{-i}) and (σ_i, s_{-i}), respectively.

Remark 5 *In a game of the extensive form with unawareness, it is not necessarily the case that for each tree $T \in \mathbf{T}$, $o(\cdot \mid \sigma_i, s_{-i})$ defines a distribution over terminal nodes $Z \subseteq T$. E.g., in the example of Figure 10, no terminal node of T occurs with the strategy of player 1 indicated by the red dashed line. Thus, $o(\cdot \mid \sigma_i, s_{-i})$ may not only be subadditive but may even assign zero to the set of all terminal nodes in a given game tree. It is though a probability distribution over nodes in the upmost tree $\bar{T}$.*

We use the notion of a node occurring to define another notion of equivalence between strategy that we dub realization-equivalent. For any player $i \in I^0$, a mixed strategy σ_i and a behavior strategy β_i are *realization-equivalent* if for every profile of opponents' strategies $s_{-i} \in S_{-i}$ and every node $n \in \mathbf{N}$ of the game of the extensive form with unawareness $o(n \mid \sigma_i, s_{-i}) = o(n \mid \beta_i, s_{-i})$. Since information sets can be viewed as functions of nodes, realization-equivalent strategies are also realization-equivalent with respect to the probability of information sets occurring. This is relevant because information sets model also the player's state of mind. We would like to assure that strategies are also equivalent with respect to the states of mind that may arise along the play.

Remark 6 *If two strategies are realization-equivalent then the same information sets occur with the same probabilities with both strategies.*

Lemma 3 *In any game of the extensive form with unawareness with perfect recall, if σ_i and β_i are equivalent to each other, then they are also realization-equivalent.*

PROOF. For any $\sigma_i \in \Delta(S_i)$, $\beta_i \in B_i$, $s_{-i} \in S_{-i}$, $T \in \mathbf{T}$, $n \in T$, $o(n \mid \sigma_i, s_{-i}) = \rho(n' \mid \sigma_i, s_{-i})$ and $o(n \mid \beta_i, s_{-i}) = \rho(n' \mid \beta_i, s_{-i})$ for $n' \in \bar{T}$ such that $(n')_T = n$. Let β_i be equivalent to σ_i. Then the conclusion follows from Theorem 1. $\qquad\square$

Theorem 1 and Lemma 3 now imply immediately the following corollary:

Corollary 2 *In every game of the extensive form with unawareness, if player i has perfect recall, then for every mixed strategy of player i there exists a realization-equivalent behavior strategy.*

4.2 T-Partial Games and T-Partial Strategies

We return to the fact that in games of the extensive form with unawareness, strategies may not only be an object of choice for a player, but also an object of belief of other players.

Let $S_i^T(n)$ be the set of all T-partial strategies of player i that reach n, where one should note that $T_n \preceq T$. That is, $s_i \in S_i^T(n)$ where $T \succeq T_n$ if and only if there exists $s_{-i} \in S_{-i}^T$ such that n is on the path of play in T_n given the profile of players' actions and nature's moves $(s_j^{T_n}(h_j(n')))_{j \in P(n')}$ in nodes $n' \in T_n$.

Note that a T-partial game is a game of the extensive form with unawareness in which the join of the join-semilattice of trees is T. Thus, Lemma 1 implies immediately the following corollary:

Corollary 3 *Consider a game of the extensive form with unawareness Γ. If Γ satisfies perfect recall (i.e., I6), then for any player $i \in I$, $n \in N$ with $h_i(n) \in H_i$, and $n' \in h_i(n)$, $S_i^T(n) = S_i^T(n')$ for any $T \succeq T_n$.*

This corollary is relevant because we can view strategies of a player as objects of beliefs of other players. Yet, their beliefs are bounded by their awareness. That is, if player i arrives at information set h_i, then her awareness level is given by T_{h_i}, the tree that contains information set h_i. Thus, she forms beliefs about player j's T_{h_i}-partial strategies.

Theorem 1 implies now immediately the version for T-partial strategies.

Corollary 4 *In every game of the extensive form with unawareness, if player i has perfect recall, then for every T-partial mixed strategy of player i there exists an equivalent T-partial behavior strategy, for $T \in \mathbf{T}$.*

The original Kuhn Theorem is now a corollary for T being a least expressive tree or $\mathbf{T}$ being singleton.

It is possible to also define a notion of node occurring with a T-partial strategy profile. Yet, such a definition is not very meaningful as the notion of a node occurring aims to characterize nodes that actually (or "objectively") occur. To determine such nodes, it is crucial to consider which nodes are reached in the upmost tree $\bar{T}$. T-partial strategies, with $T \prec \bar{T}$, are by definition silent on it. Yet, every T-partial strategy can be extended to a strategy on the entire join-semilattice of trees $\mathbf{T}$. Such an extension is typically not unique. The nodes occurring will then depend which extension is considered.

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